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Example of proof in natural style, the notion of sequent

Consider the propositional formula:
(G0) $((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C)$.

Let us for the moment disregard our theory of propositional formulae and just consider this formula at the *meta-level*. That is, we interpret the logical connectives by our mathematical intuition. In order to prove the validity of this propositional formula at the meta-level, we may apply the intuitive inference rules which are "built-in" in our brain, as the result of our mathematical training. For instance, in the proof of (G0), the first inference rule which we apply may be formulated as follows:

> In order to prove that a statement φ is equivalent to a statement ψ, first assume φ and prove ψ, second assume ψ and prove φ.

The proof of the propositional formula (G0), using such intuitive rules, may look as follows:

[1] For proving (G0), we assume:

(A1) $(A \Rightarrow C) \vee (B \Rightarrow C)$

and we prove:

(G1) $(A \wedge B) \Rightarrow C$.

[2] For proving (G1), we assume:

(A2) $A \wedge B$

and we prove:

(G2) C.

[3] From (A2) we obtain:

(A2.1) A

and

(A2.2) B.

[4] We prove (G2) by case distinction, using the disjunction (A1).

Case 1:

(A3) $A \Rightarrow C$.

[5] From (A2.1), by (A3), using modus ponens we obtain the goal (G2).

Case 2:

(A4) $B \Rightarrow C$.

[6] From (A2.2), by (A4), using modus ponens we obtain the goal (G2).

This style of proof is usually called *natural deduction*, because it uses inference rules which are naturally used by humans.

We would like to construct a mathematical formalization of this proof style. For this, let us look at this proof and try to identify the essential elements of the method which is used.

First, we notice that at each moment in the proof there is a certain "goal" which has to be proven, and a certain set of "assumptions" which can be used in the proof (forming a kind of "knowledge base" of propositions assumed to be true at that moment). For instance, at the very beginning of the proof, the goal is (G0), and there are no assumptions. After the first proof step [1], the goal is (G1) and there is only one assumption (A1). After the second proof step [2], the goal is (G2), and the assumptions are (A1) and (A2). Thus, at each moment in the course of the proof, one has a *proof situation*, composed from a set of assumptions and a goal. The proof proceeds in individual steps, each step consisting in modifying the proof situation (that is, modifying the goal, the set of assumptions, or both). In mathematical logic, the idea of proof situation is modeled by the notion of *sequent*. A sequent is a pair $\langle \Phi, \Psi \rangle$, where Φ and Psi are sets of formulae. A sequent is denoted by:

$$\Phi \vdash \Psi,$$

or more concretely by:

$$\varphi_1, \ldots, \varphi_n \ \vdash \ \psi_1, \ldots, \psi_m,$$

where $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m$ represent individual formulae.