# Symbolic Summation in Difference Fields

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## Eidesstattliche Erklärung

Ich versichere, dass ich die Dissertation selbstständig verfasst habe, andere als die angegebenen Quellen und Hilfsmittel nicht verwendet und mich auch sonst keiner unerlaubten Hilfe bedient habe.

Carsten Schneider
Hagenberg, 21. Mai 2001

## Abstract

There are implementations of the celebrated Gosper algorithm (1978) on almost any computer algebra platform. Within my PhD thesis work I implemented Karr's Summation Algorithm (1981) based on difference field theory in the Mathematica system. Karr's algorithm is, in a sense, the summation counterpart of Risch's algorithm for indefinite integration. Besides Karr's algorithm which allows us to find closed forms for a big class of multisums, we developed new extensions to handle also definite summation problems. More precisely we are able to apply creative telescoping in a very general difference field setting and are capable of solving linear recurrences in its context.
Besides this we find significant new insights in symbolic summation by rephrasing the summation problems in the general difference field setting. In particular, we designed algorithms for finding appropriate difference field extensions to solve problems in symbolic summation. For instance we deal with the problem to find all nested sum extensions which provide us with additional solutions for a given linear recurrence of any order. Furthermore we find appropriate sum extensions, if they exist, to simplify nested sums to simpler nested sum expressions. Moreover we are able to interpret creative telescoping as a special case of sum extensions in an indefinite summation problem. In particular we are able to determine sum extensions, in case of existence, to reduce the order of a recurrence for a definite summation problem.



## Zusammenfassung

In meiner Doktorarbeit implementierte ich Karrs Algorithmus (1981) in dem Computeralgebra System Mathematica. Karrs Algorithmus kann als Summations-Gegenpart zu Rischs Algorithmus für symbolische Integration angesehen werden.
Die Grundidee in Karrs Algorithmus ist, dass indefinite Summationsprobleme in Form von Differenzen-Gleichungen erster Ordnung beschrieben werden. Karrs Algorithmus kann lineare Differenzen-Gleichungen erster Ordnung in einer grossen Klasse von rekursiv aufgebauten Differenzen-Körpern in voller Allgemeinheit lösen. Dadurch kann man mit dem Computer Summationsausdrücke in symbolisch geschlossener Form vereinfachen, die aus verschachtelten Summen- und Produkt-Termen bestehen. Insbesondere erweiterte ich Karrs Algorithmus derart, dass automatisch Differenzen-Körper Erweiterungen gesucht, und im Falle ihrer Existenz auch gefunden werden können, in denen eine geschlossene Form für eine indefinite Summe existiert.
Ein wesentliches Resultat meiner Arbeit ist, dass Karrs Algorithmus Zeilbergers "creative telescoping" Methode beinhaltet. Dadurch können wir in vielen Fällen Rekurrenzen finden, die eine gegebene definite Summe als Lösung besitzen. Daraus ergibt sich die Möglichkeit, eine grosse Klasse von definiten Multisummen-Identitäten automatisch beweisen zu können. Ausgehend von dieser Erkenntnis, entwickelte ich Karrs Algorithmus weiter, um lineare Differenzen-Gleichungen beliebiger Ordnung in Differenzen-Körpern zu lösen. Insbesondere beschäftige ich mich damit, verschachtelte Summen-Erweiterungen, eine Unterklasse von d'Alembert Erweiterungen, automatisch zu finden, die zu weiteren Lösungen einer gegebenen Differenzen-Gleichung führen. Somit kann man in vielen Fällen für eine gegebene definite Summe erstmals eine geschlossene Form in algorithmischer Weise finden!

# Contents

# Introduction

Karr developed an algorithm for **indefinite** summation [Kar81, Kar85] based on the theory of difference fields [Coh65]. He introduced so called $\Pi\Sigma$-fields, in which first order linear difference equations can be solved in full generality. This algorithm cannot only deal with series of hypergeometric terms, like Gosper's algorithm [Gos78, PS95a, PS95b], series with q-hypergeometric terms, like [PR97], or holonomic series, like Chyzak's algorithm [CS98], but also with series of terms where for example the harmonic numbers can appear in the denominator. Karr's algorithm is, in a sense, the summation counterpart of Risch's algorithm [Ris69, Ris70] for indefinite integration.

I implemented this algorithm [Sch99, Sch00a, Sch00b] in the computer algebra system Mathematica and developed a user interface that dispenses the user from working explicitly with difference fields. Instead, the user can handle all summation problems in terms of sums and products.

As [Weg97, Rie01], which is based on different approaches, Karr's algorithm also allows to solve a large class of multisum problems which will be illustrated in a variety of examples in the next chapter.

In some cases appropriate difference field extensions are necessary in order to "simplify" a given summation problem. If there exists such an appropriate difference field extension, our algorithm will find it. Therefore one does not have to deal with problems concerning difference field extensions. This feature to find automatic extensions will be demonstrated in Section 1.2.3 and its theoretical background will be explained in Section 4.4.1. For example, with our implementation one can easily find the right hand sight of the following 3-fold sum identity

$$\sum_{i=1}^{N}\frac{\displaystyle\sum_{j=1}^{i}\frac{\displaystyle\sum_{k=1}^{j}\frac{1}{K+k}}{K+j}}{K+i}=3\,\mathrm{H}_K\,\mathrm{H}_{K+N}^{(2)}+\mathrm{H}_{K+N}\left(3\,\mathrm{H}_K^2-3\,\mathrm{H}_K^{(2)}+3\,\mathrm{H}_{K+N}^{(2)}\right)-2\,\mathrm{H}_K^{(3)}+2\,\mathrm{H}_{K+N}^{(3)}\right)$$

where $K$ is a positive integer and $\mathrm{H}_n^{(\alpha)} := \sum_{i=1}^{n}\frac{1}{i^\alpha}$.

Finally, I extended Karr's algorithm to handle **definite** summation problems. Although Karr's original summation algorithm was already capable of carrying out creative telescoping [Zei90], nobody has noticed this possibility until now. With creative telescoping we can compute a recurrence which has a given definite sum as a solution; therefore one can verify automatically a given definite sum identity. How creative telescoping works will be illustrated by an example in Section 1.3.3 and will be viewed in depth in Section 4.3.

Based on Bronstein's denominator bounding [Bro00], I was able to streamline Karr's ideas. Additionally, I have generalized Karr's algorithm such that linear difference equations of any order can be solved in any given $\Pi\Sigma$-field. In some sense, this generalization includes the approaches of [Abr89, Pet92, Pet94, Abr95, APP98, vH98, vH99, Wei01]. Hence we can find solutions of recurrences and thus not only prove definite sum identities, but even discover closed forms of definite sums in a very general setting. It is also possible to deal with ring extensions in form of algebraic relations, like $((-1)^k)^2 = 1$. These aspects, further observations and open problems will be introduced in Section 1.3.4 and discussed further in Chapter 3.

In order to find solutions of a given difference equation, in many cases one has to extend the underlying difference field. Starting from results in [AP94, HS99], we focus on problems how one can find appropriate difference field extensions, namely sum extensions and so called d'Alembertian extensions, to find solutions for a recurrence. In this context, our indefinite summation algorithm plays a major role to simplify this solutions further. These aspects will be made clear by examples in Sections 1.3.1 and 1.3.2, further explained in Sections 1.3.4.2 and 1.3.4.3 and in details treated in Section 4.5.

For instance, we know how to tackle definite summation problems, like [FK00]

$$\sum_{k=1}^{n} \frac{\mathrm{H}_k\ (3+k+n)!\ (-1)^k\ (-1)^{-1+n}}{(1+k)!\ (2+k)!\ (-k+n)!} + \frac{(n)!}{(3+n)!} \sum_{k=1}^{n} - \frac{(3+k+n)!\ (-1)^k\ (1-(2+n)\ (-1)^n)}{k\ (1+k)!^2\ (-k+n)!} = (2+n)(-1)^n - 2,$$

or [Kir96]

$$\begin{aligned}\sum_{k=0}^{N} \frac{\binom{N}{k}(-1)^k}{(k+K)^4} = \frac{1}{6\,K^4\binom{K+N}{K}} \big(6 - K^3\ \mathrm{H}_K^3 + 3\ K^2\ \mathrm{H}_{K+N}^2 + K^3\ \mathrm{H}_{K+N}^3 + 3\ K^2\ \mathrm{H}_K^2\ (1 + K\ \mathrm{H}_{K+N})\\ - 3\ K^2\ \mathrm{H}_K^{(2)} + 3\ K^2\ \mathrm{H}_{K+N}^{(2)} - 3\ K\ \mathrm{H}_K\ \big(2 + 2\ K\ \mathrm{H}_{K+N} + K^2\ \mathrm{H}_{K+N}^2 - K^2\ \mathrm{H}_K^{(2)} + K^2\ \mathrm{H}_{K+N}^{(2)}\big) +\\ \mathrm{H}_{K+N}\ \big(6\ K - 3\ K^3\ \mathrm{H}_K^{(2)} + 3\ K^3\ \mathrm{H}_{K+N}^{(2)}\big) - 2\ K^3\ \mathrm{H}_K^{(3)} + 2\ K^3\ \mathrm{H}_{K+N}^{(3)}\big).\end{aligned}$$

Finally we are able to find appropriate difference field extensions automatically in order to reduce the order of a recurrence. If such kind of difference field extensions exist, they will be found in our algorithm. In Section 1.3.5 we present definite summation examples which could be solved only by this feature. How we find such extensions will be described in Section 4.4.1.

### How to get the Mathematica package

The Mathematica package is available in an encoded form at

`www.risc.uni-linz.ac.at/research/combinat/risc/software/`.

### How to read this thesis

**Chapter 2:** The following chapter will explain in a variety of examples what kind of summation problems we can deal with and illustrate the usage of my Mathematica package. Headings written in *italics* are directed at readers who are not only interested in the usage of my package or how one can tackle different summation problems, but who are also interested in more details and further results of my thesis. Furthermore, headings starting with a † are for readers who are curious how the summation problems can be rephrased in terms of difference fields. These sections might be helpful for a better understanding of what will follow in the remaining chapters. Finally, the last Section 1.4 summarizes the summation problems which appear in Chapter 1 and motivates the topics discussed in detail in Chapters 2–4.
**Chapters 2–4:** The remaining chapters explain the algorithms for treating summation problems in difference fields. In order to keep this thesis to a size which is still somehow reasonable, the style of presentation is very dense and sometimes illustrative examples are omitted. Finally I want to mention that I see this thesis not as a final result of my research but as the starting point for further investigations.

### Acknowledgements

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In addition I want to thank Christian Krattenthaler who invited me to give a talk at the $10^{th}$ SIAM conference on Discrete Mathematics in Minneapolis [Sch00b], where I had a unique opportunity to present my work to a large audience of expert researchers.

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# Chapter 1

# Symbolic Summation in Difference Fields

In the book Concrete Mathematics [GKP94, exercise 6.69] the task of finding a closed form representation of

$$\sum_{k=1}^{n} k^2 \, \mathrm{H}_{n+k},$$

where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$ is the $n$-th harmonic number, is posed as a bonus problem. Knuth's solution to this problem is

$$\frac{1}{3}\, n \left(n + \frac{1}{2}\right) (n+1) \left(2\, \mathrm{H}_{2\,n} - \mathrm{H}_n\right) - \frac{1}{36}\, n \left(10\, n^2 + 9\, n - 1\right)$$

where he remarks

**"It would be nice to automate the derivation of formulas such as this."**

Inspired by Karr's algorithm [Kar81, Kar85] I developed a summation algorithm based on difference field theory which can compute the closed form of this bonus problem. The implementation is available in form of a Mathematica package called `Sigma`, in which functions are provided to define a given summation problem in the Mathematica environment.

In[1]:= << **Sigma`**

```
Sigma - A summation package by Carsten Schneider
```

In[2]:= **Problem69 = SigmaSum[k^2SigmaHNumber[n + k], {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n} \left(k^2 \, \mathrm{H}_{k+n}\right)$

The functions `SigmaSum` and `SigmaProduct` are used to define sums and products. There are several other functions available, like `SigmaHNumber`, `SigmaBinomial` or `SigmaPower` to define harmonic numbers, binomials or powers. Additional functions are provided to introduce new objects. The summation algorithm is applied to the bonus problem by calling the function `SigmaReduce`. Below the solution is simplified further by using the built-in Mathematica function `Simplify`.

In[3]:= **SigmaReduce[Problem69]//Simplify**

Out[3]= $-\frac{1}{36}\, n\, (1 + n)\, (-1 + 10\, n + 6\, (1 + 2\, n)\, \mathrm{H}_n - 12\, (1 + 2\, n)\, \mathrm{H}_{2\,n})$

## 1.1 Some Motivating Examples

In this section I want to demonstrate the different aspects of symbolic summation I can deal with and want to give a first feeling how one can find "closed forms" of a summation problem. In [Cal94] Calkin found the following curious identity

$$\sum_{k=0}^{n}\Big(\sum_{j=0}^{k}\binom{n}{j}\Big)^{3}=\frac{n}{2}\,8^{n}+8^{n}-\frac{3\,n}{4}\,2^{n}\binom{2\,n}{n}.$$

In the following I will take different variations of Calkin's multisum and try to "simplify" them.

### 1.1.1 Calkin's Identity

Let us first try to find "nice closed forms" of the sums

$$\sum_{k=0}^{a}\Big(\sum_{j=0}^{k}\binom{n}{j}\Big)^{p}$$

where $p \in \{1,2,3\}$ and[1] $a, n \in \mathbb{N}_0$.

#### The Case $p = 1$

Applying our summation algorithm to

In[4]:= **mySum** $= \sum_{\mathbf{k}=0}^{\mathbf{a}}\Big(\sum_{\mathbf{j}=0}^{\mathbf{k}}\big(\binom{\mathbf{n}}{\mathbf{j}}\big)\Big)$;

yields to the following closed form:

In[5]:= **SigmaReduce[mySum]//Simplify**

Out[5]= $\frac{1}{2}\left((-\mathtt{a}+\mathtt{n})\binom{\mathtt{n}}{\mathtt{a}}+(2+2\,\mathtt{a}-\mathtt{n})\sum_{\iota_1=0}^{\mathtt{a}}\Big(\binom{\mathtt{n}}{\iota_1}\Big)\right)$

For the specific value $a = n$ we can simplify this result further by using[2] the Binomial Theorem $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ and find

$$\boxed{\sum_{k=0}^{n}\Big(\sum_{j=0}^{k}\binom{n}{j}\Big)=(2+n)\,2^{n-1}.}$$

[1] $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of positive integers $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$.

[2] Please note that the right side $2^n$ of the Binomial Theorem can be also computed automatically by using our summation package.

### The Case $p = 2$ - A Sum Extension

Now we try to find a closed form of

In[6]:= $\mathbf{mySum} = \sum_{\mathbf{k}=0}^{\mathbf{a}}\left(\left(\sum_{\mathbf{j}=0}^{\mathbf{k}}\binom{\mathbf{n}}{\mathbf{j}}\right)^{\mathbf{2}}\right)$;

and fail:

In[7]:= $\mathbf{SigmaReduce}[\mathbf{mySum}]$

Out[7]= $\sum_{\iota_1=0}^{a}\left(\left(\sum_{\iota_2=0}^{\iota_1}\binom{n}{\iota_2}\right)^2\right)$

Loosely speaking, we are not able to find a closed form expressed by the objects $a$, $n$, $\binom{n}{a}$ and $\sum_{k=1}^{a}\binom{n}{k}$. But by choosing an appropriate sum extension we are capable of finding a closed form of the double nested sum `mySum` in terms of only single nested sums. This appropriate sum extension can be found *automatically* in our summation algorithm by setting the option `SimplifyByExt` $\rightarrow$ `Depth`.

In[8]:= $\mathbf{SigmaReduce}[\mathbf{mySum}, \mathbf{SimplifyByExt} \rightarrow \mathbf{Depth}]$

Out[8]= $(-a+n)\binom{n}{a}\sum_{\iota_1=0}^{a}\binom{n}{\iota_1}+\left(1+a-\frac{n}{2}\right)\left(\sum_{\iota_1=0}^{a}\binom{n}{\iota_1}\right)^2+\sum_{\iota_1=0}^{a}\left(-\frac{1}{2}\,n\binom{n}{\iota_1}^2\right)$

Applying our algorithm to the summation problem we have found a sum extension, namely

$$\sum_{i=0}^{a}-\frac{1}{2}\,n\binom{n}{i}^2=-\frac{1}{2}\,n\sum_{i=0}^{a}\binom{n}{i}^2,$$

which amounts algebraically to an extension of the underlying difference field, the solution space. Finally, for $a = n$ we obtain

$$\boxed{\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2=(n+1)\,4^n-\frac{n}{2}\,4^n-\frac{n}{2}\binom{2\,n}{n}}$$

by using[3] the Binomial Theorem and a variation of the Vandermonde identity $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2\,n}{n}$.

### The Case $p = 3$ - A Definite Summation Problem

If we try to simplify the sum

$$\sum_{k=0}^{a}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3$$

with our algorithm to an expression in terms of single nested sums, as in the previous examples, we fail. Replacing the upper bound $a$ with the specific value $n$ turns the indefinite summation problem to a definite one:

[3] Please note that these definite sum identities can be also computed automatically with our summation package.

In[9]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}}\left(\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{n}}{\mathbf{j}}\right)\right)^{\mathbf{3}}\right)$

For this definite summation problem we are now able to find a closed form. In a first step we generate a recurrence that is satisfied by `mySum`. The idea how to compute a recurrence is based on Zeilberger's creative telescoping method [Zei90].

In[10]:= **rec = GenerateRecurrence[mySum]**

Out[10]= $\{-16\ (1+2\ \mathtt{n})\ \mathtt{SUM[n]} - 4\ (12+7\ \mathtt{n})\ \mathtt{SUM[1+n]}$
$+4\ (1+\mathtt{n})\ \mathtt{SUM[2+n]}$
$== 8\left(-10\left(\sum_{\iota_1=0}^{\mathtt{n}}\left(\binom{\mathtt{n}}{\iota_1}\right)\right)^3 + 9\ \mathtt{n}\left(\sum_{\iota_1=0}^{\mathtt{n}}\left(\binom{\mathtt{n}}{\iota_1}\right)\right)^3\right)\}$

Using the Binomial Theorem we can simplify this recurrence further to

In[11]:= $\mathbf{rec} = \mathbf{rec[[1]]}/.\{\sum_{\iota_1=0}^{\mathbf{n}}\left(\binom{\mathbf{n}}{\iota_1}\right) \to (\mathbf{2})^{\mathbf{n}}\}$

Out[11]= $-16\ (1+2\ \mathtt{n})\ \mathtt{SUM[n]} - 4\ (12+7\ \mathtt{n})\mathtt{SUM[1+n]}$
$+4\ (1+\mathtt{n})\ \mathtt{SUM[2+n]} == 8\left(-10\ ((2)^{\mathtt{n}})^3 + 9\ \mathtt{n}\ ((2)^{\mathtt{n}})^3\right)$

Finally we solve[4] this recurrence in terms of $n$, $2^n$ and $\binom{2n}{n}$. In order to tell our algorithm to use the product extension $\binom{2n}{n}$, we set the option `Tower` $\to \{\binom{2\mathtt{n}}{\mathtt{n}}\}$.

In[12]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[n], Tower} \to \{\binom{\mathbf{2n}}{\mathbf{n}}\}\mathbf{]}$

Out[12]= $\{\{0, \mathtt{n}\binom{2\ \mathtt{n}}{\mathtt{n}}\ (2)^{\mathtt{n}}\}, \{1, \frac{1}{2}\ (2+\mathtt{n})\ ((2)^{\mathtt{n}})^3\}\}$

The result has to be interpreted as follows: the algorithm delivers one solution of the homogeneous version of the recurrence, namely $n\binom{2n}{n}2^n$ and one particular solution of the inhomogeneous recurrence itself: $\frac{1}{2}(2+n)\,2^{3n}$.
Finally the closed form of `mySum` is the linear combination of the homogeneous solutions[5] plus the particular computed inhomogeneous solution which has exactly the same initial values as `mySum`. This is also computed automatically:

In[13]:= **FindLinearCombination[recSol, mySum, 2]//Simplify**

Out[13]= $-\frac{3}{4}\ \mathtt{n}\binom{2\ \mathtt{n}}{\mathtt{n}}\ (2)^{\mathtt{n}} + \frac{1}{2}\ (2+\mathtt{n})\ ((2)^{\mathtt{n}})^3$

Therefore we obtain the identity

$$\boxed{\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = \frac{n}{2}\,8^n + 8^n - \frac{3\,n}{4}\,2^n\binom{2\,n}{n}.}$$

[4]Extending the underlying difference field with $\binom{2n}{n}$ allows us to find a solution of the homogeneous version of the recurrence. This product extension can be found automatically by using the function `FindProductExtensions` which will be introduced in Section 1.1.2.

[5]In this example there is just one solution of the homogeneous version of the recurrence.

### 1.1.2 An Alternating Version of Calkin's Identity

In [Zha99] Zhang succeeds in finding closed forms of the alternating multisums

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^p$$

where $p \in \{1,2,3\}$ and $a = n \in \mathbb{N}_0$ - with the exception $p = 3$ and $a = n$ is even. One can treat these summation problems in a similar way as in the previous subsection. For the case $p = 1$ we can even derive a more general identity, namely:

In[14]:= $\mathbf{mySum} = \sum_{\mathbf{k}=0}^{\mathbf{a}}\left((\mathbf{b})^{\mathbf{k}}\sum_{\mathbf{j}=0}^{\mathbf{k}}\binom{\mathbf{n}}{\mathbf{j}}\right);$

In[15]:= $\mathbf{res} = \mathbf{SigmaReduce}[\mathbf{mySum}, \mathbf{SimplifyByExt} \to \mathbf{Depth}]//\mathbf{Simplify}$

Out[15]= $\dfrac{-1 + \mathtt{b}\,(\mathtt{b})^{\mathtt{a}}\sum_{\iota_1=0}^{\mathtt{a}}\binom{\mathtt{n}}{\iota_1} - \sum_{\iota_1=1}^{\mathtt{a}}\left(\binom{\mathtt{n}}{\iota_1}(\mathtt{b})^{\iota_1}\right)}{-1+\mathtt{b}}$

This identity can be again simplified by the Binomial Theorem for the situation $a = n$:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} b^j\binom{n}{j} = \frac{-1 + 2^n\, b^{n+1} - (b+1)^n}{b-1}.$$

Furthermore, for $b = 1$ this gives the first instance of the alternating Calkin family.

#### The Case $p = 3$, $a = n$ and $n$ is even - A d'Alembertian Solution

There is only one more sophisticated case which I want to focus on, namely $p = 3$, $a = n$ and $n$ is even - exactly that case which could not be handled in [Zha99]:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3.$$

Since we assume that $n$ is even, we can substitute $n$ by $2n$.

In[16]:= $\mathbf{mySum} = \sum_{\mathbf{k}=0}^{2\mathbf{n}}\left((-1)^{\mathbf{k}}\left(\sum_{\mathbf{j}=0}^{\mathbf{k}}\binom{2\mathbf{n}}{\mathbf{j}}\right)^3\right);$

For this definite summation problem we compute a recurrence.

In[17]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{RecOrder} \to \mathbf{2}]//\mathbf{Simplify}$

Out[17]= $\{96\,\mathtt{n}\,(14 + 11\,\mathtt{n})\,(2 + 9\,\mathtt{n} + 9\,\mathtt{n}^2)\,\mathtt{SUM}[\mathtt{n}] + (180 + 1252\,\mathtt{n} + 2907\,\mathtt{n}^2 + 2799\,\mathtt{n}^3 + 946\,\mathtt{n}^4)\,\mathtt{SUM}[1+\mathtt{n}] + (1+\mathtt{n})^2\,(9 + 39\,\mathtt{n} + 22\,\mathtt{n}^2)\,\mathtt{SUM}[2+\mathtt{n}] == 16\,(1512 + 9884\,\mathtt{n} + 20210\,\mathtt{n}^2 + 16897\,\mathtt{n}^3 + 5005\,\mathtt{n}^4)\,(-1)^{2\,\mathtt{n}}\left(\sum_{\iota_1=0}^{2\,\mathtt{n}}\binom{2\,\mathtt{n}}{\iota_1}\right)^3\}$

The recurrence can be simplified further by applying the Binomial Theorem and the fact that $(-1)^{2n} = 1$ for any $n \in \mathbb{N}_0$.

In[18]:= $\mathbf{rec} = \mathbf{rec}/.\{\left(\sum_{\iota_1=0}^{2n}\binom{2\,n}{\iota_1}\right)^3 \to (\mathbf{64})^{\mathbf{n}}, (-\mathbf{1})^{\mathbf{2\,n}} \to \mathbf{1}\}$

Out[18]= $\{96\,n\,(14+11\,n)\,(2+9\,n+9\,n^2)\,\mathtt{SUM}[n]+$
$(180+1252\,n+2907\,n^2+2799\,n^3+946\,n^4)\,\mathtt{SUM}[1+n]+$
$(1+n)^2\,(9+39\,n+22\,n^2)\,\mathtt{SUM}[2+n] ==$
$16\,(1512+9884\,n+20210\,n^2+16897\,n^3+5005\,n^4)\,(64)^{n}\}$

Finally we solve the recurrence in terms of objects given in the recurrence.

In[19]:= **SolveRecurrence**[**rec**[[**1**]], **SUM**[**n**]]

Out[19]= $\{\{1, \frac{(64)^{n}}{2}\}\}$

Unfortunately the algorithm delivers only a particular inhomogeneous solution, which is not a sufficient solution, since it does not have the same initial values as the original summation problem.
Now we can try to extend the underlying difference field to find all solutions of the homogeneous version of the recurrence. By calling the function `FindSumSolutions` we can find *all* nested sum extensions[6] expressed in terms of objects given in the recurrence which deliver us further solutions of the recurrence.

In[20]:= **FindSumSolutions**[**rec**[[**1**]], **SUM**[**n**]]

Out[20]= $\{\{\{1, \frac{(64)^{n}}{2}\}\}, \{96\,n\,(14+11\,n)\,(2+9\,n+9\,n^2)\,\mathtt{PROD}[n]+$
$(180+1252\,n+2907\,n^2+2799\,n^3+946\,n^4)\,\mathtt{PROD}[1+n]+$
$(1+n)^2\,(9+39\,n+22\,n^2)\,\mathtt{PROD}[2+n] == 0\}\}$

The result is not very encouraging since we did not find any sum extension which constitutes some homogeneous solutions. But we point out that additionally to the already known inhomogeneous solution –as a by-product of the algorithm– we have obtained the following homogenous recurrence:

In[21]:= $\mathbf{ProdRec} = \mathbf{96\,n\,(14+11\,n)\,(2+9\,n+9\,n^2)\,PROD[n]+}$
$\mathbf{(180+1252\,n+2907\,n^2+2799\,n^3+946\,n^4)\,PROD[1+n]+}$
$\mathbf{(1+n)^2\,(9+39\,n+22\,n^2)\,PROD[2+n] == 0};$

This tells us the following: if we find a product extension which results in a solution of this recurrence then, by extending the solution space by this product extension, the function call `FindSumSolutions` will provide at least one more solution of the recurrence `rec`.
Please note that `ProdRec` is exactly the homogeneous version of the inhomogeneous recurrence `rec`. In this sense, the previous statement is trivial; finding a product extension which leads to a solution of `ProdRec` gives us at least one homogeneous solution of `rec`. But, as one can see in the following, if the function `FindSumSolutions` is applied iteratively, all solutions of the recurrence `rec` can be found.
So let us try to find a product extension which delivers us a solution of `ProdRec` by calling the function `FindProductExtensions`. This function uses M. Petkovšek's package `Hyper` [Pet92, Pet94, PWZ96] which has to be loaded first.

In[22]:= << **Hyper**`;

[6] See Sections 1.3.4.2 and 4.5 for further details.

This package is able to find all hypergeometric solutions of a linear recurrence with polynomial coefficients. In the following we obtain the following product extension:

In[23]:= $\mathbf{tower} = \mathbf{FindProductExtensions}[\mathbf{ProdRec}, \mathbf{PROD}[\mathbf{n}]]$

```
I use M. Petkovšek's package Hyper to find product extensions.
```

Out[23]= $\{\prod_{i=2}^{n}\Big(-\frac{32\ (-1+i)}{-1+2\ i}\Big)\}$

As already stated above, we try to find all nested sum extensions in terms of the objects given in the recurrence, but this time we first extend the underlying difference field by the found product.

In[24]:= $\mathbf{FindSumSolutions}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{n}], \mathbf{Tower} \to \mathbf{tower}]$

Out[24]= $\{\{\{0, \prod_{\iota_1=2}^{n}\Big(-\frac{32\ (-1+\iota_1)}{-1+2\ \iota_1}\Big)\}, \{1, \frac{(64)^{n}}{2}\}\},$

$\{-3\ (1+2\ n)\ (28+148\ n+225\ n^2+99\ n^3)\ \mathrm{PROD}[n]+$
$32\ (1+n)^3\ (3+11\ n)\ \mathrm{PROD}[1+n] == 0\}\}$

As already indicated above, we just find the homogeneous solution, exactly that one which was delivered by the package `Hyper`. But additionally the algorithm delivered a new recurrence, namely

In[25]:= $\mathbf{ProdRec} = -\mathbf{3}\ (\mathbf{1}+\mathbf{2}\ \mathbf{n})\ (\mathbf{28}+\mathbf{148}\ \mathbf{n}+\mathbf{225}\ \mathbf{n}^2+\mathbf{99}\ \mathbf{n}^3)\ \mathbf{PROD}[\mathbf{n}]+$
$\mathbf{32}\ (\mathbf{1}+\mathbf{n})^3\ (\mathbf{3}+\mathbf{11}\ \mathbf{n})\ \mathbf{PROD}[\mathbf{1}+\mathbf{n}] == \mathbf{0};$

As mentioned already above, if we find a product extension which leads to a solution of this recurrence then, by extending the solution space with this product extension, the function call `FindSumSolutions` will provide at least one more solution of the recurrence `rec`.
We first find[7] a product extension which delivers us a homogenous solution of `ProdRec`.

In[26]:= $\mathbf{FindProductExtensions}[\mathbf{ProdRec}[[\mathbf{1}]], \mathbf{PROD}[\mathbf{n}]]$

```
I use M.Petkovšek's package Hyper to find product extension.
```

Out[26]= $\{\prod_{i=1}^{n}\Big(\frac{3\ (-1+2\ i)\ (-2+3\ i)\ (-1+3\ i)}{32\ i^3}\Big)\}$

We extend the underlying difference field by this product extension

In[27]:= $\mathbf{tower} = \{\prod_{\mathbf{i}=\mathbf{2}}^{\mathbf{n}}\Big(-\frac{\mathbf{32}\ (-\mathbf{1}+\mathbf{i})}{-\mathbf{1}+\mathbf{2}\ \mathbf{i}}\Big), \prod_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}}\Big(\frac{\mathbf{3}\ (-\mathbf{1}+\mathbf{2}\ \mathbf{i})\ (-\mathbf{2}+\mathbf{3}\ \mathbf{i})\ (-\mathbf{1}+\mathbf{3}\ \mathbf{i})}{\mathbf{32}\ \mathbf{i}^3}\Big)\};$

and call the procedure in order to find all possible nested sum extensions over the underlying difference field which delivers new solutions.

In[28]:= $\mathbf{recSol} = \mathbf{FindSumSolutions}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{n}], \mathbf{Tower} \to \mathbf{tower}]$

Out[28]= $\{\{\}, \{\{0, \prod_{\iota_1=2}^{n}\Big(-\frac{32\ (-1+\iota_1)}{-1+2\ \iota_1}\Big)\}, \{0, \frac{32}{3}\left(\prod_{\iota_1=2}^{n}\Big(-\frac{32\ (-1+\iota_1)}{-1+2\ \iota_1}\Big)\right)$

$\sum_{\iota_1=1}^{n}\left(\frac{\iota_1^3\ (-8+11\ \iota_1)\ \prod_{\iota_2=1}^{\iota_1}\Big(\frac{3\ (-1+2\ \iota_2)\ (-2+3\ \iota_2)\ (-1+3\ \iota_2)}{32\ \iota_2^3}\Big)}{(-1+2\ \iota_1)\ (-2+3\ \iota_1)\ (-1+3\ \iota_1)}\right)\}, \{1, \frac{(64)^{n}}{2}\}\}\}$

Since the recurrence `rec` has order 2 and we found 2 linearly independent solutions of the homogeneous version of the recurrence, the recurrence is completely solved. Instead of using

[7] Please note that in this example the product extension can be read off directly, since the order of the recurrence is just 1.

the function `FindSumSolutions` we could also apply the function `SolveRecurrence` by setting the option `NestedSumExt`$\rightarrow \infty$; with the exception that in this case the algorithm does not deliver a recurrence to find further product extensions – in case they exist.

In[29]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{n}], \mathbf{NestedSumExt} \rightarrow \infty,$
$\mathbf{Tower} \rightarrow \mathbf{tower}]$

$$\text{Out[29]=} \Big\{\Big\{0, \prod_{\iota_1=2}^{n}\Big(-\frac{32\ (-1+\iota_1)}{-1+2\ \iota_1}\Big)\Big\}, \Big\{0, \frac{32}{3}\left(\prod_{\iota_1=2}^{n}\Big(-\frac{32\ (-1+\iota_1)}{-1+2\ \iota_1}\Big)\right)$$
$$\sum_{\iota_1=1}^{n}\left(\frac{\iota_1^3\ (-8+11\ \iota_1)\ \prod_{\iota_2=1}^{\iota_1}\Big(\frac{3\ (-1+2\ \iota_2)\ (-2+3\ \iota_2)\ (-1+3\ \iota_2)}{32\ \iota_2^3}\Big)}{(-1+2\ \iota_1)\ (-2+3\ \iota_1)\ (-1+3\ \iota_1)}\right)\Big\}, \Big\{1, \frac{(64)^{n}}{2}\Big\}\Big\}$$

Finally by calling the function `FindLinearCombination` we find the linear combination of the homogeneous solutions of the recurrence plus the particular inhomogeneous solution such that the initial values are the same as the definite sum `mySum`. By default the comparison of the initial values is started at the lower summation bound of `mySum`, in our case 0. In this concrete situation we have to start with the initial value 1 in order to find the desired linear combination. This can be achieved by setting the option `MinInitialValue`$\rightarrow$`1`.

