HOMEWORKS
Automated Theorem Proving, Timişoara, SS 2018

1. Give an example of a mathematical result which had a important impact on real life.

2. Give an example of a software failure which had an important negative impact in real life.

3. Using the induction principle from the syntactic definition of propositional formulae, define the meta-function $V[\varphi]$ which gives the set of propositional variables of the propositional formula φ.

4. Using the induction principle from the syntactic definition of propositional formulae, define the meta-function $L[\varphi]$ which gives the length of the propositional formula φ.

5. Using the induction principle from the syntactic definition of propositional formulae, define the meta-function $D[\varphi]$ which gives the depth of the propositional formula φ (that is the depth of the tree which represents the formula).

6. Using the induction principle from the syntactic definition of propositional formulae and the definitions above, prove that $D[\varphi] < L[\varphi]$ for any propositional formula φ.

7. Prove that for any propositional formulae φ, ψ, if $\varphi \equiv \psi$, then $\psi \models \varphi$. (See the style used in the lecture for proving $\varphi \models \psi$.)

8. Prove that for any propositional formulae φ, ψ, if $\varphi \models \psi$ and $\psi \models \varphi$, then $\varphi \equiv \psi$. (See the style used in the lecture for proving the opposite implication.)

9. Prove that for any propositional formulae φ, ψ, if $\varphi \Leftrightarrow \psi$ is valid, then $\varphi \equiv \psi$. (See the style used in the lecture for proving the opposite implication.)

10. Prove that for any propositional formulae $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi$, if $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$, then $(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) \Rightarrow \psi$ is valid. (See the style used in the lecture for proving the opposite implication.)

11. Prove $P,\ \neg Q \Rightarrow \neg P \models Q$ by constructing the truth table of the associated implication.

12. Prove $(A \wedge B) \Rightarrow C \equiv (A \Rightarrow C) \vee (B \Rightarrow C)$ by construncting the truth table of the two formulae.

13. Prove $(A \vee B) \Rightarrow C \equiv (A \Rightarrow C) \wedge (B \Rightarrow C)$ by construncting the truth table of the two formulae.

14. Prove $A \Rightarrow C, B \Rightarrow C \models (A \vee B) \Rightarrow C$ by construncting the truth table of the three formulae.

15. Prove in natural style: $(A \wedge B) \Rightarrow C \equiv (A \Rightarrow C) \vee (B \Rightarrow C)$. (See the style used in the lecture for proving the dual equivalence.)

16. Find the sequent rules for disjunction in the assumptions and in the goal, by studying the normal for of sequents (disjunction of negations of assumptions and copies of goals). (See the style used in the lecture for finding the sequent rules for negation and conjunction.)

17. Find the sequent rule for implication in the goal by applying sequent calculus to an equivalent formula which uses negation and disjunction. (See the style used in the lecture for finding the sequent rules for disjunction by using the rules for negation and conjunction.)

18. Find the sequent rule for equivalence in the goal by applying sequent calculus to an equivalent formula which uses conjunction and implication.

19. Define the syntax and semantics of disjunction applied to sets. Investigate the situations when the set is unary and when the set is empty. (See the style used in the lecture for conjunction.)

20. Find the sequent rules for disjunction applied to sets (in the assumptions and in the goals). (See the style used in the lecture for finding the sequent rules for conjunction applied to sets.)

21. Prove that the axioms of sequent calculus (the set of assumptions and the set of goals intersect) are valid.

22. Prove $(A \vee B) \Rightarrow C \;\equiv\; (A \Rightarrow C) \wedge (B \Rightarrow C)$ by sequent calculus (without unit propagation).

23. Find the sequent rule corresponding to "modus tollens" (similar to the one corresponding to "modus ponens").

24. Find the sequent rules for $\mathbb{F}$ in the assumptions and in the goal (duals of the rules for $\mathbb{T}$).

25. Prove $(A \vee B) \Rightarrow C \;\Leftrightarrow\; (A \Rightarrow C) \wedge (B \Rightarrow C)$ by sequent calculus with unit propagation. (See the proof of the dual statement in the lecture.)

26. Prove $(A \wedge B) \Rightarrow C \;\equiv\; (A \Rightarrow C) \vee (B \Rightarrow C)$ by reducing both formulae to normal form. (See the proof of the dual statement in the lecture.)

27. Prove $A \Rightarrow C, B \Rightarrow C \;\models\; (A \vee B) \Rightarrow C$ by refutation, reduction to CNF, and resolution.

28. Apply the DPLL algorithm to the following set of clauses:
$A \vee B \vee \overline{C},\;\; \overline{A} \vee C \vee \overline{D},\;\; B \vee C,$
$B \vee \overline{C} \vee D,\;\; \overline{A} \vee D,\;\; A \vee \overline{C} \vee \overline{D},$
$\overline{B} \vee \overline{C},\;\; \overline{A} \vee B \vee \overline{C} \vee \overline{D},\;\; A \vee \overline{B} \vee C$

29. In the previous exercise, study what happens in the DPLL algorithm if the last clause is missing.

30. Apply the resolution principle to the set of clauses from exercise 28.

31. Discover the sequent rules for the existential quantifier, by using the rules for the universal quantifier, the deMorgan equivalences, and the rules for negation.

32. Construct the sequent calculus proof for:

$$((\forall_x P[x]) \Rightarrow Q) \Leftrightarrow (\exists_x (P[x] \Rightarrow Q))$$

33-34. Derive the set of clauses needed to show by refutation that G is a semantical logical consequence of F_1, F_2:
$F_1 :\;\; \exists_x (P[x] \wedge \forall_y (D[y] \Rightarrow L[x,y]))$
$F_2 :\;\; \forall_x (P[x] \Rightarrow \forall_y (Q[y] \Rightarrow \neg L[x,y])),$
$G :\;\; \forall_x (D[x] \Rightarrow \neg Q[x]).$

35-36. Show by resolution that the set clauses obtained in the previous exercise is unsatisfiable.