Thinking Programs: Exercises

Chapter 5: Recursion

In some of the following exercises, we use the function $\mathsf{Fin}(A)$ to denote the set of all *finite* subsets of set A and $\mathsf{Inf}(A)$ to denote the set of all *infinite* subsets of A.

1. Consider the following grammar:

   $$N \in Nat$$
   $$N ::= 0 \mid \mathsf{s}(N)$$

   Intuitively, every value $\mathsf{s}^n(0) \in Nat$ (the n-fold application of constructor s to 0) represents the natural number $n \in \mathbb{N}$. Define by primitive (structural) recursion the function $+\colon Nat \times Nat \to Nat$ such that $m + n$ returns the "sum" of its arguments and the predicate $\leq\, \subseteq Nat \times Nat$ that determines whether the first argument is "less than or equal" the second one. Based upon $m + n$, define a corresponding "multiplication" function $m \cdot n$ from which you then define a corresponding "power" function m^n.

2. Consider your definition of $m + n$ from Exercise 1. First prove by structural induction $\forall n \in Nat.\ 0 + n = n + 0$. Then prove $\forall m \in Nat, n \in Nat.\ \mathsf{s}(m + n) = m + \mathsf{s}(n)$. Based upon these results, prove the commutativity property $\forall m \in Nat, n \in Nat.\ m + n = n + m$.

3. Define by an *inductive* definition for *finite* sequences $s, t \in \mathbb{N}^*$ of equal length the relation $s \leq t$ that is true if every element of s is less than or equal the corresponding element of t (you may assume the existence of operations *head* and *tail* over finite sequences). Transform the definitions into the "rule-oriented" style. Verify that your definitions are indeed well-formed by checking that the defining formulas satisfy the syntactic criteria required for upward continuity.

4. Repeat Exercise 3 but for *infinite* sequences $s, t \in \mathbb{N}^\omega$ by using *coinductive* definitions. Verify that your definitions are indeed well-formed by checking that the defining formulas satisfy the syntactic criteria required for downward continuity.

5. Define by an *inductive* definition the relation "A contains only prime numbers" for $A \in \mathsf{Fin}(\mathbb{N})$. Define by a *coinductive* definition the same relation for $A \in \mathsf{Inf}(\mathbb{N})$. Transform the definitions into the "rule-oriented" style. Verify that your definitions are indeed well-formed by checking that the defining formulas satisfy the syntactic criteria required for upward respectively downward continuity.

6. Define by an *inductive* definition the subset relation $A \subseteq B$ for *finite* sets $A \in \mathsf{Fin}(\mathbb{N})$ and $B \in \mathsf{Fin}(\mathbb{N})$. Transform the definition into the "rule-oriented" style. Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for upward continuity.

7. Define by a *coinductive* definition the subset relation $A \subseteq B$ for *infinite* sets $A \in \mathsf{Inf}(\mathbb{N})$ and $B \in \mathsf{Inf}(\mathbb{N})$. Transform the definition into the "rule-oriented" style. Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for downward continuity.

8. Define by an *inductive* function definition the union function $A \cup B$ for *finite* sets $A \in \mathsf{Fin}(\mathbb{N})$ and $B \in \mathsf{Fin}(\mathbb{N})$. Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for upward continuity.

9. Define by a *coinductive* function definition the union function $A \cup B$ for *infinite* sets $A \in \mathsf{Inf}(\mathbb{N})$ and $B \in \mathsf{Inf}(\mathbb{N})$ (please note that this definition must iterate over "both" sets in order to add elements from both sets to the result). Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for downward continuity.

10. Repeat Exercises 8 and 9 by defining the intersection function $A \cap B$ over finite respectively infinite sets A and B.

11. Let $A \in \mathsf{Fin}(\mathbb{N})$ and $n \in \mathbb{N}$. Introduce by a (possibly rule-oriented) *inductive* definition over A a predicate $A \geq n$ that states "every element of A is greater than or equal n". Prove by the principle of induction for properties defined as least fixed points (applied to the unary relation $\cdot \geq n$) that $A \geq n \Rightarrow (\forall a \in A.\ a \geq n)$ holds.

12. Repeat Exercise 11 but apply the proof principle of "fixed point induction". To justify the application of this principle, verify that the formula defining $A \geq n$ indeed satisfies the syntactic constraints of an "inclusive formula".

13. Let A be an *infinite* subset of $\mathbb{N}$ and $n \in \mathbb{N}$. Introduce by a (possibly rule-oriented) *coinductive* definition over A a predicate $A \geq n$ that states "every element of A is greater than or equal n". Furthermore, define by coinduction a function $A + n$ whose result is the set derived from A by adding n to each of its elements. Prove by the principle of coinduction for properties defined as greatest fixed points (applied to the unary relation $\cdot \geq n$) that $A + n \geq n$ holds.

14. Let $A \in \mathsf{Inf}(\mathbb{N})$. Consider the operation $A + n$ introduced in Exercise 13. Prove by the principle of coinduction that $A + 0 \sim A$ holds (here $\sim$ is the bisimilarity relation introduced by the definition of $A + n$).