In[30]:= $\mathbf{sol} = \mathbf{FindLinearCombination}[\mathbf{recSol}, \mathbf{mySum}, \mathbf{2}, \mathbf{MinInitialValue} \rightarrow \mathbf{1}]$

$$\text{Out[30]=} \frac{(64)^{n}}{2} + \frac{64}{3}\left(\prod_{\iota_1=2}^{n}\Big(-\frac{32\ (-1+\iota_1)}{-1+2\ \iota_1}\Big)\right)$$
$$\sum_{\iota_1=1}^{n}\left(\frac{\iota_1^3\ (-8+11\ \iota_1)\ \prod_{\iota_2=1}^{\iota_1}\Big(\frac{3\ (-1+2\ \iota_2)\ (-2+3\ \iota_2)\ (-1+3\ \iota_2)}{32\ \iota_2^3}\Big)}{(-1+2\ \iota_1)\ (-2+3\ \iota_1)\ (-1+3\ \iota_1)}\right)$$

By some rewriting we derive the following identity:

$$\boxed{\sum_{k=0}^{2n}(-1)^k\left(\sum_{j=0}^{k}\binom{2n}{j}\right)^3 = \frac{1}{6}\,64^n\Big(3 - \frac{(-1)^n\,4}{\binom{2n}{n}\,n}\sum_{i=1}^{n}\frac{i^3\,(11\,i-8)\binom{2i}{i}^2\binom{3i}{i}}{(2\,i-1)\,(3\,i-2)\,(3\,i-1)\,64^i}\Big)}$$

Finally we can simplify the right hand side by our summation package further to

$$\frac{64^n}{2} - \frac{(-1)^n}{16\,n}\frac{64^n}{\binom{2n}{n}}\sum_{i=0}^{n-1}(3+11\,i)\binom{2\,i}{i}^2\binom{3\,i}{i}64^{-i}$$

Note that the sum on the right hand side is indefinite. More generally, nested sums of indefinite sums are included in the class of d'Alembertian solutions, which are introduced in [AP94]; further results can be found in [Sin91]. In Section 4.5 I will consider d'Alembertian solutions and especially nested sum solutions in more details under the aspect of difference fields.

## 1.2 Indefinite Summation

### 1.2.1 Recursive Aspects - An Example from Physics

In this section I will demonstrate in more details how one can tackle multisums in a recursive way. This will be illustrated by looking at a multisum expression which appears in [EG95]. Essam and Guttmann considered multisums $S_n(p)$ for $n \in \mathbb{N}_0$ and positive integers $p$ with

$$S_n(p) = \sum_{0 \le q_1 \le \dots \le q_p \le n} w_n(q_1, \dots, q_p)$$

where

$$w_n(q_1, \dots, q_p) = \prod_{1 \le i < j \le p} (q_j - q_i + j - i) \prod_{j=1}^{p} \frac{(n+p-j)!}{(q_j+j-1)!\,(n-q_j+p-j)!}.$$

**The Case** $p = 2$

We reformulate, following [AP99], this multisum for $p = 2$ and obtain

$$S_n(2) = \sum_{k_1=0}^{n+1} \sum_{k_2=0}^{k_1} \frac{k_1 - k_2}{n+1} \binom{n+1}{k_1} \binom{n+1}{k_2}.$$

For simplicity we consider the following sum which can be easily transformed to $S_n(2)$:

$$\sum_{k_1=0}^{a} \sum_{k_2=0}^{k_1} (k_1 - k_2) \binom{n}{k_1} \binom{n}{k_2}. \tag{1.1}$$

Please note that we consider the indefinite version of the summation problem; this means the upper bound of the outermost sum is not the specific value $n$ but an arbitrary value $a$ which does not appear in the summand. The goal is to eliminate the sum quantifiers by using the two sums:

In[31]:= $\mathbf{tower} = \{ \sum_{\mathbf{k=0}}^{\mathbf{a}} ((\binom{\mathbf{n}}{\mathbf{k}}^{.})^{\mathbf{2}}), \sum_{\mathbf{k=0}}^{\mathbf{a}} (\binom{\mathbf{n}}{\mathbf{k}}^{.})\};$

In a first step we apply our indefinite summation algorithm to the innermost sum

In[32]:= $\mathbf{Sum1} = \sum_{\mathbf{k2=0}}^{\mathbf{k1}} \left( (\mathbf{k1 - k2}) \binom{\mathbf{n}}{\mathbf{k1}}^{.} \binom{\mathbf{n}}{\mathbf{k2}}^{.} \right);$

and obtain the result:

In[33]:= **summand** = **SigmaReduce**[**Sum1**, **Tower** → **tower**]

Out[33]= $-\frac{1}{2} \binom{\mathtt{n}}{\mathtt{k1}}^{.} \left( (\mathtt{k1 - n}) \binom{\mathtt{n}}{\mathtt{k1}}^{.} + (-2\,\mathtt{k1 + n}) \sum_{\iota_1=0}^{\mathtt{k1}} (\binom{\mathtt{n}}{\iota_1}^{.}) \right)$

Now we replace the innermost sum `Sum1` in the summation problem (1.1) with the result `summand` and obtain the following sum which is equal to (1.1):

In[34]:= **Sum2** = **SigmaSum**[**summand**, {**k1**, **0**, **a**}]

Out[34]= $\sum_{\mathtt{k1=0}}^{\mathtt{a}} \left( -\frac{1}{2} \binom{\mathtt{n}}{\mathtt{k1}}^{.} \left( (\mathtt{k1 - n}) \binom{\mathtt{n}}{\mathtt{k1}}^{.} + (-2\,\mathtt{k1 + n}) \sum_{\iota_1=0}^{\mathtt{k1}} (\binom{\mathtt{n}}{\iota_1}^{.}) \right) \right)$

In a second step we apply the summation algorithm to `Sum2` and receive finally a closed form (in the given context) of the summation problem (1.1).

In[35]:= **SigmaReduce[Sum2, Tower → tower]//Simplify**

$$\text{Out[35]=}\ \frac{1}{2}\left((\mathtt{a}-\mathtt{n})\ \binom{\mathtt{n}}{\mathtt{a}}^{\cdot}\ \sum_{\iota_1=0}^{\mathtt{a}}(\binom{\mathtt{n}}{\iota_1}^{\cdot})+\mathtt{n}\ \sum_{\iota_1=0}^{\mathtt{a}}((\binom{\mathtt{n}}{\iota_1}^{\cdot})^{2})\right)$$

For the specific upper bound $a = n$ we can simplify this result further by using the Binomial Theorem and $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2\,n}{n}$ and we obtain

$$\boxed{\sum_{k_1=0}^{a}\sum_{k_2=0}^{k_1}(k_1-k_2)\binom{n}{k_1}\binom{n}{k_2}=\frac{n}{2}\binom{2\,n}{n}.}$$

### The Case $p = 3$

Similarly we can deal with the case $p = 3$ by considering the sum

$$\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2}(k_1-k_2)\,(k_2-k_3)(k_1-k_3)\binom{n}{k_1}\binom{n}{k_2}\binom{n}{k_3}.$$

- Step 1:

  $$\text{In[36]:=}\ \mathbf{Sum1}=\sum_{\mathbf{k3=0}}^{\mathbf{k2}}\left((\mathbf{k1}-\mathbf{k2})\ (\mathbf{k1}-\mathbf{k3})\ (\mathbf{k2}-\mathbf{k3})\ \binom{\mathbf{n}}{\mathbf{k2}}^{\cdot}\ \binom{\mathbf{n}}{\mathbf{k3}}^{\cdot}\ \binom{\mathbf{n}}{\mathbf{k1}}^{\cdot}\right);$$

  In[37]:= **summand = SigmaReduce[Sum1, Tower → tower]**

  $$\text{Out[37]=}\ -\frac{1}{4}\ (\mathtt{k1}-\mathtt{k2})\ \binom{\mathtt{n}}{\mathtt{k1}}^{\cdot}\ \binom{\mathtt{n}}{\mathtt{k2}}^{\cdot}\ \Big((-1+2\ \mathtt{k1}-\mathtt{n})\ (\mathtt{k2}-\mathtt{n})\ \binom{\mathtt{n}}{\mathtt{k2}}^{\cdot}-$$
  $$\left(4\ \mathtt{k1}\ \mathtt{k2}+\mathtt{n}-2\ \mathtt{k1}\ \mathtt{n}-2\ \mathtt{k2}\ \mathtt{n}+\mathtt{n}^2\right)\ \sum_{\iota_1=0}^{\mathtt{k2}}(\binom{\mathtt{n}}{\iota_1}^{\cdot})\Big)$$

- Step 2:

  In[38]:= **Sum2 = SigmaSum[summand, {k2, 0, k1}];**

  In[39]:= **summand = SigmaReduce[Sum2, Tower → tower]**

  $$\text{Out[39]=}\ \frac{1}{4\ \mathtt{n}\ (-1+2\ \mathtt{n})}$$
  $$\Bigg(\binom{\mathtt{n}}{\mathtt{k1}}^{\cdot}\ \Big((\mathtt{k1}-\mathtt{n})^2\ \left(\mathtt{k1}-2\ \mathtt{k1}\ \mathtt{n}+\mathtt{n}^2+2\ \mathtt{k1}\ \mathtt{n}^2-\mathtt{n}^3\right)\ (\binom{\mathtt{n}}{\mathtt{k1}}^{\cdot})^{2}-$$
  $$\mathtt{k1}\ (\mathtt{k1}-\mathtt{n})\ \mathtt{n}\ (-1+2\ \mathtt{n})\ \binom{\mathtt{n}}{\mathtt{k1}}^{\cdot}\ \sum_{\iota_1=0}^{\mathtt{k1}}(\binom{\mathtt{n}}{\iota_1}^{\cdot})+\mathtt{n}^2$$
  $$\left(2\ \mathtt{k1}\ (1-2\ \mathtt{n})\ \mathtt{n}+(-1+\mathtt{n})\ \mathtt{n}^2+\mathtt{k1}^2\ (-2+4\ \mathtt{n})\right)\ \sum_{\iota_1=0}^{\mathtt{k1}}((\binom{\mathtt{n}}{\iota_1}^{\cdot})^{2})\Big)\Bigg)$$

- Step 3:

  In[40]:= **Sum3 = SigmaSum[summand, {k1, 0, n}];**

  In[41]:= **SigmaReduce[Sum3]//Simplify**

  $$\text{Out[41]=}\ \frac{(-1+\mathtt{n})\,\mathtt{n}^2\left(\sum_{\iota_1=0}^{\mathtt{n}}\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)\sum_{\iota_1=0}^{\mathtt{n}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot\,2}\right)}{-8+16\,\mathtt{n}}$$

By using the Binomial Theorem and $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$ we obtain

$$\boxed{\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2}(k_1-k_2)\,(k_2-k_3)(k_1-k_3)\binom{n}{k_1}\binom{n}{k_2}\binom{n}{k_3}=\frac{(n-1)\,n^2\,2^n\binom{2n}{n}}{16\,n-8}.}$$

**The Case** $p = 5$

Finally we will compute a "closed form" of the following 5-fold sum.

In[42]:= **mySum =**

$$\sum_{\mathbf{k1}=0}^{\mathbf{n}}\left(\sum_{\mathbf{k2}=0}^{\mathbf{k1}}\left(\sum_{\mathbf{k3}=0}^{\mathbf{k2}}\left(\sum_{\mathbf{k4}=0}^{\mathbf{k3}}\left(\sum_{\mathbf{k5}=0}^{\mathbf{k4}}\Big((\mathbf{k1}-\mathbf{k2})\,(\mathbf{k1}-\mathbf{k3})\,(\mathbf{k2}-\mathbf{k3})\right.\right.\right.\right.$$
$$(\mathbf{k1}-\mathbf{k4})\,(\mathbf{k2}-\mathbf{k4})\,(\mathbf{k3}-\mathbf{k4})\,(\mathbf{k1}-\mathbf{k5})\,(\mathbf{k2}-\mathbf{k5})$$
$$(\mathbf{k3}-\mathbf{k5})\,(\mathbf{k4}-\mathbf{k5})\binom{\mathbf{n}}{\mathbf{k1}}^{\cdot}\binom{\mathbf{n}}{\mathbf{k2}}^{\cdot}\binom{\mathbf{n}}{\mathbf{k3}}^{\cdot}\binom{\mathbf{n}}{\mathbf{k4}}^{\cdot}$$
$$\left.\left.\left.\left.\binom{\mathbf{n}}{\mathbf{k5}}^{\cdot}\Big)\right)\right)\right)\right);$$

Since this example is too big to unfold the summation problem step by step in this thesis, we will apply the package such that the solution of the problem is obtained in one stroke.

In[43]:= **result = SigmaReduce[mySum, Tower → tower]**

Out[43]=

$$\frac{3\,(-3+\mathtt{n})\,(-2+\mathtt{n})^2(-1+\mathtt{n})^3\,\mathtt{n}^5\left(\sum_{\iota_1=0}^{\mathtt{n}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)\right)\left(\sum_{\iota_1=0}^{\mathtt{n}}\left(\left(\binom{\mathtt{n}}{\iota_1}^{\cdot\,2}\right)\right)\right)^2}{256\,(-5+2\,\mathtt{n})\,\left(3-8\,\mathtt{n}+4\,\mathtt{n}^2\right)^2}$$

Simplifying this result further for the concrete situation $a = n$ leads to

$$\boxed{\frac{3\,(-3+n)\,(-2+n)^2\,(-1+n)^3\,n^5\binom{2n}{n}^2 2^n}{256\,(-5+2\,n)\,(3-8\,n+4\,n^2)^2}.}$$

### 1.2.2 †*Indefinite Summation and First Order Linear Difference Equations*

In this section I will give a rough outline of our approach to deal with indefinite summation problems which is based on the theory of difference fields. In the following a difference field is considered as a field $\mathbb{F}$ together with any field automorphism[8] $\sigma : \mathbb{F} \to \mathbb{F}$. In short we will write $(\mathbb{F}, \sigma)$.
As M. Karr observed in [Kar81, Kar85], a huge class of indefinite summation problems can be formalized by first order linear difference equations in difference field settings. Since our approach inspired by Karr's summation algorithm can solve first order linear difference equations in full generality, our algorithm enables to treat indefinite summation.
I will illustrate our approach by the following elementary problem: find a closed form of

In[44]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}} (\mathbf{k}\ (\mathbf{k})!.);$

With my package we can solve the problem immediately:

In[45]:= **SigmaReduce**[**mySum**]

Out[45]= $-1 + (1 + n)\ (n)!.$

#### A difference field for the problem

Let $t_1, t_2$ be indeterminates and consider the field automorphism[9] $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ canonically defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, \\ \sigma(t_1) &= t_1 + 1, \\ \sigma(t_2) &= (t_1 + 1)\, t_2. \end{aligned}$$

Note that the automorphism acts on $t_1$ and $t_2$ like the shift operator $N$ on $n$ and $n!$ via $Nn = n + 1$ and $Nn! = (n + 1)\, n!$.

#### A first order difference equation

The summation problem can be rephrased in terms of the difference field $(\mathbb{Q}(t_1, t_2), \sigma)$ as follows: find a solution $g \in \mathbb{Q}(t_1, t_2)$ of

$$\boxed{\sigma(g) - g = t_1\, t_2.}$$

Our algorithm computes the solution $g = t_2$ from which

$$(k + 1)! - k! = k\, k!$$

immediately follows.

#### The closed form

By telescoping one obtains the closed form evaluation

$$\sum_{k=0}^{n} k\, k! = (n + 1)! - 1.$$

[8] More precisely, we will consider only a subclass of difference fields, so called $\Pi\Sigma$-fields [Kar81, Kar85] which will be considered in details in Section 2.2.

[9] $\mathbb{Q}$ denotes the set of rational numbers.

### 1.2.3 Reducing the Depth of Nested Sums by Sum Extensions

One important possibility of our summation package is to find automatically appropriate sum extensions to simplify a given indefinite summation problem. Whereas the following subsections demonstrate[10] this feature and illustrate its possibilities, Section 1.2.4 will give a rough outline how the problem can be formulated in difference fields.

#### 1.2.3.1 A 3-fold Sum and Harmonic Numbers

If we apply our algorithm to the indefinite sum

$$\text{In[46]:= } \mathbf{mySum} = \sum_{\iota_1=1}^{\mathbf{N}} \left( \frac{\sum_{\iota_2=1}^{\iota_1} \left( \frac{\sum_{\iota_3=1}^{\iota_2} \left( \frac{\mathbf{1}}{\mathbf{K}+\iota_3} \right)}{\mathbf{K}+\iota_2} \right)}{\mathbf{K}+\iota_1} \right);$$

where $K \notin \{-1,-2,-3,\dots\}$ is a complex number, we cannot simplify it further. More precisely, the underlying difference field in which the sum can be expressed consists only of transcendental elements. This fact will be illustrated in more details in Section 1.2.4.1. Applying the summation algorithm with the option `SimplifyByExt→Depth`, activates the feature to search automatically for appropriate sum extensions in which `mySum` can be expressed in a simpler form.

$$\text{In[47]:= } \mathbf{SigmaReduce[mySum, SimplifyByExt \to Depth]}$$

$$\text{Out[47]= } \frac{1}{6\,\mathrm{K}^2}\left(6\sum_{\iota_1=1}^{\mathrm{N}}\left(\frac{1}{\mathrm{K}+\iota_1}\right)+6\,\mathrm{K}\left(\sum_{\iota_1=1}^{\mathrm{N}}\left(\frac{1}{\mathrm{K}+\iota_1}\right)\right)^2+\mathrm{K}^2\left(\sum_{\iota_1=1}^{\mathrm{N}}\left(\frac{1}{\mathrm{K}+\iota_1}\right)\right)^3+\right.$$
$$\left.\left(-3-3\,\mathrm{K}\sum_{\iota_1=1}^{\mathrm{N}}\left(\frac{1}{\mathrm{K}+\iota_1}\right)\right)\sum_{\iota_1=1}^{\mathrm{N}}\left(\frac{\mathrm{K}+2\,\iota_1}{(\mathrm{K}+\iota_1)^2}\right)-\mathrm{K}\sum_{\iota_1=1}^{\mathrm{N}}\left(\frac{\mathrm{K}+3\,\iota_1}{(\mathrm{K}+\iota_1)^3}\right)\right)$$

By applying partial fraction decomposition to the summands of the new sums, we obtain

$$\frac{K+2i}{(K+i)^2} = -\frac{K}{(K+i)^2} + \frac{2}{K+i}, \qquad \frac{K+3i}{(K+i)^2} = -\frac{2K}{(K+i)^3} + \frac{3}{(K+i)^2}.$$

Restricting ourself to the case that $K$ is a positive integer, this observation motivates us to go back where we started and to solve the summation problem in terms of the Harmonic numbers.

$$\text{In[48]:= } \mathbf{SigmaReduce}\left[\mathbf{mySum}, \mathbf{Tower} \to \{\{\mathbf{H_{K+N}}, \mathbf{N}\}, \{\mathbf{H^{(2)}_{K+N}}, \mathbf{N}\}, \{\mathbf{H^{(3)}_{K+N}}, \mathbf{N}\}\}\right]$$

$$\text{Out[48]= } \frac{1}{6}\left(-\mathrm{H}_{\mathrm{K}}^3-3\,\mathrm{H}_{\mathrm{K}}\,\mathrm{H}_{\mathrm{K+N}}^2+\mathrm{H}_{\mathrm{K+N}}^3+3\,\mathrm{H}_{\mathrm{K}}\,\mathrm{H}_{\mathrm{K}}^{(2)}-\right.$$
$$\left.3\,\mathrm{H}_{\mathrm{K}}\,\mathrm{H}_{\mathrm{K+N}}^{(2)}+\mathrm{H}_{\mathrm{K+N}}\left(3\,\mathrm{H}_{\mathrm{K}}^2-3\,\mathrm{H}_{\mathrm{K}}^{(2)}+3\,\mathrm{H}_{\mathrm{K+N}}^{(2)}\right)-2\,\mathrm{H}_{\mathrm{K}}^{(3)}+2\,\mathrm{H}_{\mathrm{K+N}}^{(3)}\right)$$

Now the question arises how this situation is handled for any $K \in \mathbb{N}_0$. If one looks closer at the definition of our summation object $\mathrm{H}_{N+K}$, namely

[10] The following examples are for illustrative purposes only. It might well be that the reader finds a more direct approach for obtaining their closed form evaluations.

In[49]:= $\mathbf{GetDefinition}[\mathbf{H_{K+N}}, \mathbf{N}]$

Out[49]= $\sum_{o_1=1}^{N}\left(\frac{1}{K+o_1}\right)+H_K$

and in general[11]

$$H_{N+K}^{(\alpha)}=\sum_{i=1}^{N}\frac{1}{(K+i)^\alpha}+H_K^{(\alpha)}$$

for positive integers $\alpha$, one can see how this summation problem is handled. $K$ is interpreted as an indeterminate and $H_K^{(\alpha)}$ stands for a constant which guarantees the correct evaluation for any $K \in \mathbb{N}_0$.

#### 1.2.3.2 A 3-fold Sum and $q$-Harmonic Numbers

Similarly we can simplify the following $q$-analogue of the above sum,

In[50]:= $\mathbf{mySum} = \sum_{\iota_1=1}^{N}\left(\frac{(\mathbf{q})^{\iota_1}\sum_{\iota_2=1}^{\iota_1}\left(\frac{(\mathbf{q})^{\iota_2}\sum_{\iota_3=1}^{\iota_2}\left(\frac{(\mathbf{q})^{\iota_3}}{-1+\mathbf{q}^{\mathbf{K}}(\mathbf{q})^{\iota_3}}\right)}{-1+\mathbf{q}^{\mathbf{K}}(\mathbf{q})^{\iota_2}}\right)}{-1+\mathbf{q}^{\mathbf{K}}(\mathbf{q})^{\iota_1}}\right);$

by extending the underlying difference field by appropriate sum extensions and obtain:

In[51]:= $\mathbf{SigmaReduce}[\mathbf{mySum}, \mathbf{SimplifyByExt} \to \mathbf{Depth}]$

Out[51]= $\frac{1}{6}\left(2\sum_{\iota_1=1}^{N}\left(\frac{((q)^{\iota_1})^3}{(-1+q^K(q)^{\iota_1})^3}\right)+ 3\left(\sum_{\iota_1=1}^{N}\left(\frac{((q)^{\iota_1})^2}{(-1+q^K(q)^{\iota_1})^2}\right)\right)\sum_{\iota_1=1}^{N}\left(\frac{(q)^{\iota_1}}{-1+q^K(q)^{\iota_1}}\right)+\left(\sum_{\iota_1=1}^{N}\left(\frac{(q)^{\iota_1}}{-1+q^K(q)^{\iota_1}}\right)\right)^3\right)$

If one looks closer at the sum extensions, one notices that for positive integers $K$ these sums are the so called $q$-harmonic numbers [AU85]. We have not defined these kind of objects yet. This can be achieve by calling the following function:

In[52]:= $\mathbf{DefineObject}[\mathbf{qHK}[\mathbf{a\_}, \mathbf{n\_}], \mathbf{n\_}, \sum_{\mathbf{kk}=1}^{\mathbf{n}}\left(\frac{((\mathbf{q})^{\mathbf{kk}})^{\mathbf{a}}}{(1-(\mathbf{q})^{\mathbf{K}}(\mathbf{q})^{\mathbf{kk}})^{\mathbf{a}}}\right)+\mathbf{qHK}[\mathbf{a}, \mathbf{K}], \mathbf{True}]$

Applying `GetDefinition` to **qHK**`[3,N]` we just get its definition back:

In[53]:= $\mathbf{qHK}[\mathbf{3}, \mathbf{N}]//\mathbf{GetDefinition}$

Out[53]= $\sum_{kk=1}^{N}\left(\frac{((q)^{kk})^3}{(1-(q)^K(q)^{kk})^3}\right)+qHK[3, K]$

Finally we define how **qHK**`[3,N]` is evaluated for a specific integer $N$ - it is just the evaluation of its definition in terms of sums and products:

In[54]:= $\mathbf{qHK}[\mathbf{i\_}, \mathbf{n\_Integer}] := \mathbf{GetDefinition}[\mathbf{qHK}[\mathbf{i}, \mathbf{x}]]/.\{\mathbf{x} \to \mathbf{n}\}$

Indeed, here we obtain the corresponding evaluation for $N = 4$:

In[55]:= $\mathbf{qHK}[\mathbf{3}, \mathbf{4}]$

[11] Note that $H_{N+K} = H_{N+K}^{(1)}$ and $H_{N+K} = H_{N+K}^{(1)}$.

Out[55]= $\frac{q^3}{\left(1-q\ (q)_{K.}\right)^3}+\frac{q^6}{\left(1-q^2\ (q)_{K.}\right)^3}+\frac{q^9}{\left(1-q^3\ (q)_{K.}\right)^3}+\frac{q^{12}}{\left(1-q^4\ (q)_{K.}\right)^3}+\mathrm{qHK}[3,K]$

As in the previous subsection we can now simplify our sum by the new introduced objects:

In[56]:= $\mathbf{SigmaReduce}[\mathbf{mySum}, \mathbf{Tower} \to \{\mathbf{qHK}[1,n], \mathbf{qHK}[2,n], \mathbf{qHK}[3,n]\}]$

Out[56]= $\frac{1}{6}\Big(\mathrm{qHK}[1,K]^3-3\ \mathrm{qHK}[1,K]^2\ \mathrm{qHK}[1,N]-$
$\mathrm{qHK}[1,N]^3+\mathrm{qHK}[1,N]\ (3\ \mathrm{qHK}[2,K]-3\ \mathrm{qHK}[2,N])+$
$\mathrm{qHK}[1,K]\ \left(3\ \mathrm{qHK}[1,N]^2-3\ \mathrm{qHK}[2,K]+3\ \mathrm{qHK}[2,N]\right)+$
$2\ \mathrm{qHK}[3,K]-2\ \mathrm{qHK}[3,N]\Big)$

#### 1.2.3.3 A 6-fold Sum

Here we consider the following 6-fold sum

In[57]:= $\mathbf{mySum} = \sum_{e=1}^{f}\left(\sum_{d=1}^{e}\left(\sum_{c=1}^{d}\left(\sum_{b=1}^{c}\left(\sum_{a=1}^{b}\left(\frac{1}{H_a}\right)\right)\right)\right)\right)$

which can be simplified by appropriate sums that are found automatically:

In[58]:= $\mathbf{result} = \mathbf{KReduce}[\mathbf{mySum}, \mathbf{SimplifyByExt}->\mathbf{Depth}]$

Out[58]= $\frac{1}{24}\Bigg((1+f)\ (2+f)\ (3+f)\ (4+f)\ \sum_{\iota_1=1}^{f}\left(\frac{1}{H_{\iota_1}}\right)+$
$2\ (1+f)\ \Bigg((2+f)\ \Bigg(-2\ (3+f)\ \sum_{\iota_1=1}^{f}\left(\frac{\iota_1}{H_{\iota_1}}\right)+3\ \sum_{\iota_1=1}^{f}\left(\frac{(-1+\iota_1)\ \iota_1}{H_{\iota_1}}\right)\Bigg)-$
$2\ \sum_{\iota_1=1}^{f}\left(\frac{(-2+\iota_1)\ (-1+\iota_1)\ \iota_1}{H_{\iota_1}}\right)\Bigg)+$
$\sum_{\iota_1=1}^{f}\left(\frac{(-3+\iota_1)\ (-2+\iota_1)\ (-1+\iota_1)\ \iota_1}{H_{\iota_1}}\right)\Bigg)$

Since the computed sum extensions can be expressed by the sums

In[59]:= $\mathbf{tower} = \left\{\sum_{i=1}^{f}\left(\frac{1}{H_i}\right), \sum_{i=1}^{f}\left(\frac{i}{H_i}\right), \sum_{i=1}^{f}\left(\frac{i^2}{H_i}\right), \sum_{i=1}^{f}\left(\frac{i^3}{H_i}\right), \sum_{i=1}^{f}\left(\frac{i^4}{H_i}\right)\right\};$

we obtain finally

In[60]:= $\mathbf{KReduce}[\mathbf{result}, \mathbf{f}, \mathbf{Tower}->\mathbf{tower}]$

Out[60]= $\frac{1}{24}\Bigg((1+f)\ (2+f)\ (3+f)\ (4+f)\ \sum_{\iota_1=1}^{f}\left(\frac{1}{H_{\iota_1}}\right)-2\ (5+2\ f)$
$(5+f\ (5+f))\ \sum_{\iota_1=1}^{f}\left(\frac{\iota_1}{H_{\iota_1}}\right)+(35+6\ f\ (5+f))\ \sum_{\iota_1=1}^{f}\left(\frac{\iota_1^2}{H_{\iota_1}}\right)-$
$2\ (5+2\ f)\ \sum_{\iota_1=1}^{f}\left(\frac{\iota_1^3}{H_{\iota_1}}\right)+\sum_{\iota_1=1}^{f}\left(\frac{\iota_1^4}{H_{\iota_1}}\right)\Bigg)$

#### 1.2.3.4 Some Families of Identities with Harmonic Numbers

Using the feature of finding appropriate sum extensions to simplify a given sum[12], might be useful to determine and to investigate interesting families of identities. The following two examples illustrate how one can easily detect interesting relations with my package.

**Family 1**

Looking at

In[61]:= $\mathbf{mySum3} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left(\mathbf{H_k\, H_k^{(3)}}\right);$

In[62]:= $\mathbf{SigmaReduce[mySum3, SimplifyByExt \to Depth]//Simplify}$

Out[62]= $(-n + (1+n)\, H_n)\, H_n^{(3)} - \sum_{\iota_1=1}^{n} \left(\frac{-1 + H_{\iota_1}}{\iota_1^2}\right)$

and

In[63]:= $\mathbf{mySum4} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left(\mathbf{H_k\, H_k^{(4)}}\right);$

In[64]:= $\mathbf{SigmaReduce[mySum4, SimplifyByExt \to Depth]//Simplify}$

Out[64]= $(-n + (1+n)\, H_n)\, H_n^{(4)} - \sum_{\iota_1=1}^{n} \left(\frac{-1 + H_{\iota_1}}{\iota_1^3}\right)$

and

In[65]:= $\mathbf{mySum5} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left(\mathbf{H_k\, H_k^{(5)}}\right);$

In[66]:= $\mathbf{SigmaReduce[mySum5, SimplifyByExt \to Depth]//Simplify}$

Out[66]= $(-n + (1+n)\, H_n)\, H_n^{(5)} - \sum_{\iota_1=1}^{n} \left(\frac{-1 + H_{\iota_1}}{\iota_1^4}\right)$

one discovers immediately the pattern

$$\boxed{\sum_{k=1}^{n} H_k\, H_k^{(p)} = (-n + (1+n)\, H_n)\, H_n^{(p)} + H_n^{(p-1)} + \sum_{k=1}^{n} \frac{H_k}{k^{p-1}}}$$

for $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Whereas this is in reduced representation for $p > 2$, i.e. the underlying difference field extension consists only of transcendental elements, for $p = 1, 2$ we can simplify the sums further to

In[67]:= $\mathbf{SigmaReduce}[\sum_{\mathbf{k=1}}^{\mathbf{n}} (\mathbf{H_k^2})]//\mathbf{Simplify}$

Out[67]= $2\, n - (1 + 2\, n)\, H_n + (1+n)\, H_n^2$

and

In[68]:= $\mathbf{SigmaReduce}[\sum_{\mathbf{k=1}}^{\mathbf{n}} (\mathbf{H_k\, H_k^{(2)}})]//\mathbf{Simplify}$

Out[68]= $\frac{1}{2}\left(-H_n^2 - (1 + 2\, n)\, H_n^{(2)} + 2\, H_n \left(1 + (1+n)\, H_n^{(2)}\right)\right)$

[12]See Section 4.4 for further details.

**Family 2**

Analyzing

In[69]:= $\mathbf{mySum1} = \sum_{\mathbf{k}=1}^{\mathbf{n}} \left(\frac{\mathbf{H_k}}{\mathbf{1+k}}\right);$

In[70]:= **SigmaReduce[mySum1, SimplifyByExt → Depth]**

Out[70]= $\frac{2\ \mathtt{H_n} + (1+\mathtt{n})\ \mathtt{H_n^2} + (-1-\mathtt{n})\ \sum_{\iota_1=1}^{\mathtt{n}} \left(\frac{1}{\iota_1^2}\right)}{2\ (1+\mathtt{n})}$

and

In[71]:= $\mathbf{mySum2} = \sum_{\mathbf{k}=1}^{\mathbf{n}} \left(\frac{\mathbf{H_k^{(2)}}}{(\mathbf{1+k})^{\mathbf{2}}}\right);$

In[72]:= **SigmaReduce[mySum2, SimplifyByExt → Depth]//Simplify**

Out[72]= $\frac{2\ \mathtt{H_n^{(2)}} + (1+\mathtt{n})^2\ \left(\mathtt{H_n^{(2)}}\right)^2 - (1+\mathtt{n})^2\ \sum_{\iota_1=1}^{\mathtt{n}} \left(\frac{1}{\iota_1^4}\right)}{2\ (1+\mathtt{n})^2}$

and

In[73]:= $\mathbf{mySum3} = \sum_{\mathbf{k}=1}^{\mathbf{n}} \left(\frac{\mathbf{H_k^{(3)}}}{(\mathbf{1+k})^{\mathbf{3}}}\right);$

In[74]:= **SigmaReduce[mySum3, SimplifyByExt → Depth]//Simplify**

Out[74]= $\frac{2\ \mathtt{H_n^{(3)}} + (1+\mathtt{n})^3\ \left(\mathtt{H_n^{(3)}}\right)^2 - (1+\mathtt{n})^3\ \sum_{\iota_1=1}^{\mathtt{n}} \left(\frac{1}{\iota_1^6}\right)}{2\ (1+\mathtt{n})^3}$

one can immediately find the following pattern for $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

$$\boxed{\sum_{k=1}^{n} \frac{\mathrm{H}_k^{(p)}}{(1+k)^p} = 2\,\mathrm{H}_n^{(p)} + (1+n)^p\,\mathrm{H}_n^{(p)\,2} - (1+n)^p\,\mathrm{H}_n^{(2\,p)}.}$$

### 1.2.4 †*Indefinite Summation and Difference Field Extensions*

In the following I try to give a rough outline how the problem of simplifying sums by appropriate sum extensions can be formulated in difference fields.

