Thinking Programs: Exercises

Chapter 3: The Art of Reasoning

1. Construct a proof tree for the following sequent:

   $$\vdash (\exists x.\ \forall y.\ p(x,y)) \Rightarrow (\forall y.\ \exists x.\ p(x,y))$$

   Also present this proof in informal "textbook" style.

2. Construct a proof tree for the following sequent:

   $$\vdash \big((\exists x.\ p(x)) \land (\forall x.\ p(x) \Rightarrow \exists y.\ q(x,y))\big) \Rightarrow \exists x,y.\ q(x,y)$$

   Also present this proof in informal "textbook" style.

3. Construct a proof tree for the following sequent:

   $$\vdash \big(p(a) \land (\forall x.\ p(x) \Rightarrow p(f(x))) \land (\forall x.\ p(f(x)) \Rightarrow p(g(x)))\big) \Rightarrow p(g(f(a)))$$

   Also present this proof in informal "textbook" style.

4. Construct a proof tree for the following sequent:

   $$\vdash \big((p(a) \lor q(b)) \land (\forall x.\ p(x) \Rightarrow r(x)) \land (\forall x.\ q(x) \Rightarrow r(f(x)))\big) \Rightarrow \exists x.\ r(x)$$

   Also present this proof in informal "textbook" style.

5. Construct a proof tree for the following sequent:

   $$\vdash (\forall y.\ \exists x.\ f(x) = y) \land (\forall y.\ \exists x.\ g(x) = y) \Rightarrow (\forall y.\ \exists x.\ g(f(x)) = y)$$

   Also present this proof in informal "textbook" style.

6. Construct a proof tree for the following sequent:

   $$\vdash \big((\forall x.\ r(x,x) \lor q(x)) \land (\forall x.\ (p(x) \land q(x)) \Rightarrow r(x,f(x)))\big) \Rightarrow \\ (\forall x.\ p(x) \Rightarrow \exists y.\ r(x,y))$$

   Also present this proof in informal "textbook" style.

7. Construct a proof tree from the following proof in informal "textbook" style:

   We show

   $$\forall x.\ \neg(x \in A \land x \in B) \Leftrightarrow (x \notin A \lor x \notin B)$$

   Take arbitrary x. We show $\big(\neg(x \in A \land x \in B) \Leftrightarrow (x \notin A \lor x \notin B)\big)$ (a):

- First we assume $\neg(x \in A \land x \in B)$ (1) and show $(x \notin A \lor x \notin B)$ (b). To show (b), we assume $x \in A$ (3) and show $x \notin B$ (c). To show (c), we assume $x \in B$ (4) and show a contradiction. Indeed, from (3) and (4), we know $(x \in A \land x \in B)$ which contradicts (1).
- Now we assume $(x \notin A \lor x \notin B)$ (5) and show $\neg(x \in A \land x \in B)$ (d). To show (d), we assume $(x \in A \land x \in B)$ (6) and show a contradiction. From (6), we know $x \in A$ (7) and $x \in B$ (8). From (5), we have two cases $x \in A$ and $x \in B$: however, case $x \in A$ contradicts (7) and case $x \in B$ contradicts (8), so we are done. □

Hint: in this proof the derived rule (contradict) is required.

8. Prove by induction $\forall x \in \mathbb{N}.\ n < 2^n$.

9. Prove by induction $\forall x \in \mathbb{N}.\ n \geq 4 \Rightarrow n^2 \leq 2^n$.

10. Assume $F(0) = 0,\ F(1) = 1,\ \forall n \in \mathbb{N}.\ F(n+2) = F(n) + F(n+1)$. Prove by *complete* induction the following conclusion:

$$\forall n \in \mathbb{N}.\ \sum_{i=0}^{n} F(i) = F(n+2) - 1$$

11. Consider the following grammar:

$$N \in \mathit{Nat},\ L \in \mathit{List}$$
$$N ::= \mathsf{0} \mid \mathsf{s}(N)$$
$$L ::= \mathsf{nil} \mid \mathsf{cons}(N, L)$$

We define by structural induction the functions $[\![\,.\,]\!] \colon \mathit{Nat} \to \mathbb{N}$, $s \colon \mathit{List} \to \mathbb{N}$, and $a \colon \mathit{List} \to \mathit{List}$ as follows:

$$[\![\,\mathsf{0}\,]\!] := 0,\ [\![\,\mathsf{s}(N)\,]\!] := 1 + [\![\,N\,]\!]$$
$$l(\mathsf{nil}) := 0,\ l(\mathsf{cons}(N, L)) := 1 + l(L)$$
$$s(\mathsf{nil}) := 0,\ s(\mathsf{cons}(N, L)) := [\![\,N\,]\!] + s(L)$$
$$a(\mathsf{nil}) := \mathsf{nil},\ a(\mathsf{cons}(N, L)) := \mathsf{cons}(\mathsf{s}(N), a(L))$$

Prove by structural induction the following property:

$$\forall L \in \mathit{List}.\ s(a(L)) = s(L) + l(L)$$

12. We equip the domain *Nat* introduced in Exercise 11 with the following type system:

$$\mathsf{0} \colon \mathsf{even} \qquad \frac{N \colon \mathsf{even}}{\mathsf{s}(N) \colon \mathsf{odd}} \qquad \frac{N \colon \mathsf{odd}}{\mathsf{s}(N) \colon \mathsf{even}}$$

Prove by *rule induction* the following property:

$$\forall N \in \mathit{Nat}.\ (\vdash N \colon \mathsf{even} \Rightarrow 2 \mid [\![\,N\,]\!]) \land (\vdash N \colon \mathsf{odd} \Rightarrow \neg 2 \mid [\![\,N\,]\!])$$

Please note $\forall m \in \mathbb{N}, n \in \mathbb{N}.\ m | n \Leftrightarrow \exists p \in \mathbb{N}.\ m \cdot p = n$.