# The Equality Relation. Paramodulation

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# The Equality Relation

- Equality $\approx$: A very important relation
- Reflexive
- Symmetric
- Transitive
- Substitute equals by equals
- When equality is used in a theorem, we need extra axioms which describe the properties of equality

# The Equality Relation

## Example 1

Theorem: Let $G$ be a group with the binary operation $\cdot$, the inverse ${}^{-1}$, and the identity $e$. If $x \cdot x = e$ for all $x \in G$, then $G$ is commutative.

Axioms:

1. For all $x, y \in G$, $x \cdot y \in G$.
2. For all $x, y, z \in G$, $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$.
3. For all $x \in G$, $x \cdot e \approx x$.
4. For all $x \in G$, $x \cdot x^{-1} \approx e$.

# The Equality Relation

**Example 1 (Cont.)**

Express the axioms and the theorem in first-order logic with equality:

(A1) $\forall x, y.\ \exists z.\ x \cdot y \approx z.$

(A2) $\forall x, y, z.\ (x \cdot y) \cdot z \approx x \cdot (y \cdot z).$

(A3) $\forall x.\ x \cdot e \approx x.$

(A4) $\forall x.\ x \cdot i(x) \approx e.$

(T) $\forall x.\ x \cdot x \approx e \Rightarrow \forall u, v.\ u \cdot v \approx v \cdot u.$

# The Equality Relation

## Example 1 (Cont.)

Take the conjunction of axioms and the negation of the theorem and bring it to the Skolem normal form. We obtain the set consisting of the clauses:

1. $x \cdot y \approx f(x, y)$.
2. $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$.
3. $x \cdot e \approx x$.
4. $x \cdot i(x) \approx e$.
5. $x \cdot x \approx e$
6. $\neg(a \cdot b \approx b \cdot a)$.

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5. $x \cdot x \approx e$
6. $a \cdot b \not\approx b \cdot a$.

Using resolution alone, we can not derive the contradiction here.

# The Equality Relation

## Example 1 (Cont.)

We need extra axioms to describe the properties of equality:

$$
\begin{array}{lll}
S: & x \cdot y \approx f(x, y). & x \not\approx y \vee y \not\approx z \vee x \approx z. \\
 & (x \cdot y) \cdot z \approx x \cdot (y \cdot z). & x \not\approx y \vee x \not\approx u \vee y \approx u. \\
 & x \cdot e \approx x. & x \not\approx y \vee u \not\approx x \vee y \approx u. \\
 & x \cdot i(x) \approx e. & x \not\approx y \vee f(z, x) \approx f(z, y). \\
 & x \cdot x \approx e. & x \not\approx y \vee f(x, z) \approx f(y, z). \\
 & a \cdot b \not\approx b \cdot a. & x \not\approx y \vee x \cdot z \approx y \cdot z. \\
K: & x \approx x. & x \not\approx y \vee z \cdot x \approx z \cdot y. \\
 & x \not\approx y \vee y \approx x. & x \not\approx y \vee i(x) \approx i(y).
\end{array}
$$

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 & x \cdot e \approx x. & x \not\approx y \vee u \not\approx x \vee y \approx u. \\
 & x \cdot i(x) \approx e. & x \not\approx y \vee f(z,x) \approx f(z,y). \\
 & x \cdot x \approx e. & x \not\approx y \vee f(x,z) \approx f(y,z). \\
 & a \cdot b \not\approx b \cdot a. & x \not\approx y \vee x \cdot z \approx y \cdot z. \\
K: & x \approx x. & x \not\approx y \vee z \cdot x \approx z \cdot y. \\
 & x \not\approx y \vee y \approx x. & x \not\approx y \vee i(x) \approx i(y).
\end{array}
$$

Unsatisfiability of this set can be proved by resolution.

# The Equality Relation

The described approach has several drawbacks:

- Every time equality is used, one has to provide axioms that specify reflexive, symmetric, transitive, substitutive properties of equality.
- clumsy approach.
- Generates large search space.
- Hopelessly inefficient.

# The Equality Relation

The described approach has several drawbacks:

- Every time equality is used, one has to provide axioms that specify reflexive, symmetric, transitive, substitutive properties of equality.
- clumsy approach.
- Generates large search space.
- Hopelessly inefficient.

A solution: Use a dedicated inference rule for equality.