#### 1.2.4.1 *The Underlying Difference Field of an Indefinite Sum*

In order to get a proper understanding of the problem, we first have to understand how one can construct the underlying difference field to a given summation problem. This will be illustrated with the example presented in Section 1.2.3.1.

• In a first step we try to eliminate the sum quantifier in[13]

In[75]:= $\mathbf{sum1} = \sum_{\iota_3=1}^{\iota_2}\left(\frac{\mathbf{1}}{\mathbf{K}+\iota_3}\right);$

and fail:

In[76]:= **summand1** = **SigmaReduce**[**sum1**]

Out[76]= $\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{\mathtt{K}+\iota_3}\right)$

Let $\mathbb{Q}(K)(t_1)$ be a field of rational functions over $\mathbb{Q}$. Consider the field automorphism $\sigma : \mathbb{Q}(K)(t_1) \to \mathbb{Q}(K)(t_1)$ canonically defined by

$$\sigma(c) = c \ \ \forall c \in \mathbb{Q}(K),$$
$$\sigma(t_1) = t_1 + 1.$$

Note that the automorphism $\sigma$ acts on $t_1$ like the shift operator $S$ on $\iota_2$ with $S\iota_2 = \iota_2 + 1$. Therefore we can rephrase the indefinite summation problem in terms of the difference field $(\mathbb{Q}(K)(t_1), \sigma)$: there does not exist a $g \in \mathbb{Q}(K)(t_1)$ such that

$$\sigma(g) - g = \frac{1}{K + t_1}.$$

As we show later in Section 2.4.2, we can construct a unique difference field extension $(\mathbb{Q}(K)(t_1)(t_2), \sigma)$ of $(\mathbb{Q}(K)(t_1), \sigma)$ canonically defined by

$$\sigma(t_2) = t_2 + \sigma\Big(\frac{1}{K + t_1}\Big) = t_2 + \frac{1}{K + t_1 + 1}.$$

As a result of Section 2.4.2 we obtain that $t_2$ has to be transcendental over $\mathbb{Q}(K)(t_1)$. Note that the automorphism $\sigma$ acts on $t_1$ and $t_2$ like the shift operator $S$ on $\iota_2$ and $\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)$ where $S\iota_2 = \iota_2 + 1$ and

$$S\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right) = \sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right) + \frac{1}{K+\iota_2+1}.$$

[13] In Section 1.2.3.1 we assumed that $K \notin \{-1, -2, -3 \dots\}$ is a complex number. Here we prefer to consider $K$ as a transcendental element over $\mathbb{Q}$.

• Next we try to get rid of the outermost sum of

In[77]:= $\mathbf{sum2} = \mathbf{SigmaSum}[\mathbf{summand1}/(\iota_2 + \mathbf{K}), \{\iota_2, \mathbf{1}, \iota_1\}]$

Out[77]= $\displaystyle\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{\mathtt{K}+\iota_3}\right)}{\mathtt{K}+\iota_2}\right)$

and do not succeed:

In[78]:= $\mathbf{summand2} = \mathbf{SigmaReduce}[\mathbf{sum2}]$

Out[78]= $\displaystyle\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{\mathtt{K}+\iota_3}\right)}{\mathtt{K}+\iota_2}\right)$

In terms of difference fields this means that we do not find a $g \in \mathbb{Q}(K)(t_1)(t_2)$ such that

$$\sigma(g) - g = \frac{t_2}{K + t_1}.$$

Thus we can proceed by constructing a unique difference field extension $(\mathbb{Q}(K)(t_1)(t_2)(t_3), \sigma)$ of $(\mathbb{Q}(K)(t_1)(t_2), \sigma)$ canonically defined by

$$\sigma(t_3) = t_3 + \sigma(\frac{t_2}{K + t_1})$$

where $t_3$ has to be transcendental over $\mathbb{Q}(K)(t_1)(t_2)$.

• Finally we attack the outermost sum of

In[79]:= $\mathbf{sum3} = \mathbf{SigmaSum}[\mathbf{summand2}/(\iota_1 + \mathbf{K}), \{\iota_1, \mathbf{1}, \mathbf{n}\}]$

Out[79]= $\displaystyle\sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{\mathtt{K}+\iota_3}\right)}{\mathtt{K}+\iota_2}\right)}{\mathtt{K}+\iota_1}\right)$

and fail again:

In[80]:= $\mathbf{SigmaReduce}[\mathbf{sum3}]$

Out[80]= $\displaystyle\sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{\mathtt{K}+\iota_3}\right)}{\mathtt{K}+\iota_2}\right)}{\mathtt{K}+\iota_1}\right)$

This means in terms of difference fields that we do not find a $g \in \mathbb{Q}(K)(t_1)(t_2)(t_3)$ such that

$$\sigma(g) - g = \frac{t_3}{K + t_1}.$$

Again we proceed by constructing a unique difference field extension $(\mathbb{Q}(K)(t_1)(t_2)(t_3)(t_4), \sigma)$ of $(\mathbb{Q}(K)(t_1)(t_2)(t_3), \sigma)$ canonically defined by

$$\sigma(t_4) = t_4 + \sigma(\frac{t_3}{K + t_1})$$

where $t_4$ is transcendental over $\mathbb{Q}(K)(t_1)(t_2)(t_3)$.

By this process we have constructed a difference field $(\mathbb{Q}(K)(t_1)(t_2)(t_3)(t_4), \sigma)$ where the domain $\mathbb{Q}(K, t_1, t_2, t_3, t_4)$ is a field of rational functions over $\mathbb{Q}$ and in which $t_4$ "represents" the indefinite sum `sum3`.
Actually what I have sketched here is an oversimplified process how one can construct a difference field to a given indefinite summation problem. More precisely, in my implementation I construct a difference ring homomorphism from a sub-difference ring of the difference field $(\mathbb{Q}(K)(t_1)(t_2)(t_3)(t_4), \sigma)$ to the ring of sequences $(\mathcal{S}(\mathbb{Q}(K)), S)$ in which the summation objects can be properly defined. Loosely speaking, this homomorphism will link the element $t_4$ with the summation object `sum3`. How we can construct the underlying difference field and its difference ring homomorphism for a given summation problem will be treated carefully in Section 2.5.

#### 1.2.4.2 *Appropriate Sum Extensions for an Indefinite Sum*

We will start again to build up a difference field in which we can describe the indefinite sum given in Section 1.2.3.1; but this time we try to extend the difference field step by step by avoiding nested sum extensions.

• We start again with the difference field $(\mathbb{Q}(K)(t_1), \sigma)$ and fail to eliminate the sum quantifier in

In[81]:= $\mathbf{sum1} = \sum_{\iota_3=1}^{\iota_2} \left(\frac{\mathbf{1}}{\mathbf{K} + \iota_3}\right);$

In[82]:= $\mathbf{summand1} = \mathbf{SigmaReduce}[\mathbf{sum1}, \mathbf{SimplifyByExt} \to \mathbf{Depth}]$

Out[82]= $\sum_{\iota_3=1}^{\iota_2} \left(\frac{1}{\mathtt{K} + \iota_3}\right)$

Therefore, as indicated in the previous section, we can construct a unique difference field extension $(\mathbb{Q}(K)(t_1)(t_2), \sigma)$ of $(\mathbb{Q}(t_1), \sigma)$ canonically defined by

$$\sigma(t_2) = t_2 + \sigma(\frac{1}{K + t_1})$$

where $t_2$ must be transcendental over $\mathbb{Q}(K)(t_1)$.

• Now we attempt to remove the outermost sum of

In[83]:= $\mathbf{sum2} = \mathbf{SigmaSum}[\mathbf{summand1}/(\iota_2 + \mathbf{K}), \{\iota_2, \mathbf{1}, \iota_1\}]$

Out[83]= $\sum_{\iota_2=1}^{\iota_1} \left(\frac{\sum_{\iota_3=1}^{\iota_2} \left(\frac{1}{\mathtt{K}+\iota_3}\right)}{\mathtt{K} + \iota_2}\right)$

but this time we look for an appropriate sum extension where the summand does not depend on the sum $\sum_{i=1}^{\iota_1} \frac{1}{K+i}$.

In[84]:= $\mathbf{summand2} = \mathbf{SigmaReduce}[\mathbf{sum2}, \mathbf{SimplifyByExt} \to \mathbf{Depth}]$

Out[84]= $\frac{2 \sum_{\iota_2=1}^{\iota_1} \left(\frac{1}{\mathtt{K}+\iota_2}\right) + \mathtt{K} \left(\sum_{\iota_2=1}^{\iota_1} \left(\frac{1}{\mathtt{K}+\iota_2}\right)\right)^2 - \sum_{\iota_2=1}^{\iota_1} \left(\frac{\mathtt{K}+2\ \iota_2}{(\mathtt{K}+\iota_2)^2}\right)}{2\ \mathtt{K}}$

Here my algorithm has found automatically a difference field extension $(\mathbb{Q}(K)(t_1)(t_2)(t_3'), \sigma)$ of $(\mathbb{Q}(K)(t_1)(t_2), \sigma)$ canonically defined by

$$\sigma(t_3') = t_3' + \frac{K + 2\,t_1 + 2}{(K + t_1 + 1)^2}.$$

In addition, it follows by a result of Section 2.4.2 that $t_3'$ must be transcendental over $\mathbb{Q}(K)(t_1)(t_2)$. Note that the automorphism $\sigma$ acts on $t_1$ and $t_3'$ like the shift operator $S$ on $\iota_1$ and $\sum_{\iota_2=1}^{\iota_1}\left(\frac{K+2\ \iota_2}{(K+\iota_2)^2}\right)$ where $S\iota_1 = \iota_1 + 1$ and

$$S\sum_{\iota_2=1}^{\iota_1}\left(\frac{K+2\ \iota_2}{(K+\iota_2)^2}\right) = \sum_{\iota_2=1}^{\iota_1}\left(\frac{K+2\ \iota_2}{(K+\iota_2)^2}\right) + \frac{K+2\ \iota_1+2}{(K+\iota_1+1)^2}.$$

Furthermore the algorithm finds a $g_3 \in \mathbb{Q}(t_1)(t_2)(t_3')$ such that

$$\sigma(g_3) - g_3 = \frac{t_2}{K+t_1},$$

namely

$$\sigma(g_3) = \frac{2\,t_2 + K\,t_2^2 - t_3'}{2\,K}.$$

Translating $\sigma(g_3)$ back in terms of the corresponding summation objects leads to `summand2`.

• Finally we succeed in eliminating the outermost sum of

In[85]:= $\mathbf{sum3} = \mathbf{SigmaSum}[\mathbf{summand2}/(\iota_1 + \mathbf{K}), \{\iota_1, \mathbf{1}, \mathbf{n}\}]$

Out[85]= $\sum_{\iota_1=1}^{n}\left(\frac{2\ \sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{K+\iota_2}\right) + K\left(\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{K+\iota_2}\right)\right)^2 - \sum_{\iota_2=1}^{\iota_1}\left(\frac{K+2\ \iota_2}{(K+\iota_2)^2}\right)}{2\ K\ (K+\iota_1)}\right)$

by an appropriate sum extension where the summand does not depend on $\sum_{i=1}^{n}\frac{1}{K+i}$ and $\sum_{i=1}^{n}\frac{K+2\,i}{(K+i)^2}$.

In[86]:= $\mathbf{SigmaReduce}[\mathbf{sum3}, \mathbf{SimplifyByExt} \to \mathbf{Depth}]$

Out[86]= $\frac{1}{6\ K^2}\left(6\ \sum_{\iota_1=1}^{n}\left(\frac{1}{K+\iota_1}\right) + 6\ K\left(\sum_{\iota_1=1}^{n}\left(\frac{1}{K+\iota_1}\right)\right)^2 + K^2\left(\sum_{\iota_1=1}^{n}\left(\frac{1}{K+\iota_1}\right)\right)^3 + \left(-3 - 3\ K\sum_{\iota_1=1}^{n}\left(\frac{1}{K+\iota_1}\right)\right)\sum_{\iota_1=1}^{n}\left(\frac{K+2\ \iota_1}{(K+\iota_1)^2}\right) - K\sum_{\iota_1=1}^{n}\left(\frac{K+3\ \iota_1}{(K+\iota_1)^3}\right)\right)$

More precisely, my implementation finds a difference field extension $(\mathbb{Q}(K)(t_1)(t_2)(t_3')(t_4'), \sigma)$ of $(\mathbb{Q}(K)(t_1)(t_2)(t_3'), \sigma)$ canonically defined by

$$\sigma(t_4') = t_4' + \frac{K + 3\,t_1 + 3}{(K+t_1+1)^3}$$

where $t_4'$ is transcendental over $\mathbb{Q}(K)(t_1)(t_2)(t_3')$ in which we find a $g_4 \in \mathbb{Q}(K)(t_1)(t_2)(t_3')(t_4')$ with

$$\sigma(g_4) - g_4 = \frac{\sigma(g_3)}{K+t_1}.$$

In particular the algorithm finds that

$$\sigma(g_4) = \frac{6\,t_2 + 6\,K\,t_2^2 + K^2\,t_2^3 - 3\,t_3' - 3\,K\,t_2\,t_3' - K\,t_4'}{6\,K^2}$$

which represents the closed form of `sum3`.

Looking closer at the difference fields constructed in this and the previous section one can recognize that there is a difference field isomorphism[14]

$$\boxed{(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma) \overset{\tau}{\simeq} (\mathbb{Q}(t_1)(t_2)(t_3')(t_4'),\sigma)}$$

canonically defined by

$$\begin{aligned}
\tau(t_1) &= t_1,\\
\tau(t_2) &= t_2,\\
\tau(t_3) = \sigma(g_3) &= \frac{2\,t_2 + K\,t_2^2 - t_3'}{2\,K},\\
\tau(t_4) = \sigma(g_4) &= \frac{6\,t_2 + 6\,K\,t_2^2 + K^2\,t_2^3 - 3\,t_3' - 3\,K\,t_2\,t_3' - K\,t_4'}{6\,K^2}.
\end{aligned}$$

Intuitively this means that the difference fields $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma)\simeq(\mathbb{Q}(t_1)(t_2)(t_3')(t_4'),\sigma)$ are –up to renaming– just the same. On the other side –from the simplification point of view– it matters tremendously how “the” difference field is constructed in which the summation problem is formulated.

These aspects will be worked out in Sections 4.2 and 4.4. Additionally I will consider in details in Section 4.4 how one can construct difference field extensions to simplify a given summation problem.

[14]Consider Section 2.1.1 for more details about difference field isomorphisms in general and Proposition 2.3.2 in Section 2.3.2 to understand this particular isomorphism.

## 1.3 Definite Summation

Besides indefinite summation our summation package is able to deal with definite summation. The following two Sections 1.3.1 and 1.3.2 will illustrate how one can find closed forms of definite sums with our package. Here we use the following method: first we try to compute a recurrence which is satisfied by the given definite sum and next we look for solutions of this recurrence. Then finally we make the attempt to find a closed form for the definite sum by combining the solutions of the recurrence in an appropriate way. In Sections 1.3.3 and 1.3.4 we will describe in more details how we *find recurrences* for a given definite sum and how we can *solve a given recurrence* in terms of our difference field setting.

### 1.3.1 Krattenthaler's Example

In [FK00] the following identity is used to solve a combinatorial counting problem:

$$\sum_{k=1}^{n} \frac{\mathrm{H}_k\ (3+k+n)!\ (-1)^k\ (-1)^{-1+n}}{(1+k)!\ (2+k)!\ (-k+n)!} + \frac{(n)!}{(3+n)!} \sum_{k=1}^{n} -\frac{(3+k+n)!\ (-1)^k\ (1-(2+n)\ (-1)^n)}{k\ (1+k)!^2\ (-k+n)!} = (2+n)(-1)^n - 2. \quad (1.2)$$

With my package one not only can prove this identity automatically but one even is able to *find* the closed form

$$(2+n)(-1)^n - 2.$$

In the following the two sums on the left hand side of (1.2) are considered separately.

In[87]:= $\mathbf{mySum1 = SigmaSum[SigmaPower[-1, k]\ SigmaPower[-1, n-1]}$
$\mathbf{SigmaFactorial[n+k+3]/(SigmaFactorial[k+1]}$
$\mathbf{SigmaFactorial[k+2]SigmaFactorial[n-k])}$
$\mathbf{SigmaHNumber[k], \{k, 1, n\}]}$

Out[87]= $\sum_{k=1}^{n} \left( \frac{\mathrm{H_k}\ (3+k+n)!.\ (-1)^{k.}\ (-1)^{-1+n.}}{(1+k)!.\ (2+k)!.\ (-k+n)!.} \right)$

In[88]:= $\mathbf{mySum2 = SigmaSum[-SigmaPower[-1, k]}$
$\mathbf{SigmaFactorial[n+k+3](1 - SigmaPower[-1, n](n+2))/}$
$\mathbf{(k\ SigmaFactorial[k+1]\hat{}2SigmaFactorial[n-k]),}$
$\mathbf{\{k, 1, n\}]}$

Out[88]= $\sum_{k=1}^{n} \left( -\frac{(3+k+n)!.\ (-1)^{k.}\ (1-(2+n)\ (-1)^{n.})}{k\ (1+k)!.^{2}\ (-k+n)!.} \right)$

According to the first sum, I will demonstrate the basic procedure to solve definite summation problems with my Mathematica package; whereas considering the second sum, I focus on some technical details one has to take into account to find closed forms.

### A closed form of mySum1

#### Finding a recurrence

First a recurrence is found that is satisfied by mySum1.

In[89]:= **rec1 = GenerateRecurrence[mySum1]//Simplify**

Out[89]= $\{\mathtt{n}\ (1+\mathtt{n})\ (2+\mathtt{n})\ (-1+\mathtt{n})!\ \big((2+\mathtt{n})\ (4+\mathtt{n})^2\ (27+15\ \mathtt{n}+2\ \mathtt{n}^2)\ \mathtt{SUM}[\mathtt{n}]-$
$(3+\mathtt{n})\ (4+\mathtt{n})\ (9+2\ \mathtt{n})\ (13+8\ \mathtt{n}+\mathtt{n}^2)\ \mathtt{SUM}[1+\mathtt{n}]-$
$(3+\mathtt{n})\ (4+\mathtt{n})\ (5+2\ \mathtt{n})\ (6+6\ \mathtt{n}+\mathtt{n}^2)\ \mathtt{SUM}[2+\mathtt{n}]+$
$(3+\mathtt{n})^2\ (5+\mathtt{n})\ (20+13\ \mathtt{n}+2\ \mathtt{n}^2)\ \mathtt{SUM}[3+\mathtt{n}]\big) ==$
$-2\ (-1)^{\mathtt{n}}\ (315+286\ \mathtt{n}+84\ \mathtt{n}^2+8\ \mathtt{n}^3)\ (4+\mathtt{n})!\}$

The idea how to find a recurrence is based on Zeilberger's creative telescoping method [Zei90]. Although Karr's original summation algorithm [Kar81] was already capable of carrying out creative telescoping, nobody has noticed this possibility until now.

#### Solving the recurrence

In the second step, solutions of the recurrence are computed. Here we set the option NestedSumExt→ ∞ in order to find all nested sums[15] expressed in the underlying difference ring which deliver new solutions.

In[90]:= **SolveRecurrence[rec1, SUM[n], NestedSumExt → ∞]**

Out[90]= $\{\{0,1\},\{0,(2+\mathtt{n})\ (-1)^{\mathtt{n}}\},$

$$\Big\{0,-\frac{2-\mathtt{n}+6\ \sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)+6\ \mathtt{n}\ \sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)}{6\ (1+\mathtt{n})}\Big\},$$

$$\Big\{1,\frac{1}{(1+\mathtt{n})\ (2+\mathtt{n})}\left((-1)^{\mathtt{n}}\left(3+3\ \mathtt{n}+\mathtt{n}^2+8\sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)+16\ \mathtt{n}\sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)+10\ \mathtt{n}^2\sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)+2\ \mathtt{n}^3\sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)\right)\right)\Big\}\Big\}$$

Using partial fraction decomposition, we obtain

$$\frac{1+i}{i\,(2+i)}=\frac{1}{2\,i}+\frac{1}{2\,(2+i)}$$

which tells us that the result can be reformulated by the Harmonic numbers. So in our situation we just again solve the recurrence but this time we extend the solution range, more precisely the underlying difference ring, with the Harmonic numbers by setting Tower→ $\{H_n\}$.

In[91]:= **recSol1 =**
**SolveRecurrence[rec1, SUM[n], Tower → {H_n}]**

Out[91]= $$\Big\{\{0,1\},\Big\{0,\frac{-9\ \mathtt{n}-5\ \mathtt{n}^2+8\ \mathtt{H_n}+12\ \mathtt{n}\ \mathtt{H_n}+4\ \mathtt{n}^2\ \mathtt{H_n}}{(1+\mathtt{n})\ (2+\mathtt{n})}\Big\},\{0,(2+\mathtt{n})\ (-1)^{\mathtt{n}}\},$$
$$\Big\{1,\frac{(-32\ \mathtt{n}-33\ \mathtt{n}^2-9\ \mathtt{n}^3+32\ \mathtt{H_n}+64\ \mathtt{n}\ \mathtt{H_n}+40\ \mathtt{n}^2\ \mathtt{H_n}+8\ \mathtt{n}^3\ \mathtt{H_n})\ (-1)^{\mathtt{n}}}{4\ (1+\mathtt{n})\ (2+\mathtt{n})}\Big\}\Big\}$$

[15]See Section 1.3.4.2 and 4.5 for more details.

This has to be interpreted as follows: our algorithm delivers three linear independent solutions of the homogeneous version of the recurrence, namely

$$1, \qquad \frac{-9\,n - 5\,n^2 + 8\,\mathrm{H}_n + 12\,n\,\mathrm{H}_n + 4\,n^2\,\mathrm{H}_n}{(1+n)\,(2+n)}, \qquad (2+n)\,(-1)^n$$

and one particular solution of the inhomogeneous recurrence itself:

$$\frac{(-32\,n - 33\,n^2 - 9\,n^3 + 32\,\mathrm{H}_n + 64\,n\,\mathrm{H}_n + 40\,n^2\,\mathrm{H}_n + 8\,n^3\,\mathrm{H}_n)\,(-1)^n}{4\,(1+n)\,(2+n)}.$$

### Finding the linear combination

Finally, the closed form of `mySum1` is that linear combination of the homogeneous solutions plus the inhomogeneous solution which has exactly the same initial values as `mySum1`. This is also computed automatically:

In[92]:= **solution1 =**
**FindLinearCombination[recSol1, mySum1, 3]//Simplify**

Out[92]= $-\frac{1}{(1+n)\,(2+n)}\left(5+3\,n+\left(-5-2\,n+2\,n^2+n^3\right)\,(-1)^{n}-2\,\left(2+3\,n+n^2\right)\,\mathrm{H}_n\,\left(-1+(2+n)\,(-1)^{n}\right)\right)$

## A Closed form of `mySum2`

### Finding a recurrence

Similar to `mySum1` a recurrence of order 2 for the second sum is computed.

In[93]:= **rec2 = GenerateRecurrence[mySum2, RecOrder → 2]**

55.03 Second

Out[93]= $\{n\,(1+n)\,(3+n)\,(4+n)\,(7+2\,n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(-1+n)!\,\mathrm{SUM}[n]-6\,n\,(1+n)\,(3+n)^2\,(-1+2\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(-1+n)!\,\mathrm{SUM}[1+n]-n\,(1+n)\,(2+n)\,(3+n)\,(5+2\,n)\,(-1+2\,(-1)^n+(-1)^n\,n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(-1+n)!\,\mathrm{SUM}[2+n]==2\,\left(35+24\,n+4\,n^2\right)\,\left(1-3\,(-1)^n-10\,(-1)^{2\,n}+24\,(-1)^{3\,n}-(-1)^n\,n-6\,(-1)^{2\,n}\,n+26\,(-1)^{3\,n}\,n-(-1)^{2\,n}\,n^2+9\,(-1)^{3\,n}\,n^2+(-1)^{3\,n}\,n^3\right)\,(4+n)!\}$

Here the order of the recurrence we were looking for is specified by the option `RecOrder`→2. By default - as in the previous example for `mySum1` - the algorithm starts looking for a recurrence of order one and increases the order step by step if it does not succeed in finding a recurrence of the given order.

**Solving the recurrence**

In the second step, the following solutions for the recurrence are found[16].

In[94]:= **SolveRecurrence[rec2, SUM[n], NestedSumExt → ∞, WithMinusPower → True]// Simplify**

Out[94]= $$\Big\{\big\{0, -n\,(8+6\,n+n^2)+(9+28\,n+23\,n^2+8\,n^3+n^4)\,(-1)^{n}\big\}, \big\{0, 1882+725\,n-162\,n^2-27\,n^3+(-698+756\,n+621\,n^2+216\,n^3+27\,n^4)\,(-1)^{n}\big\},$$
$$\Big\{1, \frac{1}{3}\Big(-9-55\,n-116\,n^2-105\,n^3-48\,n^4-11\,n^5-n^6- 2\,(6+11\,n+6\,n^2+n^3)\sum_{\iota_1=1}^{n}\Big(\frac{-3+\iota_1\,(2+3\,\iota_1+\iota_1^2)\,(-1)^{\iota_1}}{1+\iota_1}\Big)+ (2+n)\,(-1)^{n}\Big(n\,(4+13\,n+7\,n^2+n^3)+ 2\,(6+11\,n+6\,n^2+n^3)\sum_{\iota_1=1}^{n}\Big(\frac{-3+\iota_1\,(2+3\,\iota_1+\iota_1^2)\,(-1)^{\iota_1}}{1+\iota_1}\Big)\Big)\Big)\Big\}\Big\}$$

By partial fraction decomposition we obtain

$$\frac{-3+i\,(2+3\,i+i^2)\,(-1)^i}{1+i} = -\frac{3}{1+i}+i\,(2+i)\,(-1)^i$$

and therefore we see immediately that the solutions of the recurrence can be expressed by the Harmonic numbers:

In[95]:= **recSol2 = SolveRecurrence[rec2, SUM[n], Tower → {H_n}, WithMinusPower → True]**

Out[95]= $$\Big\{\big\{0, -8\,n-6\,n^2-n^3+9\,(-1)^{n}+ 28\,n\,(-1)^{n}+23\,n^2\,(-1)^{n}+8\,n^3\,(-1)^{n}+n^4\,(-1)^{n}\big\},$$
$$\big\{0, 114+25\,n-24\,n^2-4\,n^3-21\,(-1)^{n}+ 112\,n\,(-1)^{n}+92\,n^2\,(-1)^{n}+32\,n^3\,(-1)^{n}+4\,n^4\,(-1)^{n}\big\},$$
$$\big\{1, \frac{1}{76}\,\big(-650\,n-630\,n^2-143\,n^3+912\,H_n+1672\,n\,H_n+912\,n^2\,H_n+ 152\,n^3\,H_n-195\,(-1)^{n}+1040\,n\,(-1)^{n}+1845\,n^2\,(-1)^{n}+ 916\,n^3\,(-1)^{n}+143\,n^4\,(-1)^{n}-1824\,H_n\,(-1)^{n}-4256\,n\,H_n\,(-1)^{n}- 3496\,n^2\,H_n\,(-1)^{n}-1216\,n^3\,H_n\,(-1)^{n}-152\,n^4\,H_n\,(-1)^{n}\big)\big\}\Big\}$$

To handle this problem, I have generalized Karrs's algorithm for solving linear difference equations of any order. For this generalization a denominator bounding is used which was developed by M. Bronstein [Bro99, Bro00]. There are still unsolved problems, as I will outline in Section 1.3.4.
In addition, we have to consider the algebraic relation

$$((-1)^k)^2 = 1$$

to find all solutions for the recurrence. In order to take care of this, one can set the option `WithMinusPower→True`. Further comments on this kind of extensions can be found on page 42.

[16]The option `WithMinusPower→True` is explained below.

**Finding the linear combination of `mySum2`**

Finally the closed form of `mySum2` can be found as before:

In[96]:= **solution2 = FindLinearCombination[recSol2, mySum2, 2]//Simplify**

Out[96]= $-(3+\text{n})\,\big(-1+3\,\text{n}+2\,\text{n}^2-\big(-1+6\,\text{n}+7\,\text{n}^2+2\,\text{n}^3\big)\,(-1)^{\text{n}}+2\,\big(2+3\,\text{n}+\text{n}^2\big)\,\text{H}_{\text{n}}\,\big(-1+(2+\text{n})\,(-1)^{\text{n}}\big)\big)$

## The closed form of `mySum1`+`mySum2`

Finally, by combining the closed forms of `mySum1` and `mySum2` the closed form of the original summation problem (1.2) is computed.

In[97]:= **solution1 + solution2/((n + 1)(n + 2)(n + 3))//Simplify**

Out[97]= $-2+(2+\text{n})\,(-1)^{\text{n}}$

### 1.3.2 Kirschenhofer's Example and its $q$-Generalization

In [Kir96] P. Kirschenhofer found several families of sum identities, like for instance for the sums

$$\sum_{k=0}^{N}\frac{\binom{N}{k}(-1)^k}{(k+K)^m}$$

where $K$ and $m$ are positive integers. In the following session we will find the closed form for the particular case $m=4$.

In[98]:= **mySum** $=\sum_{\mathbf{k}=0}^{\mathbf{N}}\left(\frac{\binom{\mathbf{N}}{\mathbf{k}}\,(-1)^{\mathbf{k}}}{(\mathbf{k}+\mathbf{K})^4}\right)$;

First we find a recurrence for the sum[17].

In[99]:= **rec = GenerateRecurrence[mySum, RecOrder → 4]**

Out[99]= $\{-(1+\text{N})\,(2+\text{N})\,(3+\text{N})\,(4+\text{N})\,\text{SUM}[\text{N}]+2\,(2+\text{N})\,(3+\text{N})\,(4+\text{N})\,(5+2\,\text{K}+2\,\text{N})\,\text{SUM}[1+\text{N}]-(3+\text{N})\,(4+\text{N})\,\big(55+36\,\text{K}+6\,\text{K}^2+36\,\text{N}+12\,\text{K}\,\text{N}+6\,\text{N}^2\big)\,\text{SUM}[2+\text{N}]+(4+\text{N})\,(7+2\,\text{K}+2\,\text{N})\,\big(25+14\,\text{K}+2\,\text{K}^2+14\,\text{N}+4\,\text{K}\,\text{N}+2\,\text{N}^2\big)\,\text{SUM}[3+\text{N}]-(4+\text{K}+\text{N})^4\,\text{SUM}[4+\text{N}]==0\}$

Finally we solve the recurrence by using the following product extension[18]

In[100]:= **tower** $=\{\{\binom{\mathbf{K}+\mathbf{N}}{\mathbf{K}},\mathbf{N}\}\}$;

and find all nested sums which provide us with a solution of the recurrence.

[17] Instead of using my package to find the recurrence, one could use the much more efficient Paule-Schorn package described in [PS95a] which can deal exactly with those hypergeometric sequences as input.

[18] This product extension can be immediately found, if one first solves the problem for the more specific cases $K = 1, 2, 3...$ and recognizes from the pattern of the results the binomial $\binom{N+K}{K}$. Otherwise, one can use M. Petkovšek's package `Hyper` in combination with my function `FindProductExtensions` (Section 1.1.2) which leads immediately to the needed product extension.

In[101]:= $\mathbf{SolveRecurrence[rec[[1]], SUM[N], Tower \to tower, NestedSumExt \to \infty]}$

$$\text{Out[101]=} \Big\{\Big\{0, \frac{1}{\binom{K+N}{K}}\Big\}, \Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)}{\binom{K+N}{K}}\Big\}, \Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{K+\iota_2}\right)}{K+\iota_1}\right)}{\binom{K+N}{K}}\Big\},$$

$$\Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)}{K+\iota_2}\right)}{K+\iota_1}\right)}{\binom{K+N}{K}}\Big\}, \{1, 0\}\Big\}$$

Taking Section 1.2.3.1 into account, we know that those multisums can be expressed by the Harmonic numbers. Therefore we can solve the recurrence in one stroke using the Harmonic numbers itself.

