Thinking Programs: Exercises

Chapter 2: The Language of Logic

1. Translate the following informal sentences into closed formulas in first-order logic:
   a) Every girl is female and has a father and a mother.
   b) Every father is male and has a daughter or a son.
   c) Every uncle is male and is a sibling of some parent of someone.

   Use here predicates in standard syntax like $isGirl(x)$ ("x" is a girl) or $isFather(x, y)$ ("x is the father of y").

2. Translate the following mathematical statements into (not necessarily closed) formulas in first-order logic:
   a) For all natural numbers x and y, if y is less than x, then x equals y plus z for some natural number z.
   b) There is no natural number y greater than 1 and less than x such that any z multiplied with y gives x.
   c) Every x is an element of S if and only if x is an element of every element of T.

   Use here usual mathematical notions such as $x \in S$ ("x is an element of S"), $x \in \mathbb{N}$ ("x is a natural number") or $x \cdot y = 1$ ("x times y is one").

3. Give an abstract syntax tree for the formula F defined as $\exists x.\ p(x) \wedge \forall y. q(f(x), y)$ (considering the usual binding rules for the logical operations). Given arity $ar := [p \mapsto 1, q \mapsto 2, f \mapsto 1]$, derive the judgement $ar \vdash F : \mathsf{formula}$.

4. Give abstract syntax trees for the formulas derived in Exercises 1 and 2. Define for each formula F the arity ar of the occurring function and predicate symbols and derive the judgement $ar \vdash F : \mathsf{formula}$.

5. Show that the propositional schema $\neg(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B)$ is a tautology. Show that the propositional schemas $\neg(A \Leftrightarrow B)$ and $(A \wedge \neg B) \vee (\neg A \wedge B)$ are logically equivalent.

6. Use known logical equivalences to compute the conjunctive normal form and the disjunctive normal form of $A \Rightarrow (B \wedge C)$. Show that your results are correct from the truth table of the given formula and the truth tables of your results.

7. Consider the formulas of Exercises 1, 2, and 3 (respectively their abstract syntax trees derived in Exercise 4). Annotate each (sub)formula and (sub)term (respectively the root of the corresponding abstract syntax tree) with the set of free variables of the (sub)formula respectively (sub)term.

8. Take the formula F defined as $\forall x.\ \exists y. p(x, y)$ and the formula G defined as $\exists y.\ \forall.\ p(x, y)$. Compute the semantics of both $[\![F]\!]^I_a$ and $[\![G]\!]^I_a$ where $a = [\,]$ is the empty assignment, the domain D of the interpretation I is defined as

a) $D := \{0, 1, 2\}$,

b) $D := \mathbb{N} = \{0, 1, 2, \ldots\}$,

c) $D := \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,

and the interpretation $I(p)$ of predicate p is $I(p) := \lambda x, y.$ "x is less than or equal y". Repeat the exercise with the interpretation $I(p) := \lambda x, y.$ "x is less than y".

9. Consider the domain $D := \mathbb{N} = \{0, 1, 2, \ldots\}$ with the usual interpretations of the various symbols. For each of the following formulas give an assignment to variable x that makes the formula true:

   a) $x \neq 1 \Rightarrow 1 = 0$

   b) $\forall y.\ x \leq y$

   c) $\forall y.\ x \cdot y = y$

   d) $\forall y.\ x \cdot y = 0$

   e) $\exists y.\ y \neq 0 \land y \neq 1 \land x \cdot y = x$

10. Give a domain D and an interpretation of the various symbols that makes the following closed formula F true:

$$\forall x.\ p(x) \land (\exists y.\ q(x, y)) \land (\exists y. \neg q(x, y))$$

    Justify your choice by computing the semantics $[\![F]\!]^I_a$ for the empty assignment $a = [\ ]$.

11. Give a convincing argument why a formula of form $\exists x.\ p(x) \Rightarrow q(x)$ is usually "trivially" true. For this, explain which very special constraints the interpretations of predicates p respectively q must satisfy to make the formula false.

12. Consider the following logical formulas:

    a) $\forall x.\ (p(x) \Rightarrow \exists y. q(x, y))$

    b) $\exists x.\ (p(x) \land \forall y.\ p(y) \Rightarrow q(x, y))$

    c) $\exists x.\ (p(x) \lor \exists y.\ q(y) \land r(x, y))$

    Apply logical equivalences to transform the *negation* of each of these formulas into "negation normal forms", i.e., into a form where the connective $\neg$ is only applied to atomic formulas.

13. Justify by equivalence transformations the logical equivalence of the following two formulas:

$$\forall x.\ \forall y.\ \forall z.\ (p(x) \land q(x, y) \land q(x, z)) \Rightarrow r(y, z)$$
$$\forall x.\ p(x) \Rightarrow (\forall y.\ q(x, y) \Rightarrow (\forall z.\ q(x, z) \Rightarrow r(y, z)))$$

14. Give interpretations for p and q that demonstrate that the following pairs of formulas are *not* logically equivalent:

$$(\exists x. p(x) \land q(x)) \not\equiv (\exists x. p(x)) \land (\exists x. q(x))$$
$$(\forall x. p(x) \lor q(x)) \not\equiv (\forall x. p(x)) \lor (\forall x. q(x))$$