# Paramodulation

- An inference rule to handle equality, introduced by G. A. Robinson and L. Wos in 1969.
- It can replace the axioms concerning symmetric, transitive, substitutive properties of equality.
- Combined with resolution, paramodulation can be used to prove theorems involving equality.
- Simple, natural, and more efficient than the naive approach described in the previous slide.
- Still, search space can be large. Various improvements have been proposed to improve efficiency.

# Unsatisfiablity Under Special Class of Models

- The set $S$ in Example 1 is not unsatisfiable.
- However, it is unsatisfiable in all models of the set $K$.
- Restriction to special classes of models.

# Unsatisfiablity Under Special Class of Models

## Definition 1

Given:

- $S$: a set of clauses,
- $\mathcal{I}$: the set of all interpretations of $S$,
- $\mathcal{J}$: a nonempty subset of $\mathcal{I}$.

$S$ is said to be $\mathcal{J}$-unsatisfiable if $S$ is false in every element of $\mathcal{J}$.

# Unsatisfiablity Under Special Class of Models

How can $\mathcal{J}$ be given?

- If it is finite, just list them.
- Otherwise, it is usually defined by the axioms of a theory.
- When the axioms are axioms of the equality theory, $\mathcal{J}$-unsatisfiable sets are called also $\mathcal{E}$-unsatisfiable sets.

# Unsatisfiablity Under Special Class of Models

- In Example 1, $\mathcal{J}$ is all models of $K$.
- Since $K$ is the set of axioms of the equality theory, the set $S$ is $\mathcal{E}$-unsatisfiable.

# $\mathcal{E}$-Interpretation

Notation:

- $S$: a set of clauses,
- $I$: a Herbrand interpretation of $S$,
- $s, t, r$: terms from the Herbrand universe of $S$,
- $L$: a literal in $I$.

$I$ is called an $\mathcal{E}$-interpretation of $S$ if it satisfies the following conditions for all $s, t, r, L$:

1. $s \approx s \in I$;
2. if $s \approx t \in I$, then $t \approx s \in I$;
3. if $s \approx t \in I$ and $t \approx r \in I$, then $s \approx r \in I$;
4. if $s \approx t \in I$, $L$ contains $s$, and $L'$ is the result of replacing of one occurrence of $s$ in $L$ by $t$, then $L' \in I$.

# $\mathcal{E}$-Interpretation

## Example 2

- Let $S := \{p(a),\ \neg p(b),\ a \approx b\}$.
- Then there are 64 Herbrand interpretations of $S$.
- Among them the following six are $\mathcal{E}$-interpretations:

$$
\begin{array}{llllll}
\{\ \ p(a) & p(b) & a \approx a & b \approx b & a \approx b & b \approx a\} \\
\{\neg p(a) & \neg p(b) & a \approx a & b \approx b & a \approx b & b \approx a\} \\
\{\ \ p(a) & p(b) & a \approx a & b \approx b & a \not\approx b & b \not\approx a\} \\
\{\ \ p(a) & \neg p(b) & a \approx a & b \approx b & a \not\approx b & b \not\approx a\} \\
\{\neg p(a) & p(b) & a \approx a & b \approx b & a \not\approx b & b \not\approx a\} \\
\{\neg p(a) & \neg p(b) & a \approx a & b \approx b & a \not\approx b & b \not\approx a\}
\end{array}
$$

- $S$ is satisfiable, but $\mathcal{E}$-unsatisfiable.

# Towards Herbrand's Theorem for $\mathcal{E}$-Unsatisfiable Sets

## Definition 3

Let $S$ be a set of clauses. The set of the equality axioms for $S$ is the set consisting of the following clauses:

1. $x \approx x$.
2. $x \not\approx y \vee y \approx x$.
3. $x \not\approx y \vee y \not\approx z \vee x \approx z$.
4. $x \not\approx y \vee \neg p(x_1, \ldots, x, \ldots, x_n) \vee p(x_1, \ldots, y, \ldots, x_n)$, where $x$ and $y$ appear in the same position $i$, for all $1 \leq i \leq n$, for every $n$-ary predicate symbol $p$ appearing in $S$.
5. $x \not\approx y \vee f(x_1, \ldots, x, \ldots, x_n) \approx f(x_1, \ldots, y, \ldots, x_n)$, where $x$ and $y$ appear in the same position $i$, for all $1 \leq i \leq n$, for every $n$-ary function symbol $f$ appearing in $S$.