In[102]:= $\mathbf{tower} = \{\{\binom{\mathbf{K+N}}{\mathbf{K}}, \mathbf{N}\}, \{\mathbf{H}^{(3)}_{\mathbf{N+K}}, \mathbf{N}\}, \{\mathbf{H}^{(2)}_{\mathbf{N+K}}, \mathbf{N}\}, \{\mathbf{H}_{\mathbf{N+K}}, \mathbf{N}\}\};$

In[103]:= $\mathbf{recSol = SolveRecurrence[rec[[1]], SUM[N], Tower \to tower]}$

$$\text{Out[103]=} \Big\{\Big\{0, \frac{1}{\binom{K+N}{K}}\Big\}, \Big\{0, \frac{H_{K+N}}{\binom{K+N}{K}}\Big\}, \Big\{0, \frac{H^2_{K+N} + H^{(2)}_{K+N}}{\binom{K+N}{K}}\Big\},$$

$$\Big\{0, \frac{H^3_{K+N} + 3\, H_{K+N}\, H^{(2)}_{K+N} + 2\, H^{(3)}_{K+N}}{\binom{K+N}{K}}\Big\}, \{1, 0\}\Big\}$$

Finally we derive the following closed form:

In[104]:= $\mathbf{sol = FindLinearCombination[recSol, mySum, N, 3]//Simplify}$

$$\text{Out[104]=} \frac{1}{6\, K^4 \binom{K+N}{K}} \Big(6 - K^3\, H^3_K + 3\, K^2\, H^2_{K+N} + K^3\, H^3_{K+N} + 3\, K^2\, H^2_K\, (1 + K\, H_{K+N}) - 3\, K^2\, H^{(2)}_K + 3\, K^2\, H^{(2)}_{K+N} - 3\, K\, H_K\, \big(2 + 2\, K\, H_{K+N} + K^2\, H^2_{K+N} - K^2\, H^{(2)}_K + K^2\, H^{(2)}_{K+N}\big) + H_{K+N}\, \big(6\, K - 3\, K^3\, H^{(2)}_K + 3\, K^3\, H^{(2)}_{K+N}\big) - 2\, K^3\, H^{(3)}_K + 2\, K^3\, H^{(3)}_{K+N}\Big)$$

## A $q$-Generalization

Now let us look at the following $q$-generalization of the above sum which specialized at $K = 0$ gives a variation introduced in [Dil95, Theorem 4].

In[105]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{N}} \frac{(-\mathbf{1})^{\mathbf{k}}\, (\mathbf{q})^{\frac{1}{2}\,(-\mathbf{1}+\mathbf{k})\,\mathbf{k}} \begin{bmatrix}\mathbf{N}\\ \mathbf{k}\end{bmatrix}_{\mathbf{q}}}{(-\mathbf{1} + (\mathbf{q})^{\mathbf{K}}\, (\mathbf{q})^{\mathbf{k}})^{\mathbf{4}}};$

Using my package one can compute a recurrence for the $q$-sum

In[106]:= **rec = GenerateRecurrence[mySum, RecOrder → 4]**

Out[106]= $\{q^6(-1+q\,(q)^{N.})(-1+q^2\,(q)^{N.})(-1+q^3\,(q)^{N.})(-1+q^4\,(q)^{N.})\,\mathtt{SUM}[N]-$
$q^3(-1+q^2\,(q)^{N.})(-1+q^3\,(q)^{N.})$
$(-1+q^4\,(q)^{N.})(-1-q-q^2-q^3+4\,q^4\,(q)^{K.}\,(q)^{N.})\,\mathtt{SUM}[1+N]+$
$q\,(-1+q^3\,(q)^{N.})(-1+q^4\,(q)^{N.})$
$(1+q+2\,q^2+q^3+q^4+(-4\,q^4-4\,q^5-4\,q^6)\,(q)^{K.}\,(q)^{N.}+$
$6\,q^8\,((q)^{K.})^2\,((q)^{N.})^2)\,\mathtt{SUM}[2+N]-$
$(-1+q^4\,(q)^{N.})(-1-q+2\,q^4\,(q)^{K.}\,(q)^{N.})(1+q^2+$
$(-2\,q^4-2\,q^5)\,(q)^{K.}\,(q)^{N.}+2\,q^8\,((q)^{K.})^2\,((q)^{N.})^2)\,\mathtt{SUM}[3+N]+$
$(1-4\,q^4\,(q)^{K.}\,(q)^{N.}+6\,q^8\,((q)^{K.})^2\,((q)^{N.})^2-$
$4\,q^{12}\,((q)^{K.})^3\,((q)^{N.})^3+q^{16}\,((q)^{K.})^4\,((q)^{N.})^4)\,\mathtt{SUM}[4+N]==$
$0\}$

Here I want to comment that A. Riese's package `qZeil` [PR97] is tremendously faster than my package to find a recurrence for $q$-summation problems. Especially, Riese's package provides features to find appropriate generalizations of a given hypergeometric sequence to a $q$-hypergeometric sequence.
Using the $q$-analogue package $q$-`Hyper` [APP98], we easily find the following product extension which gives a solution of the recurrence.

In[107]:= **tower** $= \{\{\begin{bmatrix}\mathbf{K+N}\\ \mathbf{K}\end{bmatrix}_{\mathbf{q}}, \mathbf{N}\}\};$

Looking for all nested sum extensions using the underlying difference field extended by this product extension, gives us the following solutions:

In[108]:= **SolveRecurrence[rec, SUM[N], NestedSumExt → 4, Tower → tower]**

$$\mathrm{Out[108]=}\ \Big\{\Big\{0, \frac{1}{\begin{bmatrix}K+N\\ K\end{bmatrix}_q}\Big\}, \Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{(q)^{\iota_1.}}{-1+q^K\,(q)^{\iota_1.}}\right)}{q\begin{bmatrix}K+N\\ K\end{bmatrix}_q}\Big\}, \Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{(q)^{\iota_1.}\sum_{\iota_2=1}^{\iota_1}\left(\frac{(q)^{\iota_2.}}{-1+q^K\,(q)^{\iota_2.}}\right)}{-1+q^K\,(q)^{\iota_1.}}\right)}{q^2\begin{bmatrix}K+N\\ K\end{bmatrix}_q}\Big\},$$

$$\Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{(q)^{\iota_1.}\sum_{\iota_2=1}^{\iota_1}\left(\frac{(q)^{\iota_2.}\sum_{\iota_3=1}^{\iota_2}\left(\frac{(q)^{\iota_3.}}{-1+q^K\,(q)^{\iota_3.}}\right)}{-1+q^K\,(q)^{\iota_2.}}\right)}{-1+q^K\,(q)^{\iota_1.}}\right)}{q^3\begin{bmatrix}K+N\\ K\end{bmatrix}_q}\Big\}, \{1, 0\}\Big\}$$

Finally, the discussion in Section 1.2.3.2 motivates us to solve the recurrence with the $q$-Harmonic numbers

In[109]:= **tower** $= \{\{\begin{bmatrix}\mathbf{K+N}\\ \mathbf{K}\end{bmatrix}_{\mathbf{q}}, \mathbf{N}\}, \mathbf{qHK[3,N], qHK[2,N], qHK[1,N]}\};$

In[110]:= **recSol = SolveRecurrence[rec[[1]], SUM[N], Tower → tower]**

$$\text{Out[110]=}\ \Big\{\Big\{0, \frac{1}{\begin{bmatrix} K+N \\ K \end{bmatrix}_q}\Big\}, \Big\{0, \frac{\mathrm{qHK}[1,N]}{\begin{bmatrix} K+N \\ K \end{bmatrix}_q}\Big\}, \Big\{0, \frac{\mathrm{qHK}[1,N]^2+\mathrm{qHK}[2,N]}{\begin{bmatrix} K+N \\ K \end{bmatrix}_q}\Big\},$$
$$\Big\{0, \frac{\mathrm{qHK}[1,N]^3+3\ \mathrm{qHK}[1,N]\ \mathrm{qHK}[2,N]+2\ \mathrm{qHK}[3,N]}{\begin{bmatrix} K+N \\ K \end{bmatrix}_q}\Big\}, \{1,0\}\Big\}$$

which provides us with the following closed form:

In[111]:= **FindLinearCombination[recSol, mySum, 4, MinInitialValue → 1]//FullSimplify**

$$\begin{aligned}
\text{Out[111]=}\ &\frac{1}{6\ (-1+(q)^{K.})^4 \begin{bmatrix} K+N \\ K \end{bmatrix}_q}\Big(-6\ \big(-2+(q)^{K.}\big)\ (q)^{K.}\ \big(2+\big(-2+(q)^{K.}\big)\ (q)^{K.}\big)+\\
&\big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[1,K]^3-\\
&3\ \big(-1+(q)^{K.}\big)^2\ ((q)^{K.})^3\ \big(-4+3\ (q)^{K.}\big)\ \mathrm{qHK}[1,N]^2-\\
&\big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[1,N]^3+\\
&\mathrm{qHK}[1,K]^2\ \big(-3\ \big(-1+(q)^{K.}\big)^2\ ((q)^{K.})^3\ \big(-4+3\ (q)^{K.}\big)-\\
&\qquad 3\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[1,N]\big)+\\
&3\ \big(-1+(q)^{K.}\big)^2\ ((q)^{K.})^3\ \big(-4+3\ (q)^{K.}\big)\ \mathrm{qHK}[2,K]-\\
&3\ \big(-1+(q)^{K.}\big)^2\ ((q)^{K.})^3\ \big(-4+3\ (q)^{K.}\big)\ \mathrm{qHK}[2,N]+\\
&\mathrm{qHK}[1,N]\ \big(-6\ \big(-1+(q)^{K.}\big)\ ((q)^{K.})^2\ \big(6+(q)^{K.}\ \big(-8+3\ (q)^{K.}\big)\big)+\\
&\qquad 3\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[2,K]-\\
&\qquad 3\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[2,N]\big)+\\
&\mathrm{qHK}[1,K]\ \big(6\ \big(-1+(q)^{K.}\big)\ ((q)^{K.})^2\ \big(6+(q)^{K.}\ \big(-8+3\ (q)^{K.}\big)\big)+\\
&\qquad 6\ \big(-1+(q)^{K.}\big)^2\ ((q)^{K.})^3\ \big(-4+3\ (q)^{K.}\big)\ \mathrm{qHK}[1,N]+\\
&\qquad 3\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[1,N]^2-\\
&\qquad 3\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[2,K]+\\
&\qquad 3\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[2,N]\big)+\\
&2\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[3,K]-\\
&2\ \big(-1+(q)^{K.}\big)^3\ ((q)^{K.})^4\ \mathrm{qHK}[3,N]\Big)
\end{aligned}$$

### 1.3.3 *Finding Recurrences*

In this section I will describe how we can derive a recurrence for a definite summation problem by the following example: find a recurrence for

In[112]:= $\mathbf{mySum} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \left(\mathbf{H_k} \binom{\mathbf{n}}{\mathbf{k}}\right)$;

The idea of how to find a recurrence is based on Zeilberger's creative telescoping method [Zei90]. In the following I describe how creative telescoping in the difference field context can be utilized to find recurrences for a definite summation problem and how one can rephrase creative telescoping in general in terms of difference fields. Moreover I am able to interpret creative telescoping as a special case of sum extensions in an indefinite summation problem.
In the next section I describe how one can derive the following recurrence by creative telescoping

In[113]:= **rec = GenerateRecurrence[mySum]**

Out[113]= $\{4\ (1+\mathtt{n})\ \mathtt{SUM}[\mathtt{n}] - 2\ (3+2\ \mathtt{n})\ \mathtt{SUM}[1+\mathtt{n}] + (2+\mathtt{n})\ \mathtt{SUM}[2+\mathtt{n}] == 1\}$

Remark: If we solve the recurrence

In[114]:= **recSol = SolveRecurrence[rec[[1]], SUM[n],**
**Tower → {SigmaPower[2, n]}, NestedSumExt → 2]**

Out[114]= $\left\{\{0, (2)^{\mathtt{n}}\}, \left\{0, (2)^{\mathtt{n}} \sum_{\iota_1=1}^{\mathtt{n}} \left(\frac{1}{\iota_1}\right)\right\}, \left\{1, -(2)^{\mathtt{n}} \sum_{\iota_1=1}^{\mathtt{n}} \left(\frac{1}{\iota_1\ (2)^{\iota_1}}\right)\right\}\right\}$

then we find the closed form

In[115]:= **FindLinearCombination[recSol, mySum, 2]**

Out[115]= $(2)^{\mathtt{n}} \sum_{\iota_1=1}^{\mathtt{n}} \left(\frac{1}{\iota_1}\right) - (2)^{\mathtt{n}} \sum_{\iota_1=1}^{\mathtt{n}} \left(\frac{1}{\iota_1\ (2)^{\iota_1}}\right)$

#### 1.3.3.1 *Creative Telescoping*

As already mentioned we can find a recurrence for a definite summation problem by creative telescoping. In our concrete example we compute:

In[116]:= **creaSol = CreativeTelescoping[mySum]**

Out[116]= $\Big\{\{0, 0, 0, 1\}, \Big\{4\ (1+\mathtt{n}), -2\ (3+2\ \mathtt{n}),$
$2+\mathtt{n}, \left((1+\mathtt{n})\ \left(-2+\mathtt{k}-\mathtt{n}+\left(2\ \mathtt{k}-2\ \mathtt{k}^2+\mathtt{k}\ \mathtt{n}\right)\ \mathrm{H}_{\mathtt{k}}\right) \binom{\mathtt{n}}{\mathtt{k}}\right)/$
$((1-\mathtt{k}+\mathtt{n})\ (2-\mathtt{k}+\mathtt{n}))\Big\}\Big\}$

This result has to be interpreted as follows. If

$$f(n,k) := \binom{n}{k} \mathrm{H}_k$$

then

$$\begin{aligned} c_1(n) &:= 4\,(1+n)\\ c_2(n) &:= -2\,(3+2\,n)\\ c_3(n) &:= 2+n\\ g(n,k) &:= \frac{(1+n)\left(-2+k-n+\left(2\,k-2\,k^2+k\,n\right)\mathrm{H}_k\right)\binom{n}{k}\Big)}{(1-k+n)\,(2-k+n)} \end{aligned} \tag{1.3}$$

solves the *telescoping problem*

$$g(n,k+1)-g(n,k)=c_1(n)\,f(n,k)+c_2(n)\,f(n+1,k)+c_3(n)\,f(n+2,k). \tag{1.4}$$

Summing the equation from 0 to $n$ we obtain

$$g(n,n+1)-g(n,0)=c_1(n)\sum_{k=0}^{n}f(n,k)+c_2(n)\sum_{k=0}^{n}f(n+1,k)+c_2(n)\sum_{k=0}^{n}f(n+2,k). \tag{1.5}$$

By

$$\begin{aligned} \mathrm{SUM}[n] &= \sum_{k=0}^{n}f(n,k)\\ \mathrm{SUM}[n+1] &= \sum_{k=0}^{n+1}f(n+1,k)=\sum_{k=0}^{n}f(n+1,k)+f(n+1,n+1)\\ \mathrm{SUM}[n+2] &= \sum_{k=0}^{n+2}f(n+2,k)=\sum_{k=0}^{n}f(n+2,k)+f(n+2,n+1)+f(n+2,n+2) \end{aligned} \tag{1.6}$$

we finally obtain

$$\begin{aligned} g(n,n+1)-g(n,0)+c_1(n)\,f(n+1,n+1)+c_2(n)\,(f(n+2,n+1)+f(n+2,n+2))\\ =c_1(n)\,\mathrm{SUM}[n]+c_2(n)\,\mathrm{SUM}[n+1]+c_2(n)\,\mathrm{SUM}[n+2]. \end{aligned} \tag{1.7}$$

This transformation of the creative telescoping equation (1.4) into the sum recurrence (1.7) can be achieved by the function call:

In[117]:= **TransformToRecurrence**[**creaSol**, **mySum**]

Out[117]= {4 (1 + n) SUM[n] − 2 (3 + 2 n) SUM[1 + n] + (2 + n) SUM[2 + n] == 1}

#### 1.3.3.2 †*The Creative Telescoping Problem in Difference Fields*

Whereas in this section I give just the main idea of how creative telescoping works in the setting of difference fields, I regard creative telescoping in full detail in Section 4.3.

**A difference field for the problem**

Let $Q(n)(t_1, t_2, t_3)$ be the field of rational functions over $\mathbb{Q}$ and consider the field automorphism $\sigma : \mathbb{Q}(n)(t_1, t_2, t_3) \to \mathbb{Q}(n)(t_1, t_2, t_3)$ canonically defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}(n), & \sigma(t_2) &= t_2 + \frac{1}{t_1+1}, \\ \sigma(t_1) &= t_1 + 1, & \sigma(t_3) &= \frac{n-t_1}{t_1+1}\, t_3. \end{aligned}$$

Note that the automorphism acts on $t_1$, $t_2$ and $t_3$ like the shift operator $K$ on $k$, $\mathrm{H}_k$ and $\binom{n}{k}$ with $K\,k = k+1$, $K\,\mathrm{H}_k = \mathrm{H}_k + \frac{1}{k+1}$ and $K\binom{n}{k} = \frac{n-k}{k+1}\binom{n}{k}$.
Therefore $f(n,k)$ can be rephrased in terms of the difference field $(\mathbb{Q}(n)(t_1, t_2, t_3), \sigma)$ by

$$\begin{aligned} f(n,k) = \mathrm{H}_k \binom{n}{k} &\leftrightarrow t_2\, t_3 := f_1' \\ f(n+1,k) = \mathrm{H}_k \binom{n+1}{k} = \frac{(n+1)\,\mathrm{H}_k \binom{n}{k}}{n+1-k} &\leftrightarrow \frac{(n+1)\, t_2\, t_3}{n+1-t_1} := f_2' \\ f(n+2,k) = \mathrm{H}_k \binom{n+2}{k} = \frac{(n+1)\,(n+2)\,\mathrm{H}_k \binom{n}{k}}{(n+1-k)\,(n+2-k)} &\leftrightarrow \frac{(n+1)\,(n+2)\, t_2\, t_3}{(n+1-t_1)\,(n+2-t_1)} =: f_3'. \end{aligned}$$

**The Creative Telescoping Problem in Difference Fields**

The creative telescoping problem (1.4) can be reformulated in terms of the difference field $\mathbb{Q}(n)(t_1, t_2, t_3)$ as follows: find an element $g \in \mathbb{Q}(n)(t_1, t_2, t_3)$ and a vector $(0,0,0) \neq (c_1, c_2, c_3) \in \mathbb{Q}(n)^3$ such that

$$\boxed{\sigma(g) - g = c_1\, f_1' + c_2\, f_2' + c_3\, f_3'.}$$

Although Karr's original summation algorithm was already capable of carrying out creative telescoping, as will be indicated in Section 1.4, nobody has noticed this possibility until now. We find the solution

$$\begin{aligned} c_1 &:= 4\,(1+n), \\ c_2 &:= -2\,(3+2\,n), \\ c_3 &:= 2+n, \\ g &:= \frac{(1+n)\,\big(-2 + t_1 - n + \big(2\,t_1 - 2\,t_1^2 + t_1\, n\big)\, t_2\big)\, t_3\big)}{(1-t_1+n)\,(2-t_1+n)} \end{aligned}$$

which can be immediately translated back in terms of $k$, $\mathrm{H}_k$ and $\binom{n}{k}$. This results in (1.3) and leads to the solution of the creative telescoping problem (1.4).

#### 1.3.3.3 *Finding Recurrences and Sum Extensions*

In the following I demonstrate how creative telescoping can be interpreted as a special case of sum extensions in an indefinite summation problem. For this illustration consider the following sum

In[118]:= $\mathbf{mySum} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{a}} \left(\mathbf{H_k} \binom{\mathbf{2+n}}{\mathbf{k}}\right)$;

which we express in terms of

In[119]:= $\mathbf{tower} = \{\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{a}} \left(\mathbf{H_k} \binom{\mathbf{n}}{\mathbf{k}}\right), \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{a}} \left(\mathbf{H_k} \binom{\mathbf{1+n}}{\mathbf{k}}\right)\}$;

by calling the function:

In[120]:= $\mathbf{SigmaReduce[mySum, Tower \to tower]}$

Out[120]= $$\frac{1}{(2+n)\,(1-a+n)}\Bigg(1-a+n+(-1-n)\binom{n}{a}+(-2\,a+n-2\,a\,n+n^2)\,H_a\binom{n}{a}+ (-4+4\,a-8\,n+4\,a\,n-4\,n^2)\sum_{\iota_1=0}^{a}\left(H_{\iota_1}\binom{n}{\iota_1}\right)+ (6-6\,a+10\,n-4\,a\,n+4\,n^2)\sum_{\iota_1=0}^{a}\left(\frac{(1+n)\,H_{\iota_1}\binom{n}{\iota_1}}{1+n-\iota_1}\right)\Bigg)$$

In other words, if

$$\mathrm{SUM}[a,n] := \sum_{k=0}^{a} \mathrm{H}_k \binom{n}{k},$$

we find $c_1(a,n), c_2(a,n) \in \mathbb{Q}(a,n)$ and $h(a,n) \in \mathbb{Q}(a,n,\mathrm{H}_k,\binom{n}{a})$ such that

$$\mathrm{SUM}[a,n+2] = c_1(a,n)\,\mathrm{SUM}[a,n] + c_2(a,n)\,\mathrm{SUM}[a,n+1] + h(a,n).$$

Substituting $a$ by $n$ leads us to an equivalent equation of (1.5). Finally, using (1.6) results in a recurrence for the definite sum

$$\sum_{k=0}^{n} \mathrm{H}_k \binom{n}{k}.$$

| |
|---|
| Summarizing, the general problem to find a recurrence by the creative telescoping method, is just a special case of solving an indefinite summation problem in terms of appropriate sum extensions. |

We will look at this observation comprehensively in Section 4.3.2.

### 1.3.4 *Solving Linear Recurrences*

In this section we discuss several aspects which are related to solving linear recurrences in difference fields. Whereas in the following section we rephrase the problem of solving linear recurrences to the problem of finding solutions of a linear difference equation, in the next sections we will look closer at nested sum solutions and d'Alembertian solutions for a given difference equation.

#### 1.3.4.1 †*Difference Equations in Difference Fields*

In Section 1.3.2 we solved the recurrence

In[121]:= $\mathbf{rec} = \{-(\mathbf{1}+\mathbf{N})\ (\mathbf{2}+\mathbf{N})\ (\mathbf{3}+\mathbf{N})\ (\mathbf{4}+\mathbf{N})\ \mathbf{SUM}[\mathbf{N}]+ \mathbf{2}\ (\mathbf{2}+\mathbf{N})\ (\mathbf{3}+\mathbf{N})\ (\mathbf{4}+\mathbf{N})\ (\mathbf{5}+\mathbf{2\ K}+\mathbf{2\ N})\ \mathbf{SUM}[\mathbf{1}+\mathbf{N}] - (\mathbf{3}+\mathbf{N})\ (\mathbf{4}+\mathbf{N}) (\mathbf{55}+\mathbf{36\ K}+\mathbf{6\ K^2}+\mathbf{36\ N}+\mathbf{12\ K\ N}+\mathbf{6\ N^2})\ \mathbf{SUM}[\mathbf{2}+\mathbf{N}] + (\mathbf{4}+\mathbf{N}) (\mathbf{7}+\mathbf{2\ K}+\mathbf{2\ N})\ (\mathbf{25}+\mathbf{14\ K}+\mathbf{2\ K^2}+\mathbf{14\ N}+\mathbf{4\ K\ N}+\mathbf{2\ N^2})\ \mathbf{SUM}[\mathbf{3}+\mathbf{N}] - (\mathbf{4}+\mathbf{K}+\mathbf{N})^{\mathbf{4}}\ \mathbf{SUM}[\mathbf{4}+\mathbf{N}] == \mathbf{0}\}$

by using the Harmonic numbers

In[122]:= $\mathbf{tower} = \{\{\binom{\mathbf{K}+\mathbf{N}}{\mathbf{K}}, \mathbf{N}\}, \{\mathbf{H_{N+K}}, \mathbf{N}\}, \{\mathbf{H^{(2)}_{N+K}}, \mathbf{N}\}, \{\mathbf{H_{N+K}}^{(\mathbf{3})}, \mathbf{N}\}\};$

and found the result:

In[123]:= $\mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{N}], \mathbf{Tower} \to \mathbf{tower}]$

Out[123]= $\{\{0, \frac{1}{\binom{K+N}{K}}\}, \{0, \frac{H_{K+N}}{\binom{K+N}{K}}\}, \{0, \frac{H^2_{K+N} + H^{(2)}_{K+N}}{\binom{K+N}{K}}\}, \{0, \frac{H^3_{K+N} + 3\ H_{K+N}\ H^{(2)}_{K+N} + 2\ H^{(3)}_{K+N}}{\binom{K+N}{K}}\}, \{1, 0\}\}$

As already sketched in Section 1.2.3.1, we represent $\mathrm{H}^{(\alpha)}_{N+K}$ in the following way

$$\mathrm{H}^{(\alpha)}_{N+K} = \sum_{i=1}^{N} \frac{1}{(K+i)^\alpha} + \mathrm{H}^{(\alpha)}_{K}$$

where $K$ is interpreted as an indeterminate and $\mathrm{H}^{(\alpha)}_{K}$ stands for a constant which guarantees the correct evaluation for any $K \in \mathbb{N}_0$.
Therefore we assume in the following that $K$ is transcendental over $\mathbb{Q}$. Let $\mathbb{Q}(K)(t_1, \dots, t_5)$ be the field of rational functions over $\mathbb{Q}$ with the field automorphism $\sigma$ canonically defined by

$$\sigma(c) = c \ \ \forall c \in \mathbb{Q}(K), \qquad \sigma(t_3) = t_3 + \frac{1}{K+t_1+1},$$
$$\sigma(t_1) = t_1 + 1, \qquad \sigma(t_4) = t_4 + \frac{1}{(K+t_1+1)^2},$$
$$\sigma(t_2) = t_2\,\frac{K+t_1+1}{t_1+1}, \qquad \sigma(t_5) = t_5 + \frac{1}{(K+t_1+1)^3}.$$

Note that the automorphism $\sigma$ acts on $t_1$ and $t_2, t_3, t_4, t_5$ like the shift operator $S$ on $N$ and $\binom{N}{K}$, $\mathrm{H}_{N+K}$, $\mathrm{H}^{(2)}_{N+K}$, $\mathrm{H}^{(3)}_{N+K}$ with $SN = N+1$ and

$$S\binom{N+K}{K} = \frac{K+N+1}{N+1}\binom{N+K}{K}, \qquad S\,\mathrm{H}^{(2)}_{N+K} = \mathrm{H}^{(2)}_{N+K} + \frac{1}{(N+K+1)^2},$$
$$S\,\mathrm{H}_{N+K} = \mathrm{H}_{N+K} + \frac{1}{N+K+1}, \qquad S\,\mathrm{H}^{(3)}_{N+K} = \mathrm{H}^{(3)}_{N+K} + \frac{1}{(N+K+1)^3}.$$

Furthermore let

$$\begin{aligned}
a_0 &:= -(1+t_1)\ (2+t_1)\ (3+t_1)\ (4+t_1),\\
a_1 &:= 2\ (2+t_1)\ (3+t_1)\ (4+t_1)\ (5+2\ K+2\ t_1),\\
a_2 &:= -(3+t_1)\ (4+t_1)\big(55+36\ K+6\ K^2+36\ t_1+12\ K\ t_1+6\ t_1^2\big),\\
a_3 &:= (4+t_1)(7+2\ K+2\ t_1)\ \big(25+14\ K+2\ K^2+14\ t_1+4\ K\ t_1+2\ t_1^2\big),\\
a_4 &:= (4+K+t_1)^4.
\end{aligned}$$

Then the problem of solving the recurrence `rec` in terms of $N$ and $\binom{N+K}{K}$, $\mathrm{H}_{N+K}$, $\mathrm{H}^{(2)}_{N+K}$, $\mathrm{H}^{(3)}_{N+K}$ can be rephrased as the following problem in terms of difference fields: find all $g \in \mathbb{Q}(K)(t_1,\dots,t_5)$ such that

$$a_4\,\sigma^4(g) + a_3\,\sigma^3(g) + a_2\,\sigma^2(g) + a_1\,\sigma(g) + a_0\,g = 0. \tag{1.8}$$

As result we obtain four linear independent solutions over $\mathbb{Q}(K)$, namely

$$g_1 := \frac{1}{t_2} \qquad g_2 := \frac{t_3}{t_2} \qquad g_3 : \frac{t_3^2 + t_4}{t_2} \qquad g_4 := \frac{t_3^3 + 3\,t_3\,t_4 + 2\,t_5}{t_2}$$

where the set

$$\{k_1\,g_1 + k_2\,g_2 + k_3\,g_3 + k_4\,g_4 \mid k_i \in \mathbb{Q}(K)\}$$

describes *all* solutions in $\mathbb{Q}(K)(t_1,\dots,t_5)$ of the difference equation (1.8). From this result the above output of the function `SolveRecurrence` immediately follows.

### The General Problem

In general, given a difference field $(\mathbb{F},\sigma)$ with constant field

$$\mathbb{K} := \{k \in \mathbb{F} \mid \sigma(k) = k\},$$

together with $a_0,\dots,a_m \in \mathbb{F}$ and $f \in \mathbb{F}$, we look for all $g \in \mathbb{F}$ such that

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f.} \tag{1.9}$$

As a result we are interested in linearly independent solutions $g_1,\dots,g_l \in \mathbb{F}$ of the homogeneous version of the difference equation and in one particular solution $p \in \mathbb{F}$ of the difference equation (1.9) such that

$$\{p + k_1\,g_1 + \cdots + k_m\,g_m \mid k_i \in \mathbb{K}\}$$

delivers all solutions $g \in \mathbb{F}$ of the recurrence (1.9).

**An Output Observation**

In the above example the coefficients $a_i$ of the recurrence lie in $\mathbb{Q}(K)(t_1)$ and we are looking for solutions in $\mathbb{Q}(K)(t_1)(t_2, \dots, t_5)$ where $t_2$ represents the binomial $\binom{N+K}{K}$ and $t_3, t_4, t_5$ reflect the harmonic numbers $\mathrm{H}_{N+K}$, $\mathrm{H}^{(2)}_{N+K}$ and $\mathrm{H}^{(3)}_{N+K}$. Then looking at the solutions

$$g_3 := \frac{t_3^{\boxed{3}} + 3\,t_3\,t_4 + 2\,t_5}{t_2} \qquad g_2 := \frac{t_3^{\boxed{2}} + t_4}{t_2} \qquad g_1 := \frac{t_3^{\boxed{1}}}{t_2} \qquad g_0 := \frac{t_3^{\boxed{0}}}{t_2}$$

we see that the solutions are in $\mathbb{Q}(K)(t_1, t_2, t_4, t_5)[t_3]$ and that there is a decreasing sequence of the degrees in the polynomial solutions.
In general, let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-field[19] with

$$\sigma(t) = t + \beta, \quad \beta \in \mathbb{F}^*$$

where $t$ is transcendental over $\mathbb{F}$ and the constant field of $\mathbb{F}(t)$ is the same as of $\mathbb{F}$. Moreover, assume that the coefficients $a_i$ of the recurrence (1.9) are free of $t$, i.e. $a_i \in \mathbb{F}$. Then there is the following result in Section 4.1, Theorem 4.1.1: Whenever there is a solution $g \in \mathbb{F}(t)$ for the recurrence (1.9) then $g$ is a polynomial in $t$ and $\deg(g) \leq m + \deg(f)$. Additionally, there are solutions $g_0, \dots, g_{l-1} \in \mathbb{F}[t]$ for the recurrence (1.9) with $l := \deg(g) - \deg(f)$ where

$$\deg(g_i) = i.$$

In our example, for $\mathbb{F} := \mathbb{Q}(K)(t_1, t_2, t_4, t_5)$, there exists a solution $g_3 \in \mathbb{F}[t_5]$ with $\deg(g_3) = 3$ and therefore there are solutions $g_i$ with $\deg(g_i) = i$ for $i \in \{0, 1, 2\}$.

**Open Problems**

The method to compute solutions of a difference equation is based on a reduction technique. For the first order case M. Karr managed to develop a complete algorithm to solve first order difference equations in so called $\Pi\Sigma$-fields which will be introduced in Section 2.2.5.
Based on Bronstein's denominator bounding [Bro00], I was able to extend Karr's reduction technique for the general case of higher order linear difference equations.

**Boundings** One crucial step in this reduction are boundings in order to restrict the search space. Open problems still remain for the general case to solve $m$-th order difference equations, namely:
• There are still unsolved problems concerning degree boundings of some solution parts (Sections 3.1.3.1 and 3.1.3.2). Nevertheless one can find all possible solutions by an incremental strategy, i.e., increasing step by step the degree boundings for each computation attempt (Sections 3.3.1 and 3.5.2.1). By increasing the value of the `plusBound` option, these boundings are raised. Consequently the chances are higher to find more solutions. For this strategy, however, more time and space resources are required. By default - as in all examples - the value of `plusBound` is set to 1 in the functions `SolveRecurrence` or `FindSumSolutions`.
• In particular in Section 3.4, I make an effort to find a polynomial degree bound for sum extensions. Up to now I failed to prove the termination of the developed method (Implementation Note 3.4.1). Therefore I introduced an option `LoopLimitForSumBound`→*Int* which specifies the maximal amount of loops *Int* to find the bound. If the allowed maximal number of loops is reached, the computed bound might be too low which amounts to the problem that some solutions can not be found. In this situation a warning is printed out with the suggestion to increase this loop bound in order to obtain further solutions.

[19]See Section 2.2 for further details.

**Difference Rings** The reduction strategy to solve difference equations has been originally developed for difference fields, more precisely for $\Pi\Sigma$-fields. I tried to extend the reduction technique to deal also with product extensions of the type

$$\prod_{i=1}^{k}\alpha$$

where $\alpha$ is an $n$-th root of unity. Please note that in this case we do not work anymore in difference fields but in difference rings where even zero-divisors can appear. Despite the method works quite successful, there are still open problems which are described in more details in Section 3.6.

#### 1.3.4.2 *Sum Solutions*

The theoretical background how one finds all sum solutions for a difference equation is discussed carefully in Section 4.5. As already illustrated in Sections 1.3.1 and 1.3.2, we are able to find appropriate sum extensions in order to find further solutions for a given recurrence. The solutions are, loosely speaking, expressed in the form

$$d_1\sum_{i_1=0}^{n}d_2\sum_{i_2=0}^{i_1}d_3\cdots\sum_{i_l=0}^{i_{l-1}}d_{l+1} \tag{1.10}$$

where the $d_i$ are written in terms of sums and products that occur in the recurrence.
Assuming that the recurrence can be rephrased in terms of a difference field $(\mathbb{F},\sigma)$, more precisely in a $\Pi\Sigma$-field, an important result is that sum solutions for such a recurrence can be expressed in a difference field extension of $(\mathbb{F},\sigma)$; more precisely, that the solutions can be described in a $\Pi\Sigma$-field. This means that one can apply my indefinite summation algorithm, which works in $\Pi\Sigma$-fields, to simplify the found sum solutions further.

### Elimination of Algebraic Relations

In general the sum solutions are algebraic dependent, which means that one can eliminate sum quantifiers by using our indefinite summation algorithm. For instance, in Section 1.3.1 we were faced with the following recurrence

In[124]:= $\mathbf{rec1} = \mathbf{n\ (1+n)\ (2+n)\ (-1+n)!_{.}\ ((2+n)\ (4+n)^2\ (27+15\ n+2\ n^2)\ SUM[n]-}$
$\mathbf{(3+n)\ (4+n)\ (9+2\ n)\ (13+8\ n+n^2)\ SUM[1+n]-}$
$\mathbf{(3+n)\ (4+n)\ (5+2\ n)\ (6+6\ n+n^2)\ SUM[2+n]+}$
$\mathbf{(3+n)^2\ (5+n)\ (20+13\ n+2\ n^2)\ SUM[3+n])} ==$
$\mathbf{-2\ (315+286\ n+84\ n^2+8\ n^3)\ (4+n)!_{.}\ (-1)^{n_{.}}};$

which we solved by a sum extension:

In[125]:= $\mathbf{SolveRecurrence[rec1, SUM[n], NestedSumExt \to \infty]}$

Out[125]= $\Big\{\{0,1\},\{0,(2+n)\ (-1)^{n_{.}}\},$

$$\Big\{0,-\frac{2-n+6\ \sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)+6\ n\ \sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\right)}{6\ (1+n)}\Big\},$$

$$\Big\{1,\frac{1}{(1+n)\ (2+n)}\Big((-1)^{n_{.}}\ \Big(3+3\ n+n^2+8\sum_{\iota_1=1}^{n}\Big(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\Big)+$$
$$16\ n\sum_{\iota_1=1}^{n}\Big(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\Big)+10\ n^2\sum_{\iota_1=1}^{n}\Big(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\Big)+2\ n^3\sum_{\iota_1=1}^{n}\Big(\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}\Big)\Big)\Big)\Big\}\Big\}$$

Actually, the indefinite summation algorithm was applied automatically to simplify the computed sum solutions. By setting the option `AlgebraicRelationInSumSolutions→True`, the user can see the original sum solutions - without further elimination of algebraic relations:

In[126]:= **SolveRecurrence[rec1, SUM[n], NestedSumExt → ∞,**
**AlgebraicRelationInSumSolutions → True]**

$$\text{Out[126]=}\ \Big\{\{0,1\},\{0,(2+n)\,(-1)^{n}\},$$
$$\Big\{0,-\sum_{\iota_1=1}^{n}\Big((3+2\,\iota_1)\,(-1)^{\iota_1}\sum_{\iota_2=1}^{\iota_1}\Big(\frac{(-1)^{\iota_2}}{\iota_2\,(2+\iota_2)}\Big)\Big)\Big\},$$
$$\Big\{1,2\sum_{\iota_1=1}^{n}\Big((3+2\,\iota_1)\,(-1)^{\iota_1}\sum_{\iota_2=1}^{\iota_1}\Big(\frac{1+\iota_2}{\iota_2\,(2+\iota_2)}\Big)\Big)\Big\}\Big\}$$

### Simplification of Nested Sums

Although one can eliminate algebraic relations in sum solutions, which corresponds to an elimination of sum quantifiers, one still obtains sum solutions which are highly recursively defined in the form (1.10). In this case one can try to simplify the result further by finding appropriate sum extensions. In Kirschenhofer's example introduced in Section 1.3.2 we find sum extensions with a 3-nested sum for the recurrence

$$\text{In[127]:= }\mathbf{rec}=\{-(1+N)\,(2+N)\,(3+N)\,(4+N)\,\mathrm{SUM}[N]+$$
$$2\,(2+N)\,(3+N)\,(4+N)\,(5+2\,K+2\,N)\,\mathrm{SUM}[1+N]-(3+N)\,(4+N)$$
$$(55+36\,K+6\,K^2+36\,N+12\,K\,N+6\,N^2)\,\mathrm{SUM}[2+N]+(4+N)$$
$$(7+2\,K+2\,N)\,(25+14\,K+2\,K^2+14\,N+4\,K\,N+2\,N^2)\,\mathrm{SUM}[3+N]-$$
$$(4+K+N)^4\,\mathrm{SUM}[4+N]==$$
$$0\}$$

using the product

$$\text{In[128]:= }\mathbf{tower}=\Big\{\Big\{\binom{K+N}{K},N\Big\}\Big\};$$

In[129]:= **SolveRecurrence[rec[[1]], SUM[N], Tower → tower, NestedSumExt → ∞]**

$$\text{Out[129]=}\ \Big\{\Big\{0,\frac{1}{\binom{K+N}{K}}\Big\},\ \Big\{0,\frac{\sum_{\iota_1=1}^{N}\Big(\frac{1}{K+\iota_1}\Big)}{\binom{K+N}{K}}\Big\},\Big\{0,\frac{\sum_{\iota_1=1}^{N}\Big(\frac{\sum_{\iota_2=1}^{\iota_1}\Big(\frac{1}{K+\iota_2}\Big)}{K+\iota_1}\Big)}{\binom{K+N}{K}}\Big\},$$
$$\Big\{0,\frac{\sum_{\iota_1=1}^{N}\Bigg(\frac{\sum_{\iota_2=1}^{\iota_1}\Big(\frac{\sum_{\iota_3=1}^{\iota_2}\Big(\frac{1}{K+\iota_3}\Big)}{K+\iota_2}\Big)}{K+\iota_1}\Bigg)}{\binom{K+N}{K}}\Big\},\{1,0\}\Big\}$$

This result can be simplified further by using single nested sums, as can be seen in Section 1.2.3.1. By setting the option `SimplifyByExt→Depth`, this will be done automatically:

In[130]:= **SolveRecurrence[rec[[1]], SUM[N], Tower → tower,**
**NestedSumExt → ∞, SimplifyByExt → Depth]**

Out[130]=
$$\Big\{\Big\{0, \frac{1}{\binom{K+N}{K}}\Big\},$$

$$\Big\{0, \frac{\sum_{\iota_1=1}^{N}\Big(\frac{1}{K+\iota_1}\Big)}{\binom{K+N}{K}}\Big\}, \Big\{0, \frac{\sum_{\iota_1=1}^{N}\Big(\frac{1}{(K+\iota_1)^2}\Big)+\Big(\sum_{\iota_1=1}^{N}\Big(\frac{1}{K+\iota_1}\Big)\Big)^2}{2\binom{K+N}{K}}\Big\},$$

$$\Big\{0, \frac{2\sum_{\iota_1=1}^{N}\Big(\frac{1}{(K+\iota_1)^3}\Big)+3\Big(\sum_{\iota_1=1}^{N}\Big(\frac{1}{(K+\iota_1)^2}\Big)\Big)\sum_{\iota_1=1}^{N}\Big(\frac{1}{K+\iota_1}\Big)+\Big(\sum_{\iota_1=1}^{N}\Big(\frac{1}{K+\iota_1}\Big)\Big)^3}{6\binom{K+N}{K}}\Big\},$$

$$\{1,0\}\Big\}$$

#### 1.3.4.3 *d'Alembertian Solutions and Difference Fields*

In general d'Alembertian solutions not only consist of nested sum extensions but also of products over the underlying difference field. Therefore nested sum solutions are contained in d'Alembertian solutions. But whereas sum extensions can be always described in difference fields, more precisely in ΠΣ-fields, many product extensions cannot be handled in ΠΣ-fields, even worse, most of them can be only treated in difference *rings*. Therefore dealing with d'Alembertian solutions usually means to work in difference rings. But as described above, I consider it as an essential step to eliminate algebraic relations and to simplify nested sums further to simpler nested sums. Unfortunately, there does not exist such an algorithm which works also in difference fields in general or even in difference rings. Since our algorithm works only properly in ΠΣ-fields, I want to restrict just to those product extensions which can be treated in ΠΣ-fields.

In my approach I do not extend the underlying difference field automatically by product extensions, but give full control to the user about how he designs the difference field. As can be seen in Section 1.1.2, using the function `FindSumSolutions`, the user obtains, besides all sum solutions, a recurrence for which there does not exist a solution in the underlying difference field. If one finds a product extension which leads to a solution of this recurrence then extending the underlying difference field by this product extension and solving the recurrence with `FindSumSolutions` or `SolveRecurrence` with the option `NestedSumExt`→ ∞ will give at least one more solution of the recurrence.

By this strategy the user has full control about how the difference field is extended. Additionally, the user is independent of different packages to find product extensions for a recurrence. He can use M. Petkovšek's package `Hyper` [Pet92] to find hypergeometric solutions of a recurrence with polynomial coefficients, or he can use $q$-`Hyper` [APP98] to deal with $q$-recurrences. As will be shown in the next section, it would be very interesting to have further algorithms which can find product extensions over more general difference fields where for instance the Harmonic numbers are involved. One can find further information about d'Alembertian solutions in Section 4.5.

### 1.3.5 Reducing the Recurrence Order by Sum Extensions

As already motivated by previous sections, I am interested in computing a recurrence for a given definite summation problem. In the following I will give some fascinating examples which illustrate how one can reduce the order of such a recurrence by using appropriate sum extensions. The theoretical background and the algorithmical aspects will be described in Section 4.4.

#### 1.3.5.1 Example: Harmonic Numbers in a Product

We are looking for a closed form of the following curious sum[20]:

$$\text{In[131]:= } \mathbf{mySum} = \sum_{r=1}^{n}\left(\frac{\sum_{l=1}^{r}\left(\prod_{k=1}^{l}\left(\frac{H_{-k+n}}{1+H_{-k+n}}\right)\right)}{1+H_{n-r}}\right);$$

**The Naive Approach**

In a first step we find a recurrence for the sum which is definite.

$$\text{In[132]:= } \mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{RecOrder} \to \mathbf{2}]$$

822.513 Second

$$\begin{aligned}\text{Out[132]= } \{ &-\left(5+5\,n+n^2\right) H_n\,(1+H_n)\\ &\quad (1+(1+n)\,H_n)\,(2+n+(1+n)\,H_n)\,\mathrm{SUM}[n]+(2+n+(1+n)\,H_n)\\ &\quad \left(5+5\,n+n^2+\left(20+25\,n+9\,n^2+n^3\right)H_n+\left(25+40\,n+20\,n^2+3\,n^3\right)H_n^2+\right.\\ &\qquad \left.\left(10+20\,n+12\,n^2+2\,n^3\right)H_n^3\right)\mathrm{SUM}[1+n]-\\ &\quad \left(5+5\,n+n^2\right)H_n\,(1+H_n)\\ &\quad \left(4+4\,n+n^2+\left(4+6\,n+2\,n^2\right)H_n+\left(1+2\,n+n^2\right)H_n^2\right)\mathrm{SUM}[2+n]==\\ &\quad -(1+n)\left(5+5\,n+n^2\right)H_n\,(1+H_n)\,(1+(1+n)\,H_n),\\ &H_n\,(1+H_n)\,(1+(1+n)\,H_n)\,(2+n+(1+n)\,H_n)\,\mathrm{SUM}[n]-\\ &\quad (1+H_n)\,(1+2\,H_n)\,(1+(1+n)\,H_n)\,(2+n+(1+n)\,H_n)\,\mathrm{SUM}[1+n]+\\ &\quad (2+n+(1+n)\,H_n)\left((2+n)\,H_n+(3+2\,n)\,H_n^2+(1+n)\,H_n^3\right)\mathrm{SUM}[2+n]==\\ &\quad (1+n)\,H_n\,(1+H_n)\,(1+(1+n)\,H_n)\}\end{aligned}$$

In a second step we solve the recurrence by using the product[21]

$$\text{In[133]:= } \mathbf{tower} = \left\{\prod_{k=1}^{n}\left(\frac{H_k}{1+H_k}\right)\right\};$$

and find out that

[20] This example is for illustrative purposes only. It might well be that the reader finds a more direct approach for obtaining its closed form evaluation.

[21] Unfortunately there does not exist an algorithm which can find, for a given recurrence with coefficients in terms of Harmonic numbers, a product extension which delivers a solution of the recurrence. In this sense, I see my package as a motivation for further investigations to consider more general problems in symbolic summation. How I could find this particular product extension will be clear later.

In[134]:= **SolveRecurrence[rec[[1]], SUM[n], NestedSumExt → 2, Tower → tower]**

80.145 Second

$$\text{Out[134]=}\ \Big\{\Big\{0, \frac{(1+\mathtt{H_n})\ \prod_{\iota_1=1}^{n}\left(\frac{\mathtt{H}_{\iota_1}}{1+\mathtt{H}_{\iota_1}}\right)}{\mathtt{H_n}}\Big\}, \Big\{0,$$

$$\frac{(1+\mathtt{H_n})\left(\prod_{\iota_1=1}^{n}\left(\frac{\mathtt{H}_{\iota_1}}{1+\mathtt{H}_{\iota_1}}\right)\right)\ \sum_{\iota_1=1}^{n}\left(\frac{\mathtt{H}_{\iota_1}\ (-1+\mathtt{H}_{\iota_1}\ \iota_1)\ \prod_{\iota_2=1}^{\iota_1}\left(\frac{1+\mathtt{H}_{\iota_2}}{\mathtt{H}_{\iota_2}}\right)}{(1+\mathtt{H}_{\iota_1})\ (-1+\iota_1+\mathtt{H}_{\iota_1}\ \iota_1)}\right)}{\mathtt{H_n}}\Big\},$$

$$\Big\{1, \frac{1}{\mathtt{H_n}}\left((1+\mathtt{H_n})\left(\prod_{\iota_1=1}^{n}\left(\frac{\mathtt{H}_{\iota_1}}{1+\mathtt{H}_{\iota_1}}\right)\right)\right.$$

$$\left.\sum_{\iota_1=1}^{n}\left(\frac{\mathtt{H}_{\iota_1}\ (-1+\mathtt{H}_{\iota_1}\ \iota_1)\left(\prod_{\iota_2=1}^{\iota_1}\left(\frac{1+\mathtt{H}_{\iota_2}}{\mathtt{H}_{\iota_2}}\right)\right)\ \sum_{\iota_2=1}^{\iota_1}\left(\frac{\iota_2}{-1+\iota_2+\mathtt{H}_{\iota_2}\ \iota_2}\right)}{(1+\mathtt{H}_{\iota_1})\ (-1+\iota_1+\mathtt{H}_{\iota_1}\ \iota_1)}\right)\right)\Big\}\Big\}$$

### Finding a Recurrence with Lower Order by a Sum Extension

Instead of applying usual creative telescoping, I compute a recurrence of order 1 by finding an appropriate sum extension automatically, namely

In[135]:= **rec = GenerateRecurrence[mySum, SimplifyByExt → Depth]**

463.466 Second

$$\text{Out[135]=}\ \big\{(2+\mathtt{n})\ \mathtt{H_n}\ \mathtt{SUM[n]} - (2+\mathtt{n})\ (1+\mathtt{H_n})\ \mathtt{SUM[1+n]} ==$$

$$-(2+\mathtt{n})\ \mathtt{H_n}\left(1+\sum_{\iota_1=1}^{n}\left(\frac{1+\mathtt{n}-\iota_1}{2+\mathtt{n}+\mathtt{H}_{\mathtt{n}-\iota_1}+\mathtt{n}\ \mathtt{H}_{\mathtt{n}-\iota_1}-\iota_1-\mathtt{H}_{\mathtt{n}-\iota_1}\ \iota_1}\right)\right)\big\}$$

### A Definite Summation Subproblem

The found sum

$$\text{In[136]:=}\ \mathbf{defPart} = \sum_{\iota_1=1}^{n}\left(\frac{\mathbf{1+n}-\iota_1}{\mathbf{2+n+H_{n-\iota_1}+n\ H_{n-\iota_1}}-\iota_1-\mathbf{H_{n-\iota_1}}\ \iota_1}\right);$$

has to be transformed from a definite sum representation to an indefinite one. Therefore we again compute a recurrence

In[137]:= **recDefPart = GenerateRecurrence[defPart][[1]]**

15.623 Second

$$\text{Out[137]=}\ (-2-\mathtt{n}+(-1-\mathtt{n})\ \mathtt{H_n})\ \mathtt{SUM[n]} + (2+\mathtt{n}+(1+\mathtt{n})\ \mathtt{H_n})\ \mathtt{SUM[1+n]} == 1+\mathtt{n}$$

solve it

In[138]:= **recSolDefPart =**
**SolveRecurrence[recDefPart, SUM[n], NestedSumExt → 1]**

5.097 Second

$$\text{Out[138]=}\ \Big\{\{0,1\}, \Big\{1, \sum_{\iota_1=1}^{n}\left(\frac{1}{1+\mathtt{H}_{\iota_1}}\right)\Big\}\Big\}$$

and finally find the closed form:

In[139]:= **solDefPart = FindLinearCombination[recSolDefPart, defPart, 1]**

$$\text{Out[139]=}\ \sum_{\iota_1=1}^{n}\left(\frac{1}{1+\mathtt{H}_{\iota_1}}\right)$$

### Finding a Closed Form

Now we can substitute the definite sum by the indefinite representation and obtain the following recurrence which can be solved by my package.

In[140]:= **rec = rec/.{defPart → solDefPart}**

$$\text{Out[140]}= \{(2+n)\ H_n\ SUM[n] - (2+n)\ (1+H_n)\ SUM[1+n] == (-2-n)\ H_n\ \left(1+\sum_{\iota_1=1}^{n}\left(\frac{1}{1+H_{\iota_1}}\right)\right)\}$$

Here one can directly read off the product extension which gives us a solution of the homogeneous version of the recurrence, namely $\prod_{k=1}^{n}\frac{H_k}{1+H_k}$. Computing the solutions of the recurrence

In[141]:= **recSol = SolveRecurrence[rec[[1]], SUM[n], NestedSumExt → 1, Tower → tower]**

54.539 Second

$$\text{Out[141]}= \left\{\left\{0, \frac{(1+H_n)\ \prod_{\iota_1=1}^{n}\left(\frac{H_{\iota_1}}{1+H_{\iota_1}}\right)}{H_n}\right\},\right.$$
$$\left.\left\{1, \frac{1}{H_n}\left((1+H_n)\left(\prod_{\iota_1=1}^{n}\left(\frac{H_{\iota_1}}{1+H_{\iota_1}}\right)\right)\sum_{\iota_1=1}^{n}\left(\left(H_{\iota_1}\ (-1+H_{\iota_1}\ \iota_1)\left(\prod_{\iota_2=1}^{\iota_1}\left(\frac{1+H_{\iota_2}}{H_{\iota_2}}\right)\right)\left(H_{\iota_1}+(1+H_{\iota_1})\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{1+H_{\iota_2}}\right)\right)\right)\Big/ \left((1+H_{\iota_1})^2\ (-1+\iota_1+H_{\iota_1}\ \iota_1)\right)\right)\right)\right\}\right\}$$

we finally obtain the closed form of the definite sum `mySum`.

In[142]:= **FindLinearCombination[recSol, mySum, 1]**

$$\text{Out[142]}= \frac{1}{H_n}\left((1+H_n)\left(\prod_{\iota_1=1}^{n}\left(\frac{H_{\iota_1}}{1+H_{\iota_1}}\right)\right)\sum_{\iota_1=1}^{n}\left(\left(H_{\iota_1}\ (-1+H_{\iota_1}\ \iota_1)\left(\prod_{\iota_2=1}^{\iota_1}\left(\frac{1+H_{\iota_2}}{H_{\iota_2}}\right)\right)\left(H_{\iota_1}+(1+H_{\iota_1})\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{1+H_{\iota_2}}\right)\right)\right)\Big/ \left((1+H_{\iota_1})^2\ (-1+\iota_1+H_{\iota_1}\ \iota_1)\right)\right)\right)$$

Please note, that not only the result is nicer in comparison to the naive approach but also the timings are better - we needed only 539 seconds instead of 903 seconds.
By our indefinite summation algorithm `SigmaReduce` this expression can be simplified further to

$$-2-\frac{1}{H_n}-\sum_{\iota_1=1}^{n}\frac{1}{1+H_{\iota_1}}+\frac{1+H_n}{H_n}\left(\prod_{\iota_1=1}^{n}\frac{H_{\iota_1}}{1+H_{\iota_1}}\right)\left(1+\sum_{\iota_1=1}^{n}\prod_{\iota_2=1}^{\iota_1}\frac{1+H_{\iota_2}}{H_{\iota_2}}+\sum_{\iota_1=1}^{n}\left(\prod_{\iota_2=1}^{\iota_1}\frac{1+H_{\iota_2}}{H_{\iota_2}}\right)\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{1+H_{\iota_2}}\right)-\sum_{\iota_1=1}^{n}\frac{\left(\prod_{\iota_2=1}^{\iota_1}\frac{1+H_{\iota_2}}{H_{\iota_2}}\right)\sum_{\iota_2=1}^{\iota_1}\frac{1}{1+H_{\iota_2}}}{1+H_{\iota_1}}\right).$$

## A Variation of the previous example

Now we consider a small variation of the previous example

In[143]:= $\mathbf{mySum} = \sum_{r=1}^{n}\left(\frac{\sum_{l=1}^{r}\left(\prod_{k=1}^{l}\left(\frac{H_{-k+n}}{1+H_{-k+n}}\right)\right)}{(1+H_{n-r})^2}\right);$

and fail to compute a recurrence due to memory overflow:

In[144]:= **rec = GenerateRecurrence[mySum]**

```
Order:  1
Order:  2
```

Out[144]= $Aborted

Whereas trying to compute a recurrence by finding an appropriate sum extension results in a recurrence of order 1.

In[145]:= **rec = GenerateRecurrence[mySum, SimplifyByExt → Depth]**

```
1078.43 Second
```

Out[145]= $\Big\{-H_n\,\mathrm{SUM}[n] + (1+H_n)\,\mathrm{SUM}[1+n] == H_n\left(1+\sum_{\iota_1=1}^{n}\left(\frac{(1+n-\iota_1)^2}{(2+n+H_{n-\iota_1}+n\,H_{n-\iota_1}-\iota_1-H_{n-\iota_1}\,\iota_1)^2}\right)\right)\Big\}$

Applying again the definite summation method, like in the previous example, we find

$$\sum_{i=1}^{n}\frac{(1+n-i)^2}{(2+n+H_{n-i}+n\,H_{n-i}-i-H_{n-i}\,i)^2}=\sum_{i=1}^{n}\frac{1}{(1+H_i)^2}.$$

Therefore we can reformulate the above recurrence to

In[146]:= $\mathbf{rec} = \Big\{-H_n\,\mathrm{SUM}[n] + (1+H_n)\,\mathrm{SUM}[1+n] == H_n\left(1+\sum_{\iota_1=1}^{n}\left(\frac{1}{(1+H_{\iota_1})^2}\right)\right)\Big\};$

and can solve this recurrence by using the product extension[22]

In[147]:= $\mathbf{tower} = \left\{\prod_{k=1}^{n}\left(\frac{H_k}{1+H_k}\right)\right\};$

In[148]:= **recSol = SolveRecurrence[rec, SUM[n], Tower → tower, NestedSumExt → 1]**

```
58.394 Second
```

Out[148]= $\Big\{\Big\{0, \frac{(1+H_n)\prod_{\iota_1=1}^{n}\left(\frac{H_{\iota_1}}{1+H_{\iota_1}}\right)}{H_n}\Big\}, \Big\{1, \frac{1}{H_n}\left((1+H_n)\left(\prod_{\iota_1=1}^{n}\left(\frac{H_{\iota_1}}{1+H_{\iota_1}}\right)\right)\sum_{\iota_1=1}^{n}\left(\left(H_{\iota_1}\,(-1+H_{\iota_1}\,\iota_1)\left(\prod_{\iota_2=1}^{\iota_1}\left(\frac{1+H_{\iota_2}}{H_{\iota_2}}\right)\right)\left(2\,H_{\iota_1}+H_{\iota_1}^2+(1+2\,H_{\iota_1}+H_{\iota_1}^2)\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{(1+H_{\iota_2})^2}\right)\right)\right)\Big/\left((1+H_{\iota_1})^3\,(-1+\iota_1+H_{\iota_1}\,\iota_1)\right)\right)\right)\Big\}\Big\}$

[22] Here I guessed the product extension by looking at the previous example.

Finally we find the following closed form.

In[149]:= **FindLinearCombination[recSol, mySum, 1]**

```
0.29 Second
```

$$\text{Out[149]}= \frac{1}{\mathrm{H_n}}\Bigg((1+\mathrm{H_n})\left(\prod_{\iota_1=1}^{n}\left(\frac{\mathrm{H}_{\iota_1}}{1+\mathrm{H}_{\iota_1}}\right)\right)\sum_{\iota_1=1}^{n}\Bigg(\Bigg(\mathrm{H}_{\iota_1}\,(-1+\mathrm{H}_{\iota_1}\,\iota_1)\left(\prod_{\iota_2=1}^{\iota_1}\left(\frac{1+\mathrm{H}_{\iota_2}}{\mathrm{H}_{\iota_2}}\right)\right)\left(2\,\mathrm{H}_{\iota_1}+\mathrm{H}_{\iota_1}^2+\left(1+2\,\mathrm{H}_{\iota_1}+\mathrm{H}_{\iota_1}^2\right)\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{(1+\mathrm{H}_{\iota_2})^2}\right)\right)\Bigg)\Big/\left((1+\mathrm{H}_{\iota_1})^3\,(-1+\iota_1+\mathrm{H}_{\iota_1}\,\iota_1)\right)\Bigg)\Bigg)$$

By our indefinite summation algorithm `SigmaReduce` this expression can be simplified further to

$$-\frac{(1+2\,\mathrm{H_n})\left(1+\sum_{\iota_1=1}^{n}\frac{1}{(1+\mathrm{H}_{\iota_1})^2}\right)}{\mathrm{H_n}}+\frac{1+\mathrm{H_n}}{\mathrm{H_n}}\left(\prod_{\iota_1=1}^{n}\frac{\mathrm{H}_{\iota_1}}{1+\mathrm{H}_{\iota_1}}\right)\left(1+\sum_{\iota_1=1}^{n}\prod_{\iota_2=1}^{\iota_1}\frac{1+\mathrm{H}_{\iota_2}}{\mathrm{H}_{\iota_2}}-\sum_{\iota_1=1}^{n}\frac{\prod_{\iota_2=1}^{\iota_1}\frac{1+\mathrm{H}_{\iota_2}}{\mathrm{H}_{\iota_2}}}{(1+\mathrm{H}_{\iota_1})^3}+\sum_{\iota_1=1}^{n}\frac{\prod_{\iota_2=1}^{\iota_1}\frac{1+\mathrm{H}_{\iota_2}}{\mathrm{H}_{\iota_2}}}{(1+\mathrm{H}_{\iota_1})^2}+\sum_{\iota_1=1}^{n}\left(\prod_{\iota_2=1}^{\iota_1}\frac{1+\mathrm{H}_{\iota_2}}{\mathrm{H}_{\iota_2}}\right)\sum_{\iota_2=1}^{\iota_1}\frac{1}{(1+\mathrm{H}_{\iota_2})^2}\right).$$

#### 1.3.5.2 A Significant Reduction of the Recurrence Order

In [KP98, Equation 25] there appears the following sum

In[150]:= $\mathbf{mySum} = \sum_{\mathbf{j=1}}^{\mathbf{n}}\left(\sum_{\mathbf{k=1}}^{\mathbf{j}}\left(\frac{\mathbf{H_{-k+n}}}{\mathbf{k}}\right)\right)$

which can be simplified to

In[151]:= **SigmaReduce[mySum]**

Out[151]= $\mathtt{n} - \mathtt{n}\,\mathtt{H}_{\mathtt{n}} + (1+\mathtt{n})\sum_{\iota_1=1}^{\mathtt{n}}\left(\frac{\mathtt{H}_{\mathtt{n}-\iota_1}}{\iota_1}\right)$

Using my package one can easily find that

$$\sum_{k=1}^{n}\frac{\mathrm{H}_{n-k}}{k} = \mathrm{H}_n^2 - \mathrm{H}_n^{(2)}$$

and therefore we obtain

$$\sum_{j=1}^{n}\sum_{k=1}^{j}\frac{\mathrm{H}_{n-k}}{k} = n - n\,\mathrm{H}_n + (1+n)\left(\mathrm{H}_n^2 - \mathrm{H}_n^{(2)}\right).$$

Now let us look at a small variation of the definite sum:

In[152]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{n}}\left(\frac{\mathbf{H_{-k+n}}}{\mathbf{k^8}}\right);$

If we try to compute a recurrence with the naive approach, we fail due to memory overflow:

In[153]:= **rec = GenerateRecurrence[mySum]**

```
Order: 1
Order: 2
Order: 3
Order: 4
Order: 5
Order: 6
Order: 7
Order: 8
Order: 9
```

Out[153]= `$Aborted`

But we are able to compute a recurrence by finding an appropriate sum extension:

In[154]:= **rec = GenerateRecurrence[mySum, SimplifyByExt → Depth]**

Out[154]= $\left\{-\mathtt{n}^8\,\mathtt{SUM}[\mathtt{n}] + \mathtt{n}^8\,\mathtt{SUM}[1+\mathtt{n}] == 1 + \mathtt{n}^8\sum_{\iota_1=2}^{\mathtt{n}}\left(\frac{1}{(2+\mathtt{n}-\iota_1)\,(-1+\iota_1)^8}\right)\right\}$

Now we analyze the summand of the sum extension using partial fraction decomposition.

In[155]:= $\frac{\mathbf{1}}{(\mathbf{2}+\mathbf{n}-\iota_{\mathbf{1}})\ (-\mathbf{1}+\iota_{\mathbf{1}})^{\mathbf{8}}}//\mathbf{Apart}$

Out[155]= $$\frac{1}{(1+n)\ (-1+\iota_1)^8}+\frac{1}{(1+n)^2\ (-1+\iota_1)^7}+\frac{1}{(1+n)^3\ (-1+\iota_1)^6}+\frac{1}{(1+n)^4\ (-1+\iota_1)^5}+\frac{1}{(1+n)^5\ (-1+\iota_1)^4}+\frac{1}{(1+n)^6\ (-1+\iota_1)^3}+\frac{1}{(1+n)^7\ (-1+\iota_1)^2}+\frac{1}{(1+n)^8\ (-1+\iota_1)}-\boxed{\frac{1}{(1+n)^8\ (-2-n+\iota_1)}}$$

By

$$\sum_{i=2}^{n}\frac{1}{2+n-i}=\sum_{i=2}^{n}\frac{1}{i-1}$$

and summing each fraction separately, we obtain the following recurrence

$$-n^8\ \mathrm{SUM}[n]+n^8\ \mathrm{SUM}[1+n]==$$
$$1+n^8\left(\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{(-1+\iota_1)^8}\right)}{1+n}+\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{(-1+\iota_1)^7}\right)}{(1+n)^2}+\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{(-1+\iota_1)^6}\right)}{(1+n)^3}+\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{(-1+\iota_1)^5}\right)}{(1+n)^4}+\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{(-1+\iota_1)^4}\right)}{(1+n)^5}+\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{(-1+\iota_1)^3}\right)}{(1+n)^6}+\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{(-1+\iota_1)^2}\right)}{(1+n)^7}+\frac{\sum_{\iota_1=2}^{n}\left(\frac{1}{-1+\iota_1}\right)}{(1+n)^8}+\frac{-1+\sum_{\iota_1=2}^{n}\left(\frac{1}{\iota_1}\right)}{(1+n)^8}\right)$$

One can easily solve this recurrence - even by hand - and search out for a solution of the recurrence using Harmonic numbers, namely

$$1-\frac{1}{n^8}-H_n^{(9)}+\sum_{i=1}^{n}\left(\frac{H_i}{i^8}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i}{(2+i)^8}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i^{(2)}}{(2+i)^7}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i^{(3)}}{(2+i)^6}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i^{(4)}}{(2+i)^5}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i^{(5)}}{(2+i)^4}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i^{(6)}}{(2+i)^3}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i^{(7)}}{(2+i)^2}\right)+\sum_{i=0}^{-2+n}\left(\frac{H_i^{(8)}}{2+i}\right).$$

Since the initial values of this expression and the original summation problem are the same, they must be equal.

## 1.4 Symbolic Summation in Difference Fields

Below I will give a summary of the first chapter with respect to what kind of problems of symbolic summation we have to deal with in terms of difference fields. This will give a motivation for the following chapters where these problems will be treated.

**Chapter 2:** *Difference Fields*

As outlined in Chapter 1, we are mainly interested in solving difference equations. In this thesis we will develop algorithms to solve difference equations in so called $\Pi\Sigma$-fields $(\mathbb{F}, \sigma)$ with its constant field

$$\mathbb{K} := \{k \in \mathbb{F} \,|\, \sigma(k) = k\}.$$

In Chapter 2 we will show how one can constructively define such difference fields; in addition, we will explore its properties.

**Chapter 3:** *Solving Difference Equations*

As we have indicated in Section 1.2.2 we have to solve the following problem to deal with respect to **indefinite summation**.

### Telescoping

- GIVEN $f \in \mathbb{F}$,
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

In order to solve this problem, one uses a reduction process where one has to solve the following more general problem.

### Parameterized First Order Linear Difference Equations

- GIVEN $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$,
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\, \sigma(h) - a_0\, h = c_0\, f_0 + \dots + c_d\, f_d}$$

As was shown in Section 1.3.3.2, this is exactly what we need for creative telescoping in order to find a recurrence of a **definite summation problem**.

### Creative Telescoping

- GIVEN $f_i = \text{summand}(n+i,k) \in \mathbb{F}$,
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \cdots + c_d\, f_d}$$

From creative telescoping we can derive a recurrence for the definite sum as it was outlined in Section 1.3.3.1. In order to solve this recurrence we have to solve linear difference equations of higher order. This was already mentioned in Section 1.3.4.1.

### $m$-th Order Linear Difference Equations

- GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$,
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\, \sigma^m(g) + \cdots + a_0\, g = f}$$

In order to solve this problem, one uses a reduction process where one has to solve the following more general problem.