# Towards Herbrand's Theorem for $\mathcal{E}$-Unsatisfiable Sets

**Theorem 1**

*Let $S$ be a set of clauses and $E$ be the set of equality axioms for $S$. Then $S$ is $\mathcal{E}$-unsatisfiable iff $S \cup E$ is unsatisfiable.*

**Proof.**

($\Rightarrow$) Assume by contradiction that $S$ is $\mathcal{E}$-unsatisfiable but $S \cup E$ is satisfiable. Then $I \vDash S \cup E$ for some Herbrand interpretation $I$. Then $I$ satisfies $E$. Then $I$ satisfies the conditions of $\mathcal{E}$-interpretation. Then $I$ is an $\mathcal{E}$-model of $S$.
A contradiction.

# Towards Herbrand's Theorem for $\mathcal{E}$-Unsatisfiable Sets

## Theorem 1 (Cont.)

*Let $S$ be a set of clauses and $E$ be the set of equality axioms for $S$. Then $S$ is $\mathcal{E}$-unsatisfiable iff $S \cup E$ is unsatisfiable.*

## Proof.

($\Leftarrow$) Assume by contradiction that $S \cup E$ is unsatisfiable but $S$ is $\mathcal{E}$-satisfiable. Then $I \models S$ for some $\mathcal{E}$-interpretation $I$. But then $I$ satisfies $E$ as well. Then $I$ satisfies $S \cup E$.
A contradiction.

□

# Herbrand's Theorem for $\mathcal{E}$-Unsatisfiable Sets

**Theorem 2**

*A finite set $S$ of clauses is $\mathcal{E}$-unsatisfiable iff there exists a finite set $S'$ of ground instances of clauses in $S$ such that $S'$ is $\mathcal{E}$-unsatisfiable.*

**Proof.**

($\Rightarrow$) Let $E$ be the set of equality axioms of $S$. By Theorem 1, $S \cup E$ is unsatisfiable. By Herbrand's theorem, there is a finite set $S'$ of ground instances of clauses in $S$ such that $S' \cup E$ is unsatisfiable. Hence, by Theorem 1, $S'$ is $\mathcal{E}$-unsatisfiable.

($\Leftarrow$) Since $S'$ is $\mathcal{E}$-unsatisfiable, every $\mathcal{E}$-interpretation falsifies $S'$. Then every $\mathcal{E}$-interpretation falsifies $S$. Hence, $S$ is $\mathcal{E}$-unsatisfiable.

□

# Paramodulation

Example 2

Consider the clauses:

$C_1$: $p(a)$.

$C_2$: $a \approx b$.

We can substitute $b$ for $a$ in $C_1$ to obtain

$C_3$: $p(b)$.

# Paramodulation

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Consider the clauses:

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Paramodulation is an inference rule that extends this equality substitution rule.

# Paramodulation

## Example 2

Consider the clauses:

$C_1$: $p(a)$.

$C_2$: $a \approx b$.

We can substitute $b$ for $a$ in $C_1$ to obtain

$C_3$: $p(b)$.

Paramodulation is an inference rule that extends this equality substitution rule.

Notation: $A[t]$ for $A$ containing a term $t$.
$A$ can be a clause, a literal, or a term.

# Paramodulation for Ground Clauses

**Definition 4**

Given:

- A ground clause $C_1 = L[s] \vee C_1'$, where $L[s]$ is a literal containing a term $s$, and $C_1'$ is a clause,
- a ground clause $C_2 = s \approx t \vee C_2'$, where $C_2'$ is a clause.

Infer the following ground clause, called a paramodulant

$$L[t] \vee C_1' \vee C_2'.$$

# Paramodulation for Ground Clauses

## Example 5

$C_1$: $p_1(a) \vee p_2(b)$

$C_2$: $a \approx b \vee p_3(b)$

Paramodulant of $C_1$ and $C_2$: $p_1(b) \vee p_2(b) \vee p_3(b)$.

# Binary Paramodulation for General Clauses

## Definition 6

Given:

- A general clause $C_1 = L[r] \vee C_1'$, where $L[r]$ is a literal containing a term $r$, and $C_1'$ is a clause,
- a general clause $C_2 = s \approx t \vee C_2'$, where $C_2'$ is a clause, $C_1$ and $C_2$ have no variables in common, and $s$ and $r$ have an mgu $\sigma$.