### Parameterized $m$-th Order Linear Difference Equations

- GIVEN $a_0, \dots, a_m \in \mathbb{F}, f_0, \dots, f_d \in \mathbb{F}$,
- FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\, \sigma^m(g) + \cdots a_0\, g = c_0\, f_0 + \cdots c_d\, f_d}$$

This is the main problem we will focus on in Chapter 3.

## **Chapter 4:** *Summation and Difference Field Extensions*

Finally we are interested in finding appropriate sum extensions in order

- to reduce the depth of nested sums (Section 1.2.3),
- to find nested sum extensions which give rise to additional solutions of a recurrence (Section 1.3.4.2)
- and to find sum extensions to reduce the order of a creative telescoping equation (Section 1.3.5).

These problems and its solutions will be treated in the last Chapter 4. Moreover, I will derive further insight into symbolic summation by rephrasing the summation problems in the general difference field setting.

# Chapter 2

# Difference Fields

## 2.1 Basic Definitions for Difference Fields

**Definition 2.1.1.** A *difference field (resp. ring[1])* is a field (resp. ring) $\mathbb{F}$ together with a field (resp. ring) automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. $\Diamond$

**Notation 2.1.1.** The difference field (resp. ring) given by the field (resp. ring) $\mathbb{F}$ and automorphism $\sigma$ is denoted by $(\mathbb{F}, \sigma)$. $\Diamond$

**Example 2.1.1.** Consider the difference field $(\mathbb{Q}, \sigma)$ with the field automorphism

$$\forall q \in \mathbb{Q} : \sigma(q) = q.$$

It can be easily shown that this is the only field automorphism of $\mathbb{Q}$ and thus $(\mathbb{Q}, \sigma)$ is uniquely defined. $\Diamond$

Let $(\mathbb{A}, \sigma)$ be a difference ring and consider the following set

$$\mathbb{B} := \{k \in \mathbb{A} \,|\, \sigma(k) = k\}.$$

Then one can easily prove that $\mathbb{B}$ is a subring of $\mathbb{A}$. Furthermore, if $\mathbb{A}$ is a field then also $\mathbb{B}$ is a field.

**Definition 2.1.2.** Let $(\mathbb{A}, \sigma)$ be a difference ring. We define the following set

$$\text{const}_\sigma\mathbb{A} := \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

and call it the *constant ring* of $(\mathbb{A}, \sigma)$. If $\text{const}_\sigma\mathbb{A}$ is a field, we even call it the *constant field* of $(\mathbb{A}, \sigma)$. $\Diamond$

**Remark 2.1.1.** If not differently stated, in the following we assume that for any difference ring $(\mathbb{A}, \sigma)$ the constant ring $\text{const}_\sigma\mathbb{A}$ is a field, i.e. we assume that a difference ring $(\mathbb{A}, \sigma)$ has the constant field $\text{const}_\sigma\mathbb{A}$. Additionally, the constant field of any difference ring $(\mathbb{A}, \sigma)$ has **always** characteristic 0. This means

$$\mathbb{Q} \subseteq \text{const}_\sigma\mathbb{A}.$$

$\Diamond$

[1]Throughout this thesis all rings are assumed to be commutative.

**Example 2.1.2.** Consider the difference field $(\mathbb{C}, \sigma)$ with the automorphism canonically defined by

$$\sigma(a + i\,b) = a - i\,b.$$

Then this is the only nontrivial difference field and the real numbers form the constant field of $(\mathbb{C}, \sigma)$. $\diamondsuit$

**Lemma 2.1.1.** *Let* $(\mathbb{A}, \sigma)$ *be a difference ring. The* difference operator

$$\Delta : \begin{cases} \mathbb{A} & \rightarrow & \mathbb{A} \\ f & \mapsto & \sigma(f) - f \end{cases}$$

*satisfies for all* $f, g \in \mathbb{A}$ *the following* difference rules*:*

1. $\Delta(f + g) = \Delta(f) + \Delta(g)$*;*
2. $\Delta(fg) = f\Delta(g) + \Delta(f)g + \Delta(f)\Delta(g)$. *(Leibnitz)*

*Proof.* We have

$$\begin{aligned} \Delta(f\,g) &= \sigma(f\,g) - f\,g = \sigma(f)\,\sigma(g) - f\,g \\ &= f\,(\sigma(g) - g) + (\sigma(f) - f)\,g + (\sigma(f) - f)\,(\sigma(g) - g) \\ &= f\,\Delta(g) + \Delta(f)\,g + \Delta(f)\,\Delta(g); \\ \Delta(f + g) &= \sigma(f + g) - (f + g) = (\sigma(f) - f) + (\sigma(g) - g) = \Delta(f) + \Delta(g). \end{aligned}$$

$\square$

**Example 2.1.3.** Consider the polynomial ring $\mathbb{Q}[n]$ where $n$ is transcendental over $\mathbb{Q}$. We define the map

$$\sigma : \begin{cases} \mathbb{Q}[n] & \rightarrow & \mathbb{Q}[n] \\ \sum_i f_i\,n^i & \mapsto & \sum_i f_i\,(n+1)^i. \end{cases}$$

One can easily check that $\sigma$ is a ring automorphism and that $(\mathbb{Q}[n], \sigma)$ is a difference ring with constant field $\mathbb{Q}$. Clearly, the difference operator $\Delta$ on $\mathbb{Q}[n]$ satisfies the difference rules. $\diamondsuit$

### 2.1.1 Difference Field Isomorphisms

**Definition 2.1.3.** Two difference fields (resp. rings) $(\mathbb{F}, \sigma)$ and $(\tilde{\mathbb{F}}, \tilde{\sigma})$ are *isomorphic*, in symbols $(\mathbb{F}, \sigma) \simeq (\tilde{\mathbb{F}}, \tilde{\sigma})$, if there exists a field (resp. ring) isomorphism $\tau : \mathbb{F} \rightarrow \tilde{\mathbb{F}}$ with

$$\tau\,\sigma = \tilde{\sigma}\,\tau.$$

$\tau$ is called *difference field (resp. ring) isomorphism* . $\diamondsuit$

**Corollary 2.1.1.** *Let* $(\mathbb{A}, \sigma)$, $(\tilde{\mathbb{A}}, \tilde{\sigma})$ *be difference rings,* $a_0, \dots, a_m, f \in \mathbb{A}$ *and assume*

$$(\mathbb{A}, \sigma) \overset{\tau}{\simeq} (\tilde{\mathbb{A}}, \tilde{\sigma}).$$

*Then for all* $g \in \mathbb{A}$ *we have*

$$a_0\,\sigma^m(g) + \cdots + a_{m-1}\,\sigma(g) + a_m\,g = f$$

$$\Updownarrow$$

$$\tau(a_0)\,\tilde{\sigma}^m\,\tau(g) + \cdots + \tau(a_{m-1})\,\tilde{\sigma}\,\tau(g) + \tau(a_m)\,\tau(g) = \tau(f).$$

**Definition 2.1.4.** Let $(\mathbb{A}, \sigma)$, $(\tilde{\mathbb{A}}, \tilde{\sigma})$ be difference fields (resp. rings). $\tau : \mathbb{A} \to \tilde{\mathbb{A}}$ is called difference field (resp. ring) homomorphism/epimorphism/monomorphism, if $\tau$ is a field (resp. ring) homomorphism/epimorphism/monomorphism with

$$\tau\,\sigma = \tilde{\sigma}\,\tau.$$

◊

### 2.1.2 Difference Field Extensions

**Definition 2.1.5.** Let $(\mathbb{E}, \sigma_{\mathbb{E}})$, $(\mathbb{F}, \sigma_{\mathbb{F}})$ be difference fields (resp. rings). $(\mathbb{E}, \sigma_{\mathbb{E}})$ is called a *difference field (resp. ring) extension* of $(\mathbb{F}, \sigma_{\mathbb{F}})$, in symbols $(\mathbb{F}, \sigma_{\mathbb{F}}) \leq (\mathbb{E}, \sigma_{\mathbb{E}})$, if $\mathbb{F}$ is a subfield (resp. subring) of $\mathbb{E}$ and for all $f \in \mathbb{F}$ we have $\sigma_{\mathbb{F}}(f) = \sigma_{\mathbb{E}}(f)$. ◊

**Example 2.1.4.** Consider the quotient field $\mathbb{Q}(n)$ of the polynomial ring $\mathbb{Q}[n]$ from Example 2.1.3. As shown later (Lemma 2.4.5), there exists a unique difference field $(\mathbb{Q}(n), \tilde{\sigma})$ which is a difference ring extension of $(\mathbb{Q}[n], \sigma)$ with

$$\tilde{\sigma} : \begin{cases} \mathbb{Q}(n) & \to & \mathbb{Q}(n) \\ \frac{a}{b} & \mapsto & \frac{\sigma(a)}{\sigma(b)}. \end{cases}$$

◊

**Remark 2.1.2.** If $(\mathbb{E}, \tilde{\sigma})$ is a difference ring extension of $(\mathbb{A}, \sigma)$ then in the following we will not distinguish anymore that $\sigma : \mathbb{A} \to \mathbb{A}$ and $\tilde{\sigma} : \mathbb{E} \to \mathbb{E}$ are actually different automorphisms. Rather than this, we will say that $(\mathbb{E}, \sigma)$ is a difference ring extension of $(\mathbb{A}, \sigma)$. If $g \in \mathbb{E} \setminus \mathbb{A}$ then, of course, writing $\sigma(g)$ means that we use the automorphism of the difference ring $(\mathbb{E}, \sigma)$, whereas using $\sigma(g)$ with $g \in \mathbb{A}$ means that $\sigma$ can be both, the automorphism of $(\mathbb{A}, \sigma)$ or $(\mathbb{E}, \sigma)$. It will be always clear from the context which automorphism has to be used. ◊

**Example 2.1.5.** Let $(\mathbb{E}, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ in which we have an element $t \in \mathbb{E}$ with

$$\sigma(t) = \alpha\, t + \beta$$

for some[2] $\alpha \in \mathbb{A}^*$ and $\beta \in \mathbb{A}$. $\mathbb{A}[t]$, by adjoining $t$ to $\mathbb{A}$, is a subring of $\mathbb{E}$ and a ring extension of $\mathbb{A}$. As for all $f \in \mathbb{A}[t] \subseteq \mathbb{E}$ we have $\sigma(f) \in \mathbb{A}[t]$, it follows that $\sigma$ restricted on $\mathbb{A}[t]$ is a difference ring automorphism and thus $(\mathbb{A}[t], \sigma)$ is a difference ring with

$$(\mathbb{A}, \sigma) \leq (\mathbb{A}[t], \sigma) \leq (\mathbb{E}, \sigma).$$

If $\mathbb{E}$ is a field then $\mathbb{A}[t]$ is an integral domain and we can construct the *quotient field* $Q(\mathbb{A}[t])$. Furthermore - as will be shown later in Lemma 2.4.5 - there is a unique difference ring extension $(Q(\mathbb{A}[t]), \sigma)$ of $(\mathbb{A}[t], \sigma)$. If $\mathbb{A}$ is a subfield of the field $\mathbb{E}$ then we even have

$$(\mathbb{A}, \sigma) \leq (Q(\mathbb{A}[t]), \sigma) \leq (\mathbb{E}, \sigma).$$

◊

[2] Given a set $A$ we will denote by $A^*$ the set $A \setminus \{0\}$, i.e. $A^* := A \setminus \{0\}$

**Example 2.1.6.** Let $(\mathbb{F}, \sigma)$ be a difference field, $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$ and consider the polynomial ring $\mathbb{F}[t]$ with $t$ transcendental over $\mathbb{F}$. If we look at the map

$$\sigma : \begin{cases} \mathbb{F}[t] & \rightarrow & \mathbb{F}[t] \\ \sum_{i=0}^{n} f_i\, t^i & \mapsto & \sum_{i=0}^{n} \sigma(f_i)\,(\alpha\, t + \beta)^i \end{cases}$$

then one can easily verify that $\sigma : \mathbb{F}[t] \rightarrow \mathbb{F}[t]$ is a ring homomorphism. Furthermore,

$$\tau : \begin{cases} \mathbb{F}[t] & \rightarrow & \mathbb{F}[t] \\ \sum_{i=0}^{n} f_i\, t^i & \mapsto & \sum_{i=0}^{n} \sigma^{-1}(f_i)\,\left(\frac{t-\beta}{\alpha}\right)^i \end{cases}$$

is again a ring homomorphism and we have

$$\tau(\sigma(t)) = t.$$

Therefore $\tau$ is the inverse homomorphism of $\sigma$ and thus $\sigma$ is an automorphism. Consequently $(\mathbb{F}[t], \sigma)$ is a difference ring and therefore a difference ring extension of $(\mathbb{F}, \sigma)$. ♢

**Remark 2.1.3 (and Definition).** Let $(\mathbb{A}[t], \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ defined by

$$\sigma : \begin{cases} \mathbb{A}[t] & \rightarrow & \mathbb{A}[t] \\ \sum_{i=0}^{n} f_i\, t^i & \mapsto & \sum_{i=0}^{n} \sigma(f_i)\,(\alpha\, t + \beta)^i \end{cases} \tag{2.1}$$

with $\alpha \in \mathbb{A}^*$ and $\beta \in \mathbb{A}$. In the following this will be shortly expressed by saying that $(\mathbb{A}[t], \sigma)$ is a difference ring extension of $(\mathbb{A}, \sigma)$ *canonically defined by*

$$\sigma(t) = \alpha\, t + \beta.$$

If nothing more is specified then we assume that $t$ is transcendental or not transcendental over $\mathbb{A}$. ♢

The proof of the following lemma is straightforward.

**Lemma 2.1.2.** *Let $(\mathbb{A}[t], \sigma)$ be a difference ring canonically defined by*

$$\sigma(t) = \alpha\, t + \beta$$

*with $\alpha \in \mathbb{A}^*$, $\beta \in \mathbb{A}$ and $t$ transcendental over $\mathbb{A}$. Then for all $i \in \mathbb{Z}$ it follows that*

$$\deg(\sigma^i(f)) = \deg(f).$$

**Example 2.1.7.** Consider the difference ring extension $(\mathbb{F}[t], \sigma)$ of $(\mathbb{F}, \sigma)$ which we have defined in Example 2.1.6. Clearly, we can define the quotient field $\mathbb{F}(t)$ of $\mathbb{F}[t]$ and define a map $\sigma : \mathbb{F}(t) \rightarrow \mathbb{F}(t)$ canonically defined by

$$\sigma(\frac{a}{b}) = \frac{\sigma(a)}{\sigma(b)}$$

for $a \in \mathbb{F}[t]$ and $b \in \mathbb{F}[t]^*$. Then it follows that $\sigma : \mathbb{F}(t) \rightarrow \mathbb{F}(t)$ is a field automorphism and therefore $(\mathbb{F}(t), \sigma)$ a difference field with

$$(\mathbb{F}, \sigma) \leq (\mathbb{F}[t], \sigma) \leq (\mathbb{F}(t), \sigma).$$

♢

**Remark 2.1.4 (and Definition).** Let $(\mathbb{A}(t), \sigma)$ be a difference field extension of $(\mathbb{A}, \sigma)$. If the difference field $(\mathbb{A}(t), \sigma)$ is not specified further then we always assume that $t$ might be *either algebraic or transcendental* over $\mathbb{A}$. If it is algebraic then we might also write[3] $\mathbb{A}(t) = \mathbb{A}[t]$.

Furthermore, assume that $(\mathbb{A}(t), \sigma)$ is defined by

$$\sigma(\frac{a}{b}) = \frac{\sigma(a)}{\sigma(b)}$$

for $a \in \mathbb{A}[t]$, $b \in \mathbb{A}[t]^*$ and where $\sigma$ acts on $\mathbb{A}[t]$ by (2.1). In the following this will be shortly expressed by saying that $(\mathbb{A}(t), \sigma)$ is a difference field extension of $(\mathbb{A}, \sigma)$ *canonically defined by*

$$\sigma(t) = \alpha\, t + \beta.$$

$\diamondsuit$

**Definition 2.1.6.** A difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *affine*, if it is canonically defined by

$$\sigma(t) = \alpha\, t + \beta$$

for some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$.

In particular, $(\mathbb{F}(t), \sigma)$ is called *sum extension* of $(\mathbb{F}, \sigma)$, if

$$\sigma(t) = t + \beta$$

and $(\mathbb{F}(t), \sigma)$ is called *product extension* of $(\mathbb{F}, \sigma)$, if

$$\sigma(t) = \alpha\, t.$$

$\diamondsuit$

**Definition 2.1.7.** Let $(\mathbb{E}, \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$.

$t \in \mathbb{E}$ is called[4] a *sum* over $\mathbb{F}$, if $\sigma(t) - t \in \mathbb{F}$.

$t \in \mathbb{E}^*$ is called a *hyperexponential* over $\mathbb{F}$, if $\frac{\sigma(t)}{t} \in \mathbb{F}$. $\diamondsuit$

**Definition 2.1.8.** Let $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$.

$(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is called *sum extension* of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t_1, \dots, t_i)(t_{i+1}), \sigma)$ is a sum-extension of $(\mathbb{F}(t_1, \dots, t_i), \sigma)$ for all[5] $0 \leq i \leq n-1$.

$(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is called *product extension* of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t_1, \dots, t_i)(t_{i+1}), \sigma)$ is a product extension of $(\mathbb{F}(t_1, \dots, t_i), \sigma)$ for all $0 \leq i \leq n-1$.

$(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is called *product-sum extension* of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t_1, \dots, t_i)(t_{i+1}), \sigma)$ is a product extension or a sum extension of $(\mathbb{F}(t_1, \dots, t_i), \sigma)$ for all $0 \leq i \leq n-1$.

$(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is called *d'Alembertian extension* of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is a product-sum extension of $(\mathbb{F}, \sigma)$ and every product extension is hyperexponential over $\mathbb{F}$. $\diamondsuit$

[3] See for instance [Lan97] Proposition 1.4.

[4] In [Bro00] these elements are called primitive

[5] For the case $i = 0$ this means that $(\mathbb{F}(t_1), \sigma)$ is a sum extension of $(\mathbb{F}, \sigma)$.

## 2.2 ΠΣ-Fields

### 2.2.1 First Order Linear Extensions

**Definition 2.2.1.** A difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *first order linear*, if it is canonically defined by

$$\sigma(t) = \alpha\, t + \beta$$

for some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$, $t$ is transcendental over $\mathbb{F}$ and $\text{const}_\sigma \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$. ◊

**Example 2.2.1.** Let $(\mathbb{Q}(t), \sigma)$ be a difference field where $t$ is transcendental over $\mathbb{Q}$ and $\sigma$ is defined canonically on $\mathbb{Q}(t)$ by

$$\sigma(t) = 4\, t.$$

By definition, $(\mathbb{Q}, \sigma) \leq (\mathbb{Q}(t), \sigma)$ is affine and $t$ is transcendental over $\mathbb{Q}$. As will be shown later, we also have $\text{const}_\sigma \mathbb{Q}(t) = \mathbb{Q}$ and thus the extension is first order linear. ◊

**Example 2.2.2.** In Example 2.1.4 we constructed a difference field $(\mathbb{Q}(n), \sigma)$ with $\sigma(n) = n+1$ where $n$ is transcendental over $\mathbb{Q}$. As will be shown later, we also have $\text{const}_\sigma \mathbb{Q}(n) = \mathbb{Q}$ and thus the extension is first order linear. ◊

**Definition 2.2.2.** Let $\mathbb{F}[t]$ be a polynomial ring with coefficients in the field $\mathbb{F}$, $t$ is transcendental over $\mathbb{F}$, and let $\mathbb{F}(t)$ be the field of *of rational functions* over $\mathbb{F}$, this means $\mathbb{F}(t)$ is the quotient field of $\mathbb{F}[t]$. $\frac{p}{q} \in \mathbb{F}(t)$ is in *reduced representation* if $p, q \in \mathbb{F}[t]$, $\gcd(p, q) = 1$ and $q$ is monic. ◊

The proof of the following lemma is straightforward.

**Lemma 2.2.1.** *Let $(\mathbb{F}(t), \sigma)$ be a first order linear extension of $(\mathbb{F}, \sigma)$. Then $\mathbb{F}(t)$ is a field of rational functions over $\mathbb{K}$. Furthermore, $\sigma$ is an automorphism of $\mathbb{F}[t]$ and thus $(\mathbb{F}(t), \sigma)$ is a difference ring extension of $(\mathbb{F}[t], \sigma)$. Additionally, we have for all $f, g \in \mathbb{F}[t]$ that*

$$\gcd(\sigma(f), \sigma(g)) = \sigma(\gcd(f, g)),$$
$$\deg(\sigma(f)) = \deg(f)$$

*and $f$ is irreducible in $\mathbb{F}[t]$, if and only if $\sigma(f)$ is irreducible in $\mathbb{F}[t]$.*

### 2.2.2 Homogeneous and Inhomogeneous Extensions

**Definition 2.2.3.** Let $(\mathbb{E}, \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$. The difference field extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ is called *homogeneous*, if there exists an element $g \in \mathbb{E} \setminus \mathbb{F}$ such that

$$\frac{\sigma(g)}{g} \in \mathbb{F}.$$

Otherwise the extension is called *inhomogeneous*. ◊

**Example 2.2.3.** All difference field extensions $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ defined canonically by

$$\sigma(t) = \alpha\, t$$

with $\alpha \in \mathbb{F}^*$ are homogeneous. ◊

The following theorem was first stated in [Kar81, Theorem 1] and its proof is essentially the same as in [Kar85].

**Theorem 2.2.1.** *Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t + \beta$$

*for some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Then the following conditions are equivalent:*

1. *$(\mathbb{F}(t), \sigma)$ is a homogeneous extension of $(\mathbb{F}, \sigma)$.*
2. *There exists a $g \in \mathbb{F}[t] \setminus \mathbb{F}$ with $\frac{\sigma(g)}{g} \in \mathbb{F}$.*
3. *There exists a $w \in \mathbb{F}$ with $\sigma(w) = \alpha\, w + \beta$ and[6] $t \notin \mathbb{F}$.*

*Proof.* **"1 ⇒ 2"** Let $g \in \mathbb{F}(t) \setminus \mathbb{F}$ with

$$\frac{\sigma(g)}{g} \in \mathbb{F}.$$

If $t$ is algebraic then obviously[7] $\mathbb{F}(t) = \mathbb{F}[t]$ and thus $g \in \mathbb{F}[t]$. Now assume that $g$ is transcendental over $\mathbb{F}$ and write

$$g = \frac{p}{q}$$

in reduced representation, in particular $p, q \in \mathbb{F}[t]$ are relatively prime. We have

$$\frac{\sigma(g)}{g} = \frac{\sigma(p/q)}{p/q} = \frac{\sigma(p)\, q}{p\, \sigma(q)} \in \mathbb{F}$$

and because of $\gcd(\sigma(p), \sigma(q)) = \gcd(p, q) = 1$ it follows that

$$p \mid \sigma(p), \quad \sigma(q) \mid q.$$

Since $\deg(\sigma(p)) = \deg(p)$ and $\deg(\sigma(q)) = \deg(q)$, it follows that

$$\frac{\sigma(p)}{p} \in \mathbb{F}, \quad \frac{\sigma(q)}{q} \in \mathbb{F};$$

and as $g \notin \mathbb{F}$, we get $p \notin \mathbb{F}$ or $q \notin \mathbb{F}$.

**"2 ⇒ 3"** Let $g = \sum_{i=0}^{n} g_i\, t^i$ with $n \geq 1$, $g_i \in \mathbb{F}$, $g_n \neq 0$ and define

$$u := \frac{\sigma(g)}{g} \in \mathbb{F}.$$

We have

$$\sum_{i=0}^{n} \sigma(g_i)\, (\alpha\, t + \beta)^i = \sigma(g) = u\, g = u \sum_{i=0}^{n} g_i\, t^i$$

and thus, by coefficient comparison,

$$\begin{aligned} u\, g_n &= \sigma(g_n)\, \alpha^n, \\ u\, g_{n-1} &= \sigma(g_n)\, n\, \alpha^{n-1}\, \beta + \sigma(g_{n-1})\, \alpha^{n-1} = (\sigma(g_n)\, n\, \beta + \sigma(g_{n-1}))\, \alpha^{n-1}. \end{aligned} \tag{2.2}$$

[6] Please note that we corrected Karr's Theorem by adding the second condition $t \notin \mathbb{F}$.
[7] See Remark 2.1.4.

Substituting

$$\alpha^{n-1} = \frac{u\,g_n}{\alpha\,\sigma(g_n)}$$

into equation (2.2) yields

$$u\,g_{n-1} = (\sigma(g_n)\,n\,\beta + \sigma(g_{n-1}))\frac{u\,g_n}{\alpha\,\sigma(g_n)}$$

which is equivalent to

$$\frac{g_{n-1}}{g_n} = \frac{n\,\beta}{\alpha} + \sigma(\frac{g_{n-1}}{g_n})\,\frac{1}{\alpha}.$$

Thus with

$$w := -\frac{g_{n-1}}{n\,g_n} \in \mathbb{F}$$

we obtain

$$\sigma(w) - \alpha\,w = \beta.$$

**"3 ⇒ 1"** Assume there is a $w \in \mathbb{F}$ with

$$\sigma(w) - \alpha\,w = \beta.$$

Then we have

$$\sigma(t - w) - \alpha\,(t - w) = 0$$

and thus

$$\frac{\sigma(g)}{g} = \alpha \in \mathbb{F}$$

with $g := t - w \notin \mathbb{F}$. □

**Remark 2.2.1.** Let $(\mathbb{F}(t), \sigma)$ be an affine difference field extension of $(\mathbb{F}, \sigma)$ canonically defined by

$$\sigma(t) = \alpha\,t + \beta$$

for some $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$ and $t \notin \mathbb{F}$. Suppose one can solve first order linear difference equations in the difference field $(\mathbb{F}, \sigma)$; in particular this means, one can check if there exists a solution $g \in \mathbb{F}$ such that

$$\sigma(g) - \alpha\,g = \beta.$$

Then by Theorem 2.2.1 one has an algorithm which decides if $(\mathbb{F}(t), \sigma)$ is a homogeneous or an inhomogeneous extension of $(\mathbb{F}, \sigma)$. ♢

**Example 2.2.4.** Let $(\mathbb{Q}(n), \sigma)$ be the difference field defined in Example 2.1.3 with $\sigma(n) = n + 1$ and $n$ transcendental over $\mathbb{Q}$. Let $h$ be a transcendental element over $\mathbb{Q}(n)$ and define canonically the difference field extension $(\mathbb{Q}(n, h), \sigma)$ of $(\mathbb{Q}(n), \sigma)$ by

$$\sigma(h) = h + \frac{1}{n+1}.$$

As will be shown later, one can check automatically that there does not exist a $g \in \mathbb{Q}(n)$ such that

$$\sigma(g) - g = \frac{1}{n+1}$$

and therefore $(\mathbb{Q}(n, h), \sigma)$ is an inhomogeneous extension of $(\mathbb{Q}(n), \sigma)$. ♢

In [Kar81] Karr remarked that one can "change a basis" so that a homogeneous difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ canonically defined by

$$\sigma(t)=\alpha\,t+\beta$$

for some $\alpha\in\mathbb{F}^*$ and $\beta\in\mathbb{F}$ can be rewritten to a difference field extension $(\mathbb{F}(x),\sigma)$ of $(\mathbb{F},\sigma)$ canonically defined by

$$\sigma(x)=\alpha\,x.$$

The following corollary makes this statement more precise.

**Corollary 2.2.1.** *Let $(\mathbb{F}(t),\sigma)$ and $(\mathbb{F}(x),\sigma)$ be homogeneous difference field extensions of $(\mathbb{F},\sigma)$ canonically defined by*

$$\begin{aligned}\sigma(t)&=\alpha\,t+\beta,\\ \sigma(x)&=\alpha\,x\end{aligned}$$

*where $\alpha\in\mathbb{F}^*$, $\beta\in\mathbb{F}$ and $t,x$ are transcendental over $\mathbb{F}$. Then we have*

$$(\mathbb{F}(t),\sigma)\simeq(\mathbb{F}(x),\sigma).$$

*Proof.* As $(\mathbb{F}(t),\sigma)$ is a homogeneous difference field extension of $(\mathbb{F},\sigma)$, by Theorem 2.2.1 there is an element $w\in\mathbb{F}$ such that

$$\sigma(w)-\alpha\,w=\beta.$$

Now consider the canonical map

$$\tau:\begin{cases}\mathbb{F}(t) & \to\quad \mathbb{F}(x)\\ t & \mapsto\quad x+w.\end{cases}$$

Since $t$ and $x$ are transcendental over $\mathbb{F}$, $\tau$ is a field isomorphism. By

$$\begin{aligned}\tau\,\sigma(t)=&\tau(\alpha\,t+\beta)=\alpha(x+w)+\beta=\alpha\,x+\beta+\alpha\,w,\\ \sigma\,\tau(t)=&\sigma(x+w)=\alpha\,x+\sigma(w)=\alpha\,x+\beta+\alpha\,w\end{aligned}$$

it follows that $\tau\,\sigma=\sigma\,\tau$ and thus $\tau$ is a difference field isomorphism. □

**Example 2.2.5.** Let $(\mathbb{Q}(n),\sigma)$ be the difference field defined in Example 2.1.4 with $\sigma(n)=n{+}1$. Let $t$ be transcendental over $\mathbb{Q}(n)$ and consider the difference field extensions $(\mathbb{Q}(n,t),\sigma)$ and $(\mathbb{Q}(n,t),\tilde{\sigma})$ canonically defined by

$$\begin{aligned}\sigma(t)&=n\,t+1-n,\\ \tilde{\sigma}(t)&=n\,t.\end{aligned}$$

Since $w=1$ is a solution of

$$\sigma(w)=nw+1-n,$$

it follows that $(\mathbb{Q}(n,t),\sigma)$ and $(\mathbb{Q}(n,t),\tilde{\sigma})$ are homogeneous extensions and $\tau:\mathbb{F}(t)\to\mathbb{F}(t)$ canonically defined by

$$\sigma(t)=t+1$$

is a difference field isomorphism. Thus

$$(\mathbb{Q}(n,t),\sigma)\overset{\tau}{\simeq}(\mathbb{Q}(n,t),\tilde{\sigma}).$$

◊

### 2.2.3 $\Pi$-Extensions

Let $(\mathbb{F}(t), \sigma)$ be a first order linear extension of $(\mathbb{F}, \sigma)$. Then Corollary 2.2.1 motivates us to consider only those homogeneous extensions with

$$\sigma(t) = \alpha\, t.$$

**Definition 2.2.4.** $(\mathbb{F}(t), \sigma)$ is a $\Pi$-*extension* of $(\mathbb{F}, \sigma)$ if

1. $(\mathbb{F}(t), \sigma)$ is first order linear
2. and $\sigma(t) = \alpha\, t$ with $\alpha \in \mathbb{F}^*$.

$\diamondsuit$

**Remark 2.2.2.** Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$. Then $(\mathbb{F}(t), \sigma)$ is canonically defined by

$$\sigma(t) = \alpha\, t$$

for some $\alpha \in \mathbb{F}^*$, $t$ is transcendental over $\mathbb{F}$ and $\mathrm{const}_\sigma \mathbb{F}(t) = \mathrm{const}_\sigma \mathbb{F}$. $\diamondsuit$

**Definition 2.2.5.** Let $(\mathbb{F}(t_1, \ldots, t_n), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$.
$(\mathbb{F}(t_1, \ldots, t_n), \sigma)$ is called $\Pi$-*extension* of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t_1, \ldots, t_i)(t_{i+1}), \sigma)$ is a $\Pi$-extension of $(\mathbb{F}(t_1, \ldots, t_i), \sigma)$ for all $0 \leq i \leq n-1$. Moreover, if $(\mathbb{H}, \sigma)$ is a sub-difference field of $(\mathbb{F}, \sigma)$ and all $t_i$ for $1 \leq i \leq n$ are hyperexponentials over $\mathbb{H}$, i.e. $\frac{\sigma(t_i)}{t_i} \in \mathbb{H}$, then $(\mathbb{F}(t_1, \ldots, t_n), \sigma)$ is called a $\Pi$-*extension* of $(\mathbb{F}, \sigma)$ *over* $\mathbb{H}$. $\diamondsuit$

**Definition 2.2.6.** The *homogeneous group* of the difference field $(\mathbb{F}, \sigma)$ is defined by

$$\mathrm{H}_{(\mathbb{F},\sigma)} := \left\{ \frac{\sigma(g)}{g} \,|\, g \in \mathbb{F}^* \right\}.$$

$\diamondsuit$

The following theorem was essentially stated in [Kar81, Theorem 2] and its proof is given in [Kar85, Theorem 2.2]. For the reason of completeness we will repeat the proof.

**Theorem 2.2.2.** *Let $(\mathbb{F}(t), \sigma)$ be a difference extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t$$

*for some $\alpha \in \mathbb{F}^*$ and*[8] *$t \neq 0$. Then $(\mathbb{F}(t), \sigma)$ is first order linear if and only if for all $n > 0$ we have*

$$\alpha^n \notin \mathrm{H}_{(\mathbb{F},\sigma)}.$$

*Proof.* "$\Rightarrow$" Suppose there is an $n > 0$ such that

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)},$$

i.e.

$$\sigma(w) = \alpha^n w$$

[8] In [Kar85] the Theorem 1 of [Kar81] was corrected by including the second condition $t \neq 0$.

for some $w \in \mathbb{F}^*$. By

$$\sigma(t^n) = \alpha^n t^n$$

we obtain

$$\sigma(\frac{t^n}{w}) = \frac{\sigma(t^n)}{\sigma(w)} = \frac{\alpha^n t^n}{\alpha^n w} = \frac{t^n}{w}$$

and thus

$$t^n/w \in \mathrm{const}_\sigma \mathbb{F}(t).$$

So either $\mathrm{const}_\sigma \mathbb{F}(t) \neq \mathrm{const}_\sigma \mathbb{F}$ or $t$ is algebraic over $\mathbb{F}$.
**"$\Leftarrow$" Suppose $t$ is algebraic over** $\mathbb{F}$, i.e. there is an irreducible

$$g(x) = \sum_{i=0}^{m} g_i x^i \in \mathbb{F}[x]$$

where $m \geq 1$, $g_m = 1$ and

$$g(t) = 0.$$

It follows that

$$0 = \sigma(g(t)) = \sum_{i=0}^{m} \sigma(g_i)\, \alpha^i\, t^i.$$

For

$$h(x) = \sum_{i=0}^{m} \sigma(g_i) \alpha^i\, x^i \in \mathbb{F}[x]$$

we have $h(t) = 0$ and thus

$$g|h.$$

Because of $\deg(g) = \deg(h)$ and $g_m = 1$ it follows that

$$\underbrace{\mathrm{lc}(h)}_{\alpha^m} g = h$$

and thus

$$\forall i : \alpha^m g_i = \sigma(g_i)\alpha^i.$$

If $g = x$, it follows that $t = 0$ which contradicts to the assumption. Otherwise, if $g \neq x$, there exists a $k$ with $k < m$ and $g_k \neq 0$ and thus

$$\frac{\sigma(g_k)}{g_k} = \alpha^{m-k}$$

which is equivalent to $\alpha^{m-k} \in \mathrm{H}_{(\mathbb{F},\sigma)}$.
**Now suppose that $t$ is transcendental over $\mathbb{F}$ but**

$$\mathrm{const}_\sigma \mathbb{F} \neq \mathrm{const}_\sigma \mathbb{F}(t).$$

Thus there is a $g \in \mathbb{F}(t) \setminus \mathbb{F}$ with $\sigma(g) = g$. Let

$$g = \frac{p}{q}$$

be in reduced representation. Because of

$$\frac{\sigma(p)}{\sigma(q)} = \frac{p}{q}$$

and $\gcd(\sigma(p), \sigma(q)) = \gcd(p, q) = 1$ it follows that

$$\frac{\sigma(p)}{p}, \frac{\sigma(q)}{q} \in \mathbb{F}. \tag{2.3}$$

Now either $p$ or $q$ has degree greater than 0, say $\deg(q) = m > 0$, i.e.

$$q = \sum_{i=0}^{m} q_i\, t^i, \quad q_m \neq 0.$$

By (2.3) there is a $u \in \mathbb{F}$ with

$$u\, q = \sigma(q) = \sum_{i=0}^{m} \sigma(q_i)\, \alpha^i\, t^i$$

and thus by coefficient comparison we get

$$u\, q_i = \sigma(q_i)\, \alpha^i$$

for all $0 \leq i \leq m$. If there is some $q_k \neq 0$ for $0 \leq k < m$ then it follows that

$$\frac{q_m}{q_k} = \frac{\sigma(q_m)}{\sigma(q_k)} \alpha^{m-k}.$$

Thus for $w := q_m/q_k \in \mathbb{F}$ we get

$$\frac{\sigma(w)}{w} = \alpha^{m-k}$$

where $m - k > 0$ and we are done. Otherwise, we have

$$q = q_m\, t^m.$$

As $\gcd(p, q) = 1$, it follows that the constant term of $p$ is not zero. If $\deg(p) > 0$ then we can argue as with $q$. Otherwise, we have to consider the remaining case $p \in \mathbb{F}$ and $q = q_m\, t^m$ with $m > 0$. We have

$$\frac{p}{q_m\, t^m} = \frac{p}{q} = g = \sigma(g) = \frac{\sigma(p)}{\sigma(q)} = \frac{\sigma(p)}{\sigma(q_m)\, \alpha^m\, t^m}$$

and thus

$$\frac{\sigma(p/q_m)}{p/q_m} = \alpha^m$$

and consequently $\frac{\sigma(w)}{w} = \alpha^m$ for $w := p/q_m \in \mathbb{F}$ with $m > 0$. $\square$

**Corollary 2.2.2.** *Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t$$

*where $\alpha \in \mathbb{F}^*$ and $t \neq 0$. Then $(\mathbb{F}(t), \sigma)$ is a* $\Pi$-extension *of $(\mathbb{F}, \sigma)$ if and only if there does not exist an $n > 0$ such that*

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

**Example 2.2.6.** Let $(\mathbb{Q}(t), \sigma)$ be the difference field from Example 2.2.1, this means $t$ is transcendental over $\mathbb{Q}$ and $\sigma$ is canonically defined by

$$\sigma(t) = 4\,t.$$

Since there does not exist an $n > 0$ such that

$$4^n \in \mathrm{H}_{(\mathbb{Q},\sigma)} = \{1\},$$

it follows by Corollary 2.2.2 that $(\mathbb{Q}(t), \sigma)$ is a Π-extension of $(\mathbb{Q}, \sigma)$; in particular this means that $\mathrm{const}_\sigma \mathbb{Q}(t) = \mathbb{Q}$ as already stated in Example 2.2.1. $\diamondsuit$

**Remark 2.2.3.** Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ canonically defined by

$$\sigma(t) = \alpha\,t$$

for some $\alpha \in \mathbb{F}^*$ and suppose that one can check if there exists an $n > 0$ such that

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}. \tag{2.4}$$

Then there exists an algorithm which decides if $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$, in particular, if $t$ is transcendental over $\mathbb{F}$ and the constant field is not extended. Please note that in Section 2.2.5 we will define so called ΠΣ-fields in which one can check if there exists an $n > 0$ with the property (2.4). $\diamondsuit$

### 2.2.4 $\Sigma$-extensions

The following theorem was first stated in [Kar81, Theorem 3] and its proof is essentially the same as in [Kar85].

**Theorem 2.2.3.** *Let $(\mathbb{F}(t),\sigma)$ be an inhomogeneous extension of $(\mathbb{F},\sigma)$ canonically defined by*

$$\sigma(t)=\alpha\,t+\beta$$

*with $\alpha\in\mathbb{F}^*$, $\beta\in\mathbb{F}$ and*[9] *$t\notin\mathbb{F}$. Then $(\mathbb{F}(t),\sigma)$ is first order linear.*

*Proof.* Suppose the constant field is extended. Then there exists a $g\in\mathbb{F}(t)\setminus\mathbb{F}$ with $\frac{\sigma(g)}{g}=1$ and thus by Theorem 2.2.1 the extension is homogeneous, contrary to the assumption. Otherwise, suppose $t$ is algebraic over $\mathbb{F}$, i.e. there is an irreducible polynomial

$$g(x)=\sum_{i=0}^{m}g_ix^i\in\mathbb{F}[x]$$

where $m\geq 1$, $g_m=1$ and

$$g(t)=0.$$

It follows that

$$0=\sigma(g(t))=\sum_{i=0}^{m}\sigma(g_i)(\alpha\,t+\beta)^i.$$

For

$$h(x):=\sum_{i=0}^{m}\sigma(g_i)(\alpha\,x+\beta)^i\in\mathbb{F}[x]$$

we have $h(t)=0$ and thus

$$g\mid h.$$

Because of $\deg(g)=\deg(h)$ and $g_m=1$ it follows that

$$\underbrace{\mathrm{lc}(h)}_{\alpha^m}g=h.$$

We have

$$h=\sum_{i=0}^{m}\sigma(g_i)\,(\alpha\,x+\beta)^i=\alpha^m\,g=\alpha^m\sum_{i=0}^{m}g_i\,x^i$$

and thus matching coefficients at degree $m-1$ we get

$$\alpha^{m-1}\,\sigma(g_{m-1})+m\,\alpha^{m-1}\,\beta=\alpha^m\,g_{m-1}$$

which is equivalent to

$$\sigma(\frac{g_{m-1}}{m})=\alpha\,\frac{g_{m-1}}{m}+\beta.$$

Since $t\notin\mathbb{F}$ by assumption, the extension is homogeneous by Theorem 2.2.1, a contradiction. □

[9] Please note that we corrected Karr's Theorem by adding the condition $t\notin\mathbb{F}$.

**Definition 2.2.7.** Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with

$$\sigma(t) = \alpha\, t + \beta$$

where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. $(\mathbb{F}(t), \sigma)$ is a Σ-*extension* of $(\mathbb{F}, \sigma)$ if

1. $(\mathbb{F}(t), \sigma)$ is an inhomogeneous extension of $(\mathbb{F}, \sigma)$ with $t \notin \mathbb{F}$
2. and for all $n \in \mathbb{Z}^*$ we have

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)} \Rightarrow \alpha \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

◊

**Remark 2.2.4.** Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$. Then $(\mathbb{F}(t), \sigma)$ is canonically defined by

$$\sigma(t) = \alpha\, t + \beta$$

for some $\alpha, \beta \in \mathbb{F}^*$, $t$ is transcendental over $\mathbb{F}$ and $\mathrm{const}_\sigma\mathbb{F}(t) = \mathrm{const}_\sigma\mathbb{F}$. ◊

The first condition of the definition of a Σ-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ can be easily checked by the following proposition, if one can solve first order difference equations in the difference field $(\mathbb{F}, \sigma)$.

**Lemma 2.2.2.** *Let $(\mathbb{F}(t), \sigma)$ be difference field extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t + \beta$$

*for some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Then this extension is inhomogeneous and $t \notin \mathbb{F}$ if and only if there does not exist a $g \in \mathbb{F}$ with*

$$\sigma(g) = \alpha\, g + \beta.$$

*Proof.* By Theorem 2.2.1 the extension is inhomogeneous if and only if there does not exist a $g \in \mathbb{F}$ with

$$\sigma(g) = \alpha\, g + \beta$$

or $t \in \mathbb{F}$. Therefore, if the extension is inhomogeneous and $t \notin \mathbb{F}$ then there does not exist a $g \in \mathbb{F}$ with $\sigma(g) = \alpha\, g + \beta$. Reversely, assume there does not exist a $g \in \mathbb{F}$ with $\sigma(g) = \alpha\, g + \beta$. Then clearly the extension is inhomogeneous. But of course, we cannot have $t \notin \mathbb{F}$, since otherwise take $g := t$. □

**Corollary 2.2.3.** *Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t + \beta$$

*where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Then $(\mathbb{F}(t), \sigma)$ is a* Σ-extension *of $(\mathbb{F}, \sigma)$ if and only if*

1. *there does not exist a $g \in \mathbb{F}$ with*

$$\sigma(g) - \alpha\, g = \beta$$

   *and*

2. *for all $n \in \mathbb{Z}^*$ we have*

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)} \Rightarrow \alpha \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

**Remark 2.2.5.** The second condition in Definition 2.2.7 or Corollary 2.2.3 will not be properly motivated in this thesis. Loosely speaking, it is a technical condition which is necessary to guarantee some properties which are needed to compute the so called $\sigma$-factorization which is defined in [Kar81]. This $\sigma$-factorization[10] is an essential step to compute the denominator bounding in Section 3.5.3; in addition, we need $\sigma$-factorization to decide if there exists an $n > 0$ with the property (2.4) which is necessary to decide if an extension is a $\Pi$-extension.

The algorithmically aspects how one can compute the $\sigma$-factorization are considered in details in [Kar81, Kar85]. Although $\sigma$-factorization plays a fundamental role in the summation theory under construction, for this thesis I decided to focus more on new aspects how one can solve linear difference equations of any order in so called $\Pi\Sigma$-fields (see the next section). $\diamondsuit$

**Definition 2.2.8.** Let $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$.

$(\mathbb{F}(t_1,\dots,t_n),\sigma)$ is called $\Sigma$- *extension* of $(\mathbb{F},\sigma)$, if $(\mathbb{F}(t_1,\dots,t_i)(t_{i+1}),\sigma)$ is a $\Sigma$-extension of $(\mathbb{F}(t_1,\dots,t_i),\sigma)$ for all $0 \le i \le n-1$. $\diamondsuit$

The main goal in this thesis is to deal with symbolic summation in difference fields. Therefore we will mainly focus on $\Pi$-extensions with which we can describe products and on those $\Sigma$-extensions with which we can describe sums. More precisely, we will consider in more details those $\Sigma$-extensions which are canonically defined by $\sigma(t) = t + \beta$. Since this is an important subclass of $\Sigma$-extensions, we introduce the following definition.

**Definition 2.2.9.** Let $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$.

$(\mathbb{F}(t_1,\dots,t_n),\sigma)$ is called a *proper sum extension* of $(\mathbb{F},\sigma)$, if $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ is a sum and $\Sigma$-extension of $(\mathbb{F},\sigma)$. Moreover, if $(\mathbb{H},\sigma)$ is a sub-difference field of $(\mathbb{F},\sigma)$ and all $t_i$ for $1 \le i \le n$ are sums over $\mathbb{H}$, i.e. $\sigma(t_i) - t_i \in \mathbb{H}$, then $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ is called a *proper sum extension* of $(\mathbb{F},\sigma)$ *over* $\mathbb{H}$. $\diamondsuit$

**Corollary 2.2.4.** *Let $(\mathbb{F}(t),\sigma)$ be a sum extension of $(\mathbb{F},\sigma)$, i.e. $(\mathbb{F}(t),\sigma)$ is canonically defined by*

$$\sigma(t) = t + \beta$$

*for some $\beta \in \mathbb{F}$. Then $(\mathbb{F}(t),\sigma)$ is a proper sum extension, if and only if there does not exist a $g \in \mathbb{F}$ with*

$$\sigma(g) - g = \beta.$$

**Example 2.2.7.** Let $(\mathbb{Q}(n),\sigma)$ be the difference field from Example 2.2.2, this means $n$ is transcendental over $\mathbb{Q}$ and $\sigma$ is canonically defined by

$$\sigma(n) = n + 1.$$

Since there does not exist a $g \in \mathbb{Q}$ such that

$$\sigma(g) - g = 1,$$

it follows by Corollary 2.2.4 that $(\mathbb{Q}(n),\sigma)$ is a proper sum extension of $(\mathbb{Q},\sigma)$; in particular this means that $\mathrm{const}_\sigma\mathbb{Q}(n) = \mathbb{Q}$ as stated already in Example 2.2.2. $\diamondsuit$

[10] Here P. Paule's greatest factorial factorization introduced in [Pau95] may play an important role to avoid factorizations.

### 2.2.5 ΠΣ-Extensions and ΠΣ-Fields

**Definition 2.2.10.** Let $(\mathbb{F}(t_1,\ldots,t_n),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$.

$(\mathbb{F}(t_1,\ldots,t_n),\sigma)$ is called ΠΣ-*extension* of $(\mathbb{F},\sigma)$ if $(\mathbb{F}(t_1,\ldots,t_i),\sigma)$ is either a Π- or a Σ-extension of $(\mathbb{F}(t_1,\ldots,t_{i-1}),\sigma)$ for all $1 \le i \le n$.

$(\mathbb{F}(t_1,\ldots,t_n),\sigma)$ is called *reduced product-sum extension* of $(\mathbb{F},\sigma)$, if $(\mathbb{F}(t_1,\ldots,t_n),\sigma)$ is a product-sum extension and a ΠΣ-extension of $(\mathbb{F},\sigma)$.

$(\mathbb{F}(t_1,\ldots,t_n),\sigma)$ is called *reduced d'Alembertian extension* of $(\mathbb{F},\sigma)$, if $(\mathbb{F}(t_1,\ldots,t_n,\sigma)$ is a d'Alembertian extension and a ΠΣ-extension of $(\mathbb{F},\sigma)$.

Furthermore $(\mathbb{F},\sigma)$ is called a ΠΣ-*field* over $\mathbb{K}$, if $\mathbb{F}=\mathbb{K}(t_1,\ldots,t_n)$ and $(\mathbb{K}(t_1,\ldots,t_n),\sigma)$ is a ΠΣ-extension of $(\mathbb{K},\sigma)$ with $\mathrm{const}_\sigma\mathbb{F}=\mathbb{K}$.

$\Diamond$

Let $(\mathbb{F},\sigma)=(\mathbb{K}(t_1,\ldots,t_n),\sigma)$ be a ΠΣ-field over the constant field $\mathbb{K}$.

PROBLEMS:

1. GIVEN $\alpha\in\mathbb{F}^*$

   FIND all $n$:
   $$\boxed{\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}}$$

2. GIVEN $a_1,a_2,f_1,\ldots,f_n\in\mathbb{F}$ with $a_1\neq 0\neq a_2$

   FIND all $c_1,\ldots,c_n\in\mathbb{K}$, $g\in\mathbb{F}$:
   $$\boxed{a_1\sigma(g)+a_2 g = c_1 f_1+\cdots+c_n f_n}$$

These problems are COMPUTABLE, if

- for any $k\in\mathbb{K}$ one can decide, if $k\in\mathbb{Z}$,
- polynomials in $\mathbb{K}[t_1,\ldots,t_n]$ may be factored over $\mathbb{K}$ and
- there exists an algorithm which finds for all $(c_1,\ldots,c_k)\in\mathbb{K}^k$ all $(n_1,\ldots,n_k)\in\mathbb{Z}^k$ such that
  $$c_1^{n_1}\cdots c_k^{n_k}=1.$$

**Implementation Note 2.2.1.** Let $\mathbb{K} := \mathbb{Q}(n_1, \dots, n_r)$ be a field of rational functions and let $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{K}(t_1, \dots, t_l)$ be a $\Pi\Sigma$-field over the constant field $\mathbb{K}$ canonically defined by

$$\sigma(t_i) = \alpha_i\, t_i + \beta_i, \quad \alpha_i \in \mathbb{K}(t_1, \dots, t_{i-1})^*,\ \beta_i \in \mathbb{K}(t_1, \dots, t_{i-1})$$

for $1 \leq i \leq l$. Let $\alpha, \beta \in \mathbb{F}$. Then the function call

$$\texttt{CheckSigmaExtension}[\{\texttt{t}_{\texttt{l+1}}, \alpha_{\texttt{l+1}}, \beta_{\texttt{l+1}}\}, \{\{\texttt{t}_1, \alpha_1, \beta_1\}, \dots, \{\texttt{t}_{\texttt{l}}, \alpha_{\texttt{l}}, \beta_{\texttt{l}}\}\}]$$

checks if one can construct a $\Sigma$-extension $(\mathbb{F}(t_{l+1}), \sigma)$ canonically defined by

$$\sigma(t_{l+1}) = \alpha_{l+1}\, t_{l+1} + \beta_{l+1}.$$

In particular in this function call Corollary 2.2.3 is applied and one checks if

1. there does not exist a $g \in \mathbb{F}$ with
$$\sigma(g) - \alpha_{l+1}\, g = \beta_{l+1}$$
and
2. for all $n \in \mathbb{Z}^*$ we have
$$\alpha_{l+1}^n \in \mathrm{H}_{(\mathbb{F},\sigma)} \Rightarrow \alpha_{l+1} \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

Similarly the function call

$$\texttt{CheckPiExtension}[\{\texttt{t}_{\texttt{l+1}}, \alpha_{\texttt{l+1}}, \texttt{0}\}, \{\{\texttt{t}_1, \alpha_1, \beta_1\}, \dots, \{\texttt{t}_{\texttt{l}}, \alpha_{\texttt{l}}, \beta_{\texttt{l}}\}\}]$$

checks if one can construct a $\Pi$-extension $(\mathbb{F}(t_{l+1}), \sigma)$ canonically defined by

$$\sigma(t_{l+1}) = \alpha_{l+1}\, t_{l+1}.$$

In particular in this function call Corollary 2.2.2 is applied and one checks if there does not exist an $n > 0$ such that

$$\alpha_{l+1}^n \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

### 2.2.6 The Period in a ΠΣ-Extension

**Definition 2.2.11.** Let $(\mathbb{E},\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$. The *period* of $f\in\mathbb{E}^*$ is defined by

$$\mathrm{per}_{(\mathbb{F},\sigma)}(f) := \begin{cases} 0 & \text{if } \forall p>0: \frac{\sigma^p(f)}{f}\notin\mathbb{F} \\ \min\{p>0 \,|\, \sigma^p(f)/f\in\mathbb{F}\} & \text{otherwise.}\end{cases}$$

◊

**Remark 2.2.6.** Let $(\mathbb{F},\sigma)$ be a difference field. We have

$$\forall f\in\mathbb{F}^*:\ \mathrm{per}_{(\mathbb{F},\sigma)}(f)=1.$$

◊

**Example 2.2.8.** Consider the Σ-extension $(\mathbb{Q}(n),\sigma)$ of $(\mathbb{Q},\sigma)$ canonically defined by

$$\sigma(n)=n+1.$$

Since for all $f\in\mathbb{Q}(n)\setminus\mathbb{Q}$ we have

$$\forall p>0:\ \frac{\sigma^p(f)}{f}\notin\mathbb{Q},$$

it follows that

$$\forall f\in\mathbb{Q}(n):\ \mathrm{per}_{(\mathbb{Q},\sigma)}(f)=0.$$

◊

**Example 2.2.9.** Consider the Π-extension $(\mathbb{Q}(t),\sigma)$ of $(\mathbb{Q},\sigma)$ canonically defined by

$$\sigma(t)=2\,t.$$

We have for all $i\in\mathbb{Z}$

$$\frac{\sigma(t^i)}{t^i}=2^i\in\mathbb{Q}$$

and consequently

$$\forall i\in\mathbb{Z}:\ \mathrm{per}_{(\mathbb{Q},\sigma)}(t^i)=1.$$

◊

The main goal of this chapter is to prove Theorem 2.2.4 which states that all elements in a difference field have either period 0 or 1, and characterizes the elements which have period 0 or 1. This Theorem is included in [Kar81, Theorem 4] and is essentially the same as [Kar85, Lemma 3.2]. I chose the proof given in [Bro00] which is quite a simplified version of Karr's proof.

**Definition 2.2.12.** Let $(\mathbb{A},\sigma)$ be a difference ring. The *σ-factorial* of $f\in\mathbb{A}$ is defined by

$$(f)_k := \prod_{i=0}^{k-1}\sigma^i(f)$$

for $k\in\mathbb{Z}$. ◊

**Lemma 2.2.3.** *Let $(\mathbb{F}(t)),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ and let $p\in\mathbb{F}[t]\setminus\mathbb{F}$ such that*

$$\frac{\sigma^n(p)}{p}\in\mathbb{F}$$

*for some $n>0$. Then for $g:=(p)_n$ we have*

$$\frac{\sigma(g)}{g}\in\mathbb{F}.$$

*Proof.* We have

$$\frac{\sigma(g)}{g}=\frac{\prod_{i=0}^{n-1}\sigma^{i+1}(p)}{\prod_{i=0}^{n-1}\sigma^i(p)}=\frac{\prod_{i=0}^{n-1}\sigma^i(p)}{\prod_{i=0}^{n-1}\sigma^i(p)}\frac{\sigma^n(p)}{p}=\frac{\sigma^n(p)}{p}\in\mathbb{F}.$$

□

The following Lemma and its proof is essentially the same as in [Bro00, Corollary 1].

**Lemma 2.2.4.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Sigma$-extension of $(\mathbb{F},\sigma)$. Then for all $f\in\mathbb{F}[t]\setminus\mathbb{F}$ we have*

$$\operatorname{per}_{(\mathbb{F},\sigma)}(f)=0$$

*and*

$$\left\{f\in\mathbb{F}[t]^*\,|\,\operatorname{per}_{(\mathbb{F},\sigma)}(f)=1\right\}=\mathbb{F}^*.$$

*Proof.* Assume there is an $f\in\mathbb{F}[t]\setminus\mathbb{F}$ with $\operatorname{per}_{(\mathbb{F},\sigma)}(f)>0$, i.e.

$$\frac{\sigma^p(f)}{f}\in\mathbb{F}$$

where $p>0$ is minimal. Then by Lemma 2.2.3 it follows that

$$\frac{\sigma((f)_p)}{(f)_p}\in\mathbb{F}.$$

By Theorem 2.2.1 it follows that $(\mathbb{F}(t),\sigma)$ is a homogeneous extension of $(\mathbb{F},\sigma)$ which is a contradiction. The statement

$$\left\{f\in\mathbb{F}[t]^*\,|\,\operatorname{per}_{(\mathbb{F},\sigma)}(f)=1\right\}=\mathbb{F}^*$$

follows immediately by Remark 2.2.6. □

The following Lemma and its proof is essentially the same as in [Bro00, Corollary 2].

**Lemma 2.2.5.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$. Then for all $f\in\mathbb{F}[t]^*$ we have*

$$\operatorname{per}_{(\mathbb{F},\sigma)}(f)\in\{0,1\}$$

*and*

$$\left\{f\in\mathbb{F}[t]^*\,|\,\operatorname{per}_{(\mathbb{F},\sigma)}(f)=1\right\}=\{h\,t^m\,|\,h\in\mathbb{F}^*\ \&\ m\geq 0\}.$$

*Proof.* Assume there is an $f \in \mathbb{F}[t] \setminus \mathbb{F}$ with $\mathrm{per}_{(\mathbb{F},\sigma)}(f) = 1$, i.e.

$$\frac{\sigma(f)}{f} \in \mathbb{F}.$$

For $g := f/\mathrm{lc}(f)$ we get

$$\frac{\sigma(g)}{g} = \frac{\sigma(f)}{f} \frac{\mathrm{lc}(f)}{\sigma(\mathrm{lc}(f))}$$

and thus $\frac{\sigma(g)}{g} \in \mathbb{F}$. Now let

$$g = t^m + \sum_{i=0}^{m-1} g_i\, t^i$$

where $g_i \in \mathbb{F}$ and consider

$$\sigma(g) = \alpha^m\, t^m + \sum_{i=0}^{m-1} \sigma(g_i)\, \alpha^i\, t^i = u\, g$$

where $u \in \mathbb{F}^*$. Then by matching coefficients we obtain

$$u = \alpha^m$$

and, by using this information,

$$\sigma(g_i)\, \alpha^i = \alpha^m\, g_i$$

for all $0 \leq i < m$. As $(\mathbb{F}(t), \sigma)$ is a Π-extension, there does not exist an $h \in \mathbb{F}^*$ with

$$\frac{\sigma(h)}{h} = \alpha^i$$

for any $i > 0$. Consequently $g_i = 0$ for all $0 \leq i < m$ and we get

$$f = v\, t^m$$

for some $v \in \mathbb{F}^*$. Therefore we have

$$\left\{ f \in \mathbb{F}[t]^* \,|\, \mathrm{per}_{(\mathbb{F},\sigma)}(f) = 1 \right\} \subseteq \{h\, t^m \,|\, h \in \mathbb{F}^* \;\&\; m \geq 0\}.$$

Conversely, we have

$$\frac{\sigma(v\, t^m)}{v\, t^m} = \frac{\alpha^m\, \sigma(v)}{v} \in \mathbb{F}$$

for any $m \geq 0$ and $v \in \mathbb{F}^*$ and therefore the sets are equal.

Finally, assume there is an $f \in \mathbb{F}[t] \setminus \mathbb{F}$ with $\mathrm{per}_{(\mathbb{F},\sigma)}(f) = p$, i.e.

$$\frac{\sigma^p(f)}{f} \in \mathbb{F}$$

with $p > 0$ minimal. Then by Lemma 2.2.3 we obtain

$$\frac{\sigma((f)_p)}{(f)_p} \in \mathbb{F}$$

and thus, from above, we have $(f)_p = t^m\, v$ for some $m \geq 0$ and $v \in \mathbb{F}$. As

$$f \mid (f)_p = t^m\, v$$

it follows that

$$f = t^n\, u$$

for some $n \geq 0$ and $u \in \mathbb{F}^*$ and thus $\mathrm{per}_{(\mathbb{F},\sigma)}(f) = 1$. □

The following theorem contains Lemmas 2.2.4 and 2.2.5 and extends them slightly from the polynomial case to the rational function case.

**Theorem 2.2.4.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Sigma$-extension of $(\mathbb{F},\sigma)$. Then for all $f \in \mathbb{F}(t)\setminus\mathbb{F}$ we have*

$$\operatorname{per}_{(\mathbb{F},\sigma)}(f) = 0$$

*and*

$$\left\{f \in \mathbb{F}(t)^* \,|\, \operatorname{per}_{(\mathbb{F},\sigma)}(f) = 1\right\} = \mathbb{F}^*.$$

*Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$. Then for all $f \in \mathbb{F}(t)^*$ we have*

$$\operatorname{per}_{(\mathbb{F},\sigma)}(f) \in \{0,1\}$$

*and*

$$\left\{f \in \mathbb{F}(t)^* \,|\, \operatorname{per}_{(\mathbb{F},\sigma)}(f) = 1\right\} = \{h\,t^m \,|\, h \in \mathbb{F}^* \;\&\; m \in \mathbb{Z}\}.$$

*Proof.* Let $f \in \mathbb{F}(t)^*$ with $f = \frac{p}{q}$ in reduced representation and assume

$$\frac{\sigma^n(f)}{f} \in \mathbb{F}$$

for some $n \in \mathbb{Z}$. Then

$$\frac{\sigma^n(p)}{\sigma^n(q)} = \sigma^n(\frac{p}{q}) = \frac{p}{q}\,u$$

for some $u \in \mathbb{F}$. Since $\gcd(p,q) = \gcd(\sigma^n(p),\sigma^n(q)) = 1$, it follows that

$$\begin{aligned}\sigma^n(p) &= p\,u_1,\\ \sigma^n(q) &= q\,u_2\end{aligned}$$

for some $u_1,u_2 \in \mathbb{F}$ with $u = u_1/u_2$. Thus the theorem follows immediately by the Lemmas 2.2.4 and 2.2.5. □

**Corollary 2.2.5.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $\alpha \in \mathbb{F}^*$ and $g \in \mathbb{F}(t)^*$ with*

$$\frac{\sigma(g)}{g} = \alpha.$$

*Then $g = w\,t^k$ where $w \in \mathbb{F}^*$ and*

$$k \in \begin{cases} \{0\} & \text{if } t \text{ is a } \Sigma\text{-extension} \\ \mathbb{Z} & \text{if } t \text{ is a } \Pi\text{-extension.}\end{cases}$$

**Corollary 2.2.6.** *Let $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ where for any $\Pi$-extension with*

$$\sigma(t_i) = \alpha_i\,t_i$$

*we have $\alpha_i \in \mathbb{F}^*$. Let $\alpha \in \mathbb{F}^*$ and suppose there is a $g \in \mathbb{F}(t_1,\dots,t_n)^*$ such that*

$$\frac{\sigma(g)}{g} = \alpha.$$

*Then $g = w\,t_1^{k_1}\cdots t_n^{k_n}$ where $w \in \mathbb{F}^*$ and*

$$k_i \in \begin{cases} \{0\} & \text{if } t_i \text{ is a } \Sigma\text{-extension} \\ \mathbb{Z} & \text{if } t_i \text{ is a } \Pi\text{-extension.}\end{cases}$$

*Proof.* For $n = 0$ nothing has to be proven. Now let $g \in \mathbb{F}(t_1, \dots, t_n)(t_{n+1})$ such that

$$\frac{\sigma(g)}{g} = \alpha.$$

Applying Corollary 2.2.5 we get

$$g = w\, t_{n+1}^{k_{n+1}}$$

where $w \in \mathbb{F}(t_1, \dots, t_n)$. If $t_{n+1}$ is a $\Sigma$-extension it follows that $k_{n+1} = 0$ and thus $g \in \mathbb{F}(t_1, \dots, t_n)$. Applying the induction assumption we get the result. If $t_{n+1}$ is a $\Pi$-extension we have

$$\alpha = \frac{\sigma(g)}{g} = \frac{\sigma(w)}{w}\, \alpha_{n+1}^{k_{n+1}}$$

and thus

$$\frac{\sigma(w)}{w} = \frac{\alpha}{\alpha_{n+1}^{k_{n+1}}} \in \mathbb{F}.$$

Consequently by the induction assumption we get $w = \tilde{w}\, t_1^{k_1} \cdots t_n^{k_n}$ where $\tilde{w} \in \mathbb{F}$ and $k_i = 0$ if $t_i$ is a $\Sigma$-extension or $k_i \in \mathbb{Z}$ if $t_i$ is a $\Pi$-extension. $\square$

### 2.2.7 The Spread in a ΠΣ-extension

The following definition of the spread function will play an essential role in Section 3.5.3 to bound parts of the denominator of solutions of a given difference equations.

**Definition 2.2.13.** Let $(\mathbb{A}[t], \sigma)$ be a difference ring with $t$ transcendental over $\mathbb{A}$ and $a, b \in \mathbb{A}[t]^*$. We define the *spread* of $a$ and $b$ w.r.t. $\sigma$ as

$$\operatorname{spread}_\sigma(a, b) = \{m \geq 0 \mid \deg(\gcd(a, \sigma^m(b))) > 0\}.$$

$\diamondsuit$

Given a $\Pi\Sigma$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$, the following theorem states when the set $\operatorname{spread}_\sigma(a, b)$ for $a, b \in \mathbb{F}[t]$ is finite.