Infer the following clause, called a binary paramodulant of the parent clauses $C_1$ and $C_2$:

$$L\sigma[t\sigma] \vee C_1'\sigma \vee C_2'\sigma.$$

The literals $L$ and $s \approx t$ are called the literals paramodulated upon. Sometimes one also says that paramodulation has been applied from $C_2$ into $C_1$.

# Binary Paramodulation for General Clauses

Example 7

- $C_1$: $p_1(g(f(x))) \vee p_2(x)$.
- $C_2$: $f(g(b)) \approx a \vee p_3(g(c))$.
- An mgu of $f(x)$ and $f(g(b))$: $\sigma = \{x \mapsto g(b)\}$.
- Paramodulant of $C_1$ and $C_2$: $p_1(g(a)) \vee p_2(g(b)) \vee p_3(g(c))$.
- The literals paramodulated upon are $p_1(g(f(x)))$ and $f(g(b)) \approx a$.

# Putting Things Together: The Inference system $\mathcal{RP}$

Binary Resolution: $$\frac{A \lor C \quad \neg B \lor D}{(C \lor D)\sigma}, \qquad \sigma = mgu(A, B)$$

Positive Factoring: $$\frac{A \lor B \lor C}{(A \lor C)\sigma}, \qquad \sigma = mgu(A, B)$$

Binary Paramodulation: $$\frac{s \approx t \lor C \quad L[r] \lor D}{(L[t] \lor C \lor D)\sigma}, \qquad \sigma = mgu(s, r)$$

Reflexivity Resolution: $$\frac{s \not\approx t \lor C}{C\sigma}, \qquad \sigma = mgu(s, t)$$

$A, B$ atomic formulas, $C, D$ clauses, $L$ literal, $s, t, r$ terms.

# Completeness of $\mathcal{RP}$

**Theorem 3**

*If $S$ is an $\mathcal{E}$-unsatisfiable set of clauses, then the empty clause can be generated from $S$ using the rules in $\mathcal{RP}$.*

# Resolution and Paramodulation

## Example 8

$$
\begin{array}{ll}
(1) & q(a) \\
(2) & \neg q(a) \vee f(x) \approx x \\
(3) & p(x) \vee p(f(a)) \\
(4) & \neg p(x) \vee \neg p(f(x))
\end{array}
$$

# Resolution and Paramodulation

## Example 8

| | | |
|---|---|---|
| (1) | $q(a)$ | |
| (2) | $\neg q(a) \vee f(x) \approx x$ | |
| (3) | $p(x) \vee p(f(a))$ | |
| (4) | $\neg p(x) \vee \neg p(f(x))$ | |
| (5) | $f(x) \approx x$ | Resolution (1,2) |

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| (5) | $f(x) \approx x$ | Resolution (1,2) |
| (6) | $\neg p(f(f(a)))$ | Resolution (factor 3,4) |

# Resolution and Paramodulation

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| (5) | $f(x) \approx x$ | Resolution (1,2) |
| (6) | $\neg p(f(f(a)))$ | Resolution (factor 3,4) |
| (7) | $\neg p(f(a))$ | Paramodulation (5,6) |

# Resolution and Paramodulation

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| (5) | $f(x) \approx x$ | Resolution (1,2) |
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| (4) | $\neg p(x) \vee \neg p(f(x))$ | |
| (5) | $f(x) \approx x$ | Resolution (1,2) |
| (6) | $\neg p(f(f(a)))$ | Resolution (factor 3,4) |
| (7) | $\neg p(f(a))$ | Paramodulation (5,6) |
| (8) | $\square$ | Resolution (factor 3,7) |

# Restriction of Paramodulation

- Unrestricted use of paramodulation can make the inference system too inefficient.
- For instance, from an equation $f(a) \approx a$ it can generate infinitely many new equations:
  $f(f(a)) \approx a, \; f(f(f(a))) \approx a, \ldots$.
- History of paramodulation-based proving: Restrict applications of the paramodulation rule.
- Important restrictions:
  - Prohibit paramodulation into a variable.
  - The use of reduction orderings.
  - The basic strategy of paramodulation.
  - Simplification.