**Theorem 2.2.5.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $a, b \in \mathbb{F}[t]^*$. Then $\operatorname{spread}_\sigma(a, b)$ is an infinite set if and only if $(\mathbb{F}(t), \sigma)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma)$ and $t \mid \gcd(a, b)$.*

*Proof.* By [Bro00, Theorem 6] it follows that $\operatorname{spread}_\sigma(a, b)$ is an infinite set if and only if $b$ has a nontrivial factor $p \in \mathbb{F}[t] \setminus \mathbb{F}$ with $\operatorname{per}_{(\mathbb{F},\sigma)}(p) \neq 0$ such that $\sigma^n(p) \mid a$ for some $n \geq 0$. By Theorem 2.2.4 this is possible if and only if $(\mathbb{F}(t), \sigma)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma)$ and $t \mid \gcd(a, b)$. $\square$

## 2.3 Some Difference Field Isomorphisms in Difference Fields

### 2.3.1 The Summand Isomorphism for Proper Sum Extensions

**Lemma 2.3.1.** *Let* $(\mathbb{F}(s), \sigma)$ *and* $(\mathbb{F}(t), \sigma)$ *be difference field extensions of* $(\mathbb{F}, \sigma)$ *with*

$$\sigma(s) = s + \beta,$$
$$\sigma(t) = t + \beta + \sigma(g) - g$$

*for some* $\beta, g \in \mathbb{F}$*. Then*

$$(\mathbb{F}(s), \sigma) \text{ is a proper sum extension of } (\mathbb{F}, \sigma)$$
$$\Updownarrow$$
$$(\mathbb{F}(t), \sigma) \text{ is a proper sum extension of } (\mathbb{F}, \sigma).$$

*Proof.* We have

$$\begin{gathered}
(\mathbb{F}(s), \sigma) \text{ is not a proper sum extension of } (\mathbb{F}, \sigma) \\
\Updownarrow \quad \text{Cor. 2.2.4} \\
\exists f \in \mathbb{F} : \sigma(f) - f = \beta \\
\Updownarrow \\
\exists h \in \mathbb{F} : \sigma(h - g) - (h - g) = \beta \\
\Updownarrow \\
\exists h \in \mathbb{F} : \sigma(h) - (h) = \beta + \sigma(g) - g \\
\Updownarrow \quad \text{Cor. 2.2.4} \\
(\mathbb{F}(t), \sigma) \text{ is not a proper sum extension of } (\mathbb{F}, \sigma).
\end{gathered}$$

□

**Proposition 2.3.1.** *Let* $(\mathbb{F}(s), \sigma)$ *and* $(\mathbb{F}(t), \sigma)$ *be sum extensions of* $(\mathbb{F}, \sigma)$ *with*

$$\sigma(s) = s + \beta,$$
$$\sigma(t) = t + \beta + \sigma(g) - g$$

*for some* $g, \beta \in \mathbb{F}$*. If* $(\mathbb{F}(s), \sigma)$ *is a proper sum extension of* $(\mathbb{F}, \sigma)$ *then also* $(\mathbb{F}(t), \sigma)$ *is a proper sum extension of* $(\mathbb{F}, \sigma)$ *and*

$$(\mathbb{F}(s), \sigma) \simeq (\mathbb{F}(t), \sigma).$$

*Proof.* By Lemma 2.3.1 it follows immediately that $(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$. Therefore $s$ and $t$ are transcendental over $\mathbb{F}$ and consequently

$$\tilde{\tau} : \mathbb{F}(s) \to \mathbb{F}(t)$$

canonically defined by

$$\tilde{\tau}(s) = t - g,$$
$$\tilde{\tau}(z) = \tau(z) \quad \forall z \in \mathbb{F}$$

is a field isomorphism. Furthermore we have

$$\begin{aligned}\tau\sigma(s) &= \tau(s+\beta) = t-g+\beta,\\ \sigma\tau(s) &= \sigma(t-g) = t+\beta+\sigma(g)-g-\sigma(g) = t-g+\beta\end{aligned}$$

and thus

$$\tau\sigma = \sigma\tau.$$

Consequently $\tau$ is a difference field isomorphism. □

### 2.3.2 The Indefinite Summation Isomorphism

The following proposition needs Corollary 4.1.2 which is one of the consequences of Chapter 3. Due to the completeness of this section I already state here the proposition.

**Proposition 2.3.2.** *Let $(\mathbb{F}(s),\sigma)$ be a proper sum extension of $(\mathbb{F},\sigma)$ and $(\mathbb{F}(t),\sigma)$ be a sum extension of $(\mathbb{F},\sigma)$. Assume there is a $g\in\mathbb{F}(s)\setminus\mathbb{F}$ such that*

$$\sigma(g)-g=\sigma(t)-t.$$

*Then $(\mathbb{F}(t),\sigma)$ is a proper sum extension of $(\mathbb{F},\sigma)$ and*

$$(\mathbb{F}(s),\sigma)\simeq(\mathbb{F}(t),\sigma).$$

*Proof.* Let

$$\sigma(g)-g=\sigma(t)-t=:\beta\in\mathbb{F}.$$

By Corollary 4.1.2, which arises naturally but in an other context, there are a $c\in\mathrm{const}_\sigma\mathbb{F}$ and a $w\in\mathbb{F}$ such that

$$g=c\,s+w.$$

Since $s$ is transcendental over $\mathbb{F}$, also $g$ is transcendental over $\mathbb{F}$ and therefore $\tau:\mathbb{F}(t)\to\mathbb{F}(g)$ canonically defined by

$$\tau(t)=g$$

is a field isomorphism. We have

$$\begin{aligned}\tau\sigma(t) &= \tau(t+\beta) = g+\beta,\\ \sigma\tau(t) &= \sigma(g) = g+\beta\end{aligned}$$

and thus $\tau$ is a difference field isomorphism. Since $\mathbb{F}(g)=\mathbb{F}(s)$, the proposition follows. □

### 2.3.3 A Recursively Induced Isomorphism for $\Pi\Sigma$-Fields

**Lemma 2.3.2.** *Let $(\mathbb{F},\sigma)$ and $(\mathbb{G},\sigma)$ be difference fields with*

$$(\mathbb{F},\sigma)\overset{\tau}{\simeq}(\mathbb{G},\sigma).$$

*Let $(\mathbb{F}(x),\sigma)$ and $(\mathbb{G}(y),\sigma)$ be difference field extensions of $(\mathbb{F},\sigma)$ and $(\mathbb{G},\sigma)$ with*

$$\begin{aligned}\sigma(x) &= \alpha\,x+\beta,\\ \sigma(y) &= \tau(\alpha)\,y+\tau(\beta).\end{aligned}$$

*Then*

$$\begin{gathered}(\mathbb{F}(x),\sigma)\text{ is a }\Pi\Sigma\text{-extension of }(\mathbb{F},\sigma)\\ \Updownarrow\\ (\mathbb{G}(y),\sigma)\text{ is a }\Pi\Sigma\text{-extension of }(\mathbb{G},\sigma).\end{gathered}$$

*Proof.* Let $a \in \mathbb{F}$ and $n \in \mathbb{Z}$. We have

$$\begin{aligned}
\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)} \quad &\Leftrightarrow \quad \exists g \in \mathbb{F}: \ \frac{\sigma(g)}{g} = \alpha^n \\
&\Leftrightarrow \quad \exists g \in \mathbb{F}: \ \sigma(g) - \alpha^n\, g = 0 \\
&\overset{Cor.\ 2.1.1}{\Leftrightarrow} \quad \exists g \in \mathbb{G}: \ \sigma(g) - \tau(\alpha^n)\, g = 0 \\
&\Leftrightarrow \quad \exists g \in \mathbb{G}: \ \frac{\sigma(g)}{g} = \tau(\alpha)^n \\
&\Leftrightarrow \quad \tau(\alpha)^n \in \mathrm{H}_{(\mathbb{G},\sigma)}.
\end{aligned}$$

Therefore looking at Corollaries 2.2.2 and 2.2.3 it follows immediately that $(\mathbb{F}(x), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ if and only if $(\mathbb{G}(y), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$. $\square$

**Proposition 2.3.3.** *Let $(\mathbb{F}, \sigma)$ and $(\mathbb{G}, \sigma)$ be difference fields with*

$$(\mathbb{F}, \sigma) \overset{\tau}{\simeq} (\mathbb{G}, \sigma).$$

*Let $(\mathbb{F}(x), \sigma)$ and $(\mathbb{G}(y), \sigma)$ be difference field extensions of $(\mathbb{F}, \sigma)$ and $(\mathbb{G}, \sigma)$ with*

$$\begin{aligned}
\sigma(x) &= \alpha\, x + \beta, \\
\sigma(y) &= \tau(\alpha)\, y + \tau(\beta).
\end{aligned}$$

*If $(\mathbb{F}(x), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ then also $(\mathbb{G}(y), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$ and*

$$(\mathbb{F}(x), \sigma) \simeq (\mathbb{G}(y), \sigma).$$

*Proof.* By Lemma 2.3.2, $(\mathbb{F}(x), \sigma)$ and $(\mathbb{G}(y), \sigma)$ are $\Pi\Sigma$-extensions of $(\mathbb{F}, \sigma)$ and $(\mathbb{G}, \sigma)$ and therefore $x$ is transcendental over $\mathbb{F}$ and $y$ is transcendental over $\mathbb{G}$. Thus

$$\tilde{\tau} : \mathbb{F}(x) \to \mathbb{G}(y)$$

canonically defined by

$$\begin{aligned}
\tilde{\tau}(x) &= y, \\
\tilde{\tau}(z) &= \tau(z) \quad \forall z \in \mathbb{F}
\end{aligned}$$

is a field isomorphism. Additionally we have

$$\begin{aligned}
\tilde{\tau}(\sigma(x)) &= \tilde{\tau}(\alpha\, x + \beta) = \tau(\alpha)\, y + \tau(\beta), \\
\sigma(\tilde{\tau}(x)) &= \sigma(y) = \tau(\alpha)\, y + \tau(\beta)
\end{aligned}$$

and thus

$$\tilde{\tau}\sigma = \sigma\tilde{\tau}.$$

Consequently

$$(\mathbb{F}(x), \sigma) \overset{\tilde{\tau}}{\simeq} (\mathbb{G}(y), \sigma).$$

$\square$

## 2.4 Construction of Difference Rings and Fields

### 2.4.1 Some Simple Constructions of Difference Fields

The following lemma restates what was already described in Example 2.1.7.

**Lemma 2.4.1.** *Let $(\mathbb{F},\sigma)$ be a difference field, $t$ be transcendental over $\mathbb{F}$ and $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$. Then there exists a difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t + \beta.$$

**Lemma 2.4.2.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ canonically defined by $\sigma(t) = \alpha\, t+\beta$ $(\alpha \in \mathbb{F}^*, \beta \in \mathbb{F})$ and assume*

$$(\mathbb{F},\sigma) \overset{\tau}{\simeq} (\mathbb{G},\tilde{\sigma})$$

*for some difference field $(\mathbb{G},\tilde{\sigma})$. Then there is a $\Pi\Sigma$-extension $(\mathbb{G}(x),\tilde{\sigma})$ canonically defined by $\tilde{\sigma}(x) = \tau(\alpha)\, x + \tau(\beta)$ and*

$$(\mathbb{F}(t),\sigma) \overset{\tau}{\simeq} (\mathbb{G}(x),\tilde{\sigma}).$$

*Proof.* Let $x$ be transcendental over $\mathbb{G}$. Then by Lemma 2.4.1 there is a difference field extension $(\mathbb{G}(x),\tilde{\sigma})$ of $(\mathbb{F},\sigma)$ canonically defined by

$$\tilde{\sigma}(x) = \tau(\alpha)\, x + \tau(\beta).$$

The lemma follows immediately by Proposition 2.3.3. □

### 2.4.2 Construction of Sum Extensions without Changing the Constant Field

**Proposition 2.4.1.** *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and $\beta \in \mathbb{F}$. If there does not exist a $g \in \mathbb{F}$ with*

$$\sigma(g) - g = \beta$$

*then there is - up to a difference field isomorphism - a unique difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ canonically defined by*

$$\sigma(t) = t + \beta.$$

*Moreover, $(\mathbb{F}(t),\sigma)$ is a proper sum extension of $(\mathbb{F},\sigma)$ and $\mathrm{const}_\sigma\mathbb{F}(t) = \mathbb{K}$.*

*Proof.* By Lemma 2.4.1 we can construct a difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ where $t$ is transcendental over $\mathbb{F}$ and

$$\sigma(t) = t + \beta.$$

Since there does not exist a $g \in \mathbb{F}$ such that

$$\sigma(g) - g = \beta,$$

$(\mathbb{F}(t),\sigma)$ is a proper sum extension of $(\mathbb{F},\sigma)$ by Corollary 2.2.4 and therefore $\mathrm{const}_\sigma\mathbb{F}(t) = \mathrm{const}_\sigma\mathbb{F}$. If there exists an other difference field extension $(\mathbb{F}(x),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(x) = x+\beta$ then by Proposition 2.3.3 they are isomorphic. □

**Corollary 2.4.1.** *Let* $(\mathbb{F}, \sigma)$ *be a difference field with constant field* $\mathbb{K}$ *and* $\beta \in \mathbb{F}$*. Then there is - up to a difference field isomorphism - a unique difference field extension* $(\mathbb{F}(t), \sigma)$ *of* $(\mathbb{F}, \sigma)$ *with*

$$\sigma(t) = t + \beta$$

*and* $\mathrm{const}_\sigma \mathbb{F}(t) = \mathbb{K}$.

*Proof.* If there does not exist a $g \in \mathbb{F}$ with

$$\sigma(g) = g + \beta$$

then the corollary follows by Proposition 2.4.1. Otherwise, if there exists such a $g \in \mathbb{F}$ then clearly for $t := g$ we have $(\mathbb{F}(t), \sigma) = (\mathbb{F}, \sigma)$ and $\mathrm{const}_\sigma \mathbb{F}(t) = \mathbb{K}$. If there exists an other difference field extension $(\mathbb{F}(x), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(x) = x + \beta$ and $\mathrm{const}_\sigma \mathbb{F}(x) = \mathbb{K}$ then

$$\sigma(t - x) = (t - x)$$

and therefore $t - x \in \mathbb{K}$. Thus $\mathbb{F}(x) = \mathbb{F} = \mathbb{F}(t)$. □

### 2.4.3 Construction of Product Extensions without Changing the Constant Field

**Proposition 2.4.2.** *Let* $(\mathbb{F}, \sigma)$ *be a difference field and* $\alpha \in \mathbb{F}^*$*. Assume there does not exist an* $n > 0$ *such that*

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

*Then there is - up to a difference field isomorphism - a unique difference field extension* $(\mathbb{F}(t), \sigma)$ *with*

$$\sigma(t) = \alpha\, t.$$

*Moreover,* $(\mathbb{F}(t), \sigma)$ *is a* $\Pi$*-extension of* $(\mathbb{F}, \sigma)$ *and* $\mathrm{const}_\sigma \mathbb{F}(t) = \mathbb{K}$.

*Proof.* By Lemma 2.4.1 we can construct a difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ where $t$ is transcendental over $\mathbb{F}$ and

$$\sigma(t) = \alpha\, t.$$

If there does not exist an $n > 0$ with $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$ then by Corollary 2.2.2 it follows that $(\mathbb{F}(t), \sigma)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma)$. Assume there is an other difference field extension of $(\mathbb{F}(x), \sigma)$ with $\sigma(x) = \alpha\, x$ then by Proposition 2.3.3 they are isomorphic. □

**Corollary 2.4.2.** *Let* $(\mathbb{F}, \sigma)$ *be a difference field and* $\alpha \in \mathbb{F}^*$*. Assume there does not exist an* $n > 1$ *such that*

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

*Then there is - up to a difference field isomorphism - a unique difference field extension* $(\mathbb{F}(t), \sigma)$ *with*

$$\sigma(t) = \alpha\, t$$

*and* $\mathrm{const}_\sigma \mathbb{F}(t) = \mathbb{K}$.

*Proof.* If there does not exist an $n \geq 0$ with $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$ then the corollary follows by Proposition 2.4.2. If $\alpha \in \mathrm{H}_{(\mathbb{F},\sigma)}$ then there exists a $g \in \mathbb{F}$ with

$$\frac{\sigma(g)}{g} = \alpha.$$

Then for $t := g$ we have $(\mathbb{F}(t), \sigma) = (\mathbb{F}, \sigma)$ and $\mathrm{const}_\sigma \mathbb{F}(t) = \mathbb{K}$. If there exists an other difference field extension $(\mathbb{F}(x), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(x) = \alpha\, x$ and $\mathrm{const}_\sigma \mathbb{F}(x) = \mathbb{K}$ then

$$\sigma(\frac{t}{x}) = \frac{t}{x}$$

and therefore $t/x \in \mathbb{K}$. Thus $\mathbb{F}(x) = \mathbb{F} = \mathbb{F}(t)$. □

### 2.4.4 A Proper Sum Representation of a Sum Extension

**Proposition 2.4.3.** *Let $(\mathbb{E}, \sigma)$ be a sum extension of $(\mathbb{F}, \sigma)$ with $\mathrm{const}_\sigma \mathbb{E} = \mathrm{const}_\sigma \mathbb{F}$. Then there is a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with*

$$(\mathbb{G}, \sigma) \simeq (\mathbb{E}, \sigma).$$

*Proof.* Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}(s_1, \dots, s_n)$ be a sum extension of $(\mathbb{F}, \sigma)$. We will do the proof by induction on the number $n$ of sum extensions. For $n = 0$ nothing has to be proven. Now assume that the statement holds for $n \geq 0$ sum extensions and let $(\mathbb{E}(s), \sigma)$ with $\mathbb{E} = \mathbb{F}(s_1, \dots, s_n)$ be a sum extension of $(\mathbb{E}, \sigma)$ canonically defined by

$$\sigma(s) = s + \beta$$

and $\mathrm{const}_\sigma \mathbb{E}(s) = \mathrm{const}_\sigma \mathbb{F}$. By the induction assumption there is a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$(\mathbb{G}, \sigma) \overset{\tau}{\simeq} (\mathbb{E}, \sigma).$$

Assume there is a $g \in \mathbb{E}$ such that

$$\sigma(g) - g = \beta. \tag{2.5}$$

Then we obtain

$$\sigma(g - s) = g - s$$

and thus there is a $k \in \mathrm{const}_\sigma \mathbb{F}$ with

$$s = g + k.$$

Therefore

$$(\mathbb{E}(s), \sigma) = (\mathbb{E}, \sigma) \simeq (\mathbb{G}, \sigma).$$

Now suppose there does not exist such a $g \in \mathbb{E}(s)$ with (2.5). Then $(\mathbb{E}(s), \sigma)$ is a proper sum extension of $(\mathbb{E}, \sigma)$. By Lemma 2.4.1 we can construct a difference field extension $(\mathbb{G}(t), \sigma)$ of $(\mathbb{G}, \sigma)$ with

$$\sigma(t) = t + \tau(\beta).$$

By Proposition 2.3.3 $(\mathbb{G}(t), \sigma)$ is a proper sum extension of $(\mathbb{G}, \sigma)$ with

$$(\mathbb{E}(s), \sigma) \simeq (\mathbb{G}(t), \sigma).$$

□

### 2.4.5 Embeddings of Proper Sum Extensions in a Reduced Product-Sum Extension

**Lemma 2.4.3.** *Let $(\mathbb{F}(t_1)(t_2), \sigma)$ be a reduced product-sum extension*[11] *of $(\mathbb{F}, \sigma)$. Then it follows that $(\mathbb{F}(t_2)(t_1), \sigma)$ is a reduced product-sum extension of $(\mathbb{F}, \sigma)$ and*

$$(\mathbb{F}(t_1)(t_2), \sigma) \simeq (\mathbb{F}(t_2)(t_1), \sigma).$$

*Proof.* Let $(\mathbb{F}(t_1)(t_2), \sigma)$ be a reduced product-sum extension of $(\mathbb{F}, \sigma)$. Since $(\mathbb{F}(t_2), \sigma)$ is a sub-difference field of $(\mathbb{F}(t_1)(t_2), \sigma)$, it follows immediately that $(\mathbb{F}(t_2), \sigma)$ is a reduced product-sum extension of $(\mathbb{F}, \sigma)$. Since $\mathbb{F}(t_1, t_2)$ is the field of rational functions with coefficients in $\mathbb{F}$, $t_1$ is transcendental over $\mathbb{F}(t_2)$ and therefore $(\mathbb{F}(t_2)(t_1), \sigma)$ is a product-sum extension of $(\mathbb{F}, \sigma)$. Since additionally we have

$$\mathrm{const}_\sigma \mathbb{F}(t_1)(t_2) = \mathrm{const}_\sigma \mathbb{F}(t_2)(t_1)$$

it follows immediately that $(\mathbb{F}(t_2)(t_1), \sigma)$ is first order linear and thus a reduced product-sum extension of $(\mathbb{F}(t_1), \sigma)$. Additionally, there is the following trivial difference field isomorphism $\tau : \mathbb{F}(t_1)(t_2) \to \mathbb{F}(t_2)(t_1)$ canonically defined by

$$\tau(t_1) = t_1, \qquad \tau(t_2) = t_2.$$

□

**Proposition 2.4.4.** *Let $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ be a reduced product-sum extension of $(\mathbb{F}, \sigma)$ and $(\mathbb{F}(t_1, \dots, t_n)(x), \sigma)$ be a proper sum extension of $(\mathbb{F}(t_1, \dots, t_n), \sigma)$. Then $(\mathbb{F}(x)(t_1, \dots, t_n), \sigma)$ is a reduced product-sum extension of $(\mathbb{F}, \sigma)$ and*

$$(\mathbb{F}(x)(t_1, \dots, t_n), \sigma) \simeq (\mathbb{F}(t_1, \dots, t_n)(x), \sigma).$$

*Proof.* We will prove the theorem by induction on $n$. Clearly the induction base $n = 0$ holds. Let $(\mathbb{F}(t_1, \dots, t_{n+1})(x), \sigma)$ be a proper sum extension of $(\mathbb{F}(t_1, \dots, t_{n+1}), \sigma)$ being a reduced product-sum extension of $(\mathbb{F}, \sigma)$. By Lemma 2.4.3 it follows that $(\mathbb{F}(t_1, \dots, t_n)(x)(t_{n+1}), \sigma)$ is a reduced product-sum extension of $(\mathbb{F}, \sigma)$ and

$$(\mathbb{F}(t_1, \dots, t_n)(t_{n+1})(x), \sigma) \simeq (\mathbb{F}(t_1, \dots, t_n)(x)(t_{n+1}), \sigma).$$

By the induction hypothesis we may assume that $(\mathbb{F}(x)(t_1, \dots, t_n), \sigma)$ is a reduced product-sum extension of $(\mathbb{F}, \sigma)$ and

$$(\mathbb{F}(x)(t_1, \dots, t_n), \sigma) \simeq (\mathbb{F}(t_1, \dots, t_n)(x), \sigma).$$

Thus by Proposition 2.3.3 and the fact that $(\mathbb{F}(t_1, \dots, t_n)(x)(t_{n+1}), \sigma)$ is a reduced product-sum extension of $(\mathbb{F}, \sigma)$ we conclude that $(\mathbb{F}(x)(t_1, \dots, t_n)(t_{n+1}), \sigma)$ is a reduced product sum extension of $(\mathbb{F}, \sigma)$ and

$$(\mathbb{F}(x)(t_1, \dots, t_n)(t_{n+1}), \sigma) \simeq (\mathbb{F}(t_1, \dots, t_n)(x)(t_{n+1}), \sigma).$$

Finally, we get that $(\mathbb{F}(x)(t_1, \dots, t_{n+1}), \sigma)$ is a reduced product-sum extension of $(\mathbb{F}, \sigma)$ and

$$(\mathbb{F}(x)(t_1, \dots, t_{n+1}), \sigma) \simeq (\mathbb{F}(t_1, \dots, t_{n+1})(x), \sigma).$$

□

[11] See Definition 2.2.10 on page 71.

**Proposition 2.4.5.** *Let $(\mathbb{F}(x),\sigma)$ be a proper sum extension of $(\mathbb{F},\sigma)$ and $(\mathbb{F}(x)(t_1,\dots,t_n),\sigma)$ be a reduced product-sum extension of $(\mathbb{F}(x),\sigma)$. Then $(\mathbb{F}(t_1,\dots,t_n)(x),\sigma)$ is a reduced product-sum extension of $(\mathbb{F},\sigma)$ and*

$$(\mathbb{F}(x)(t_1,\dots,t_n),\sigma)\simeq(\mathbb{F}(t_1,\dots,t_n)(x),\sigma).$$

*Proof.* The proof is similar to the proof of Proposition 2.4.4. □

**Corollary 2.4.3.** *Let $(\mathbb{E},\sigma)$ be a reduced d'Alembertian extension*[12] *of $(\mathbb{F},\sigma)$. Then there is a reduced d'Alembertian extension $(\mathbb{F}(h_1,\dots,h_m)(s_1,\dots,s_n),\sigma)$ of $(\mathbb{F},\sigma)$ in which*

1. *$h_i$ is hyperexponential over $\mathbb{F}(h_1,\dots,h_{i-1})$ for $1\le i\le m$*
2. *and $s_i$ is a sum over $\mathbb{F}(h_1,\dots,h_m)(s_1,\dots,s_{i-1})$ for $1\le i\le n$*

*such that*

$$(\mathbb{E},\sigma)\simeq(\mathbb{F}(h_1,\dots,h_m)(s_1,\dots,s_n),\sigma)$$

*Proof.* Let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1,\dots,t_l)$ be a reduced d'Alembertian extension of $(\mathbb{F},\sigma)$. We will prove the corollary by induction on the number of extensions $t_i$. For the case $l=0$ nothing has to be proven. By the induction assumption there exists a difference field extension $\mathbb{F}(h_1,\dots,h_k)(s_1,\dots,s_{l-k})$ of $(\mathbb{F},\sigma)$ as stated above with

$$(\mathbb{E},\sigma)\overset{\tau}{\simeq}(\underbrace{\mathbb{F}(h_1,\dots,h_k)(s_1,\dots,s_{l-k})}_{=:\mathbb{G}},\sigma).$$

Let $(\mathbb{E}(t_{l+1}),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{E},\sigma)$ where $t_{l+1}$ is hyperexponential over $\mathbb{F}$ or a sum over $\mathbb{E}$ with

$$\sigma(t_{l+1})=\alpha\, t_{l+1}+\beta.$$

By Lemma 2.4.1 we construct a difference field extension $(\mathbb{G}(x),\sigma)$ of $(\mathbb{G},\sigma)$ with

$$\sigma(x)=\tau(\alpha)\,x+\tau(\beta).$$

Then by Proposition 2.3.3 $(\mathbb{G}(x),\sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and

$$(\mathbb{E}(t),\sigma)\simeq(\mathbb{G}(x),\sigma).$$

If $t_{l+1}$ is a sum over $\mathbb{E}$, $x$ is also a sum over $\mathbb{F}$ and thus the induction hypothesis is proven. Otherwise $x$ is hyperexponential over $\mathbb{G}$. Then by Proposition 2.4.4 it follows that

$$(\mathbb{F}(h_1,\dots,h_k)(s_1,\dots,s_{l-k})(x),\sigma)\simeq(\mathbb{F}(h_1,\dots,h_k)(x)(s_1,\dots,s_{l-k}),\sigma)$$

and consequently the induction hypothesis is proven. □

[12] See Definition 2.2.10 on page 71.

### 2.4.6 Changing the Order of Sum Extensions

**Corollary 2.4.4.** *Let* $(\mathbb{F}, \sigma)$ *and* $(\mathbb{G}, \sigma)$ *be difference fields with*

$$(\mathbb{F}, \sigma) \overset{\tau}{\simeq} (\mathbb{G}, \sigma)$$

*and constant field* $\mathbb{K}$*. Let* $(\mathbb{F}(x), \sigma)$ *and* $(\mathbb{G}(y), \sigma)$ *be difference field extensions of* $(\mathbb{F}, \sigma)$ *and* $(\mathbb{G}, \sigma)$ *with*

$$\sigma(x) = x + \beta,$$
$$\sigma(y) = y + \tau(\beta)$$

*and* $\mathrm{const}_\sigma \mathbb{F}(x) = \mathrm{const}_\sigma \mathbb{G}(y) = \mathbb{K}$*. Then*

$$(\mathbb{F}(x), \sigma) \simeq (\mathbb{G}(y), \sigma).$$

*Proof.* If $(\mathbb{F}(x), \sigma)$ is a proper sum-extension then the corollary follows by Proposition 2.3.3. Otherwise, $(\mathbb{F}(x), \sigma)$ is not a proper sum-extension of $(\mathbb{F}, \sigma)$. By Corollary 2.2.4 there is a $g \in \mathbb{F}$ such that

$$\sigma(g) - g = \beta,$$
$$\sigma(\tau(g)) - \tau(g) = \tau(\beta).$$

Consequently $g - x, \tau(g) - y \in \mathbb{K}$ and thus

$$x = g + k,$$
$$y = \tau(g) + k'$$

for some $k, k' \in \mathbb{K}$. By $\tau(x) = \tau(g) + k'$ we finally get

$$\tau(x) = y + k''$$

for some $k'' \in \mathbb{K}$ which provides the desired isomorphism. □

**Lemma 2.4.4.** *Let* $(\mathbb{F}(s)(t), \sigma)$ *be a difference field extension of* $(\mathbb{F}, \sigma)$ *where* $s, t$ *are sums over* $\mathbb{F}$ *and* $\mathrm{const}_\sigma \mathbb{F}(s)(t) = \mathrm{const}_\sigma \mathbb{F}$*. Then* $(\mathbb{F}(t)(s), \sigma)$ *is a difference field extension of* $(\mathbb{F}, \sigma)$ *with* $\mathrm{const}_\sigma \mathbb{F}(s)(t) = \mathrm{const}_\sigma \mathbb{F}$ *and*

$$(\mathbb{F}(s)(t), \sigma) \simeq (\mathbb{F}(t)(s), \sigma).$$

*Proof.* Let $(\mathbb{F}(s)(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ where $\mathrm{const}_\sigma \mathbb{F}(s)(t) = \mathrm{const}_\sigma \mathbb{F}$ and $\sigma(s) = s + \beta$, $\sigma(t) = t + \gamma$ with $\beta, \gamma \in \mathbb{F}$.

If $(\mathbb{F}(s), \sigma)$ is not a proper sum extension of $(\mathbb{F}, \sigma)$ then there is a $g \in \mathbb{F}$ with

$$\sigma(g) - g = \beta$$

and thus $g - s \in \mathbb{K}$. Consequently $s \in \mathbb{F}$ and therefore

$$(\mathbb{F}(s)(t), \sigma) = (\mathbb{F}(t), \sigma) = (\mathbb{F}(t)(s), \sigma).$$

Now assume, $(\mathbb{F}(s), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$. If also $(\mathbb{F}(s)(t), \sigma)$ is a proper sum extension of $(\mathbb{F}(s), \sigma)$ then the lemma follows by Lemma 2.4.3. Otherwise, assume that $(\mathbb{F}(s)(t), \sigma)$ is not a proper sum extension of $(\mathbb{F}(s), \sigma)$. Then there is a $g \in \mathbb{F}(s)$ with

$$\sigma(g) - g = \gamma$$

and thus

$$g - t \in \mathbb{K}. \tag{2.6}$$

If $g \in \mathbb{F}$ then $t \in \mathbb{F}$ and thus

$$(\mathbb{F}(s)(t), \sigma) = (\mathbb{F}(s), \sigma) = (\mathbb{F}(t)(s), \sigma).$$

Otherwise assume $g \in \mathbb{F}(s) \setminus \mathbb{F}$. Since $(\mathbb{F}(t), \sigma)$ is a sub-difference field of $(\mathbb{F}(s)(t), \sigma)$ and $g \in \mathbb{F}(s) \setminus \mathbb{F}$, by Proposition 2.3.2, $(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$ with

$$(\mathbb{F}(s), \sigma) \simeq (\mathbb{F}(t), \sigma).$$

Furthermore $(\mathbb{F}(t)(s), \sigma)$ cannot be a proper sum extension of $(\mathbb{F}(s), \sigma)$, since otherwise also $(\mathbb{F}(s)(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$ by Lemma 2.4.3. Consequently there is a $g \in \mathbb{F}(t)$ with

$$\sigma(g) - g = \beta,$$

therefore $g - s \in \mathbb{K}$ and thus $s \in \mathbb{F}(t)$. By (2.6) we have $t \in \mathbb{F}(s)$ and thus

$$(\mathbb{F}(s)(t), \sigma) = (\mathbb{F}(s), \sigma) \simeq (\mathbb{F}(t), \sigma) = (\mathbb{F}(t)(s), \sigma).$$

□

**Proposition 2.4.6.** *Let $(\mathbb{F}(t_1, \dots, t_n)(s), \sigma)$ be a sum extension of $(\mathbb{F}, \sigma)$ where the constant field is not extended. Then $(\mathbb{F}(s)(t_1, \dots, t_n), \sigma)$ is a sum extension of $(\mathbb{F}, \sigma)$ and*

$$(\mathbb{F}(t_1, \dots, t_n)(s), \sigma) \simeq (\mathbb{F}(s)(t_1, \dots, t_n), \sigma).$$

*Proof.* The theorem follows by induction on the number of extensions using Corollary 2.4.4 and Lemma 2.4.4. (See the proof of Proposition 2.4.4.) □

### 2.4.7 Construction of Difference Rings of Fractions

Let $\mathbb{A}$ be a ring and $M$ be a multiplicative subset of $\mathbb{A}$ - a multiplicative monoid. Then we can construct the *ring of fractions*[13] of $\mathbb{A}$ by $M$ as follows. We define a relation

$$(a, m) \sim (a', m')$$

on the product set $\mathbb{A} \times M$, by the condition that there exists an element $s \in M$ such that

$$(a\,m' - m\,a')\,s = 0. \tag{2.7}$$

Then one can check that the relation is an equivalence relation; the equivalence class containing $(a, m)$ is denoted by $\frac{a}{m}$. We can define addition and multiplication by the rules

$$\frac{a}{m} + \frac{a}{m'} = \frac{a\,m' + m\,a'}{m\,m'}, \quad \frac{a}{m}\,\frac{a'}{m'} = \frac{a\,a'}{m\,m'}. \tag{2.8}$$

One can verify that these operations are well defined and that the set $Q(\mathbb{A}, M)$ of all equivalence classes forms a ring under the operations (2.8), with $\frac{0}{1}$ as zero and $\frac{1}{1}$ as unit element. The natural mapping $\lambda : \mathbb{A} \to Q(\mathbb{A}, M)$ given by

$$\lambda : x \mapsto \frac{x}{1}$$

is a ring homomorphism. Furthermore $\lambda$ maps all elements of $M$ to units in $Q(\mathbb{A}, M)$, since

$$\frac{s}{1}\frac{1}{s} = 1.$$

In the following we will assume that $M$ consists only of units in $\mathbb{A}$. Then condition (2.7) simplifies to

$$(a, m) \sim (a', m') \;\Leftrightarrow\; a\,m' - m\,a' = 0.$$

Additionally, $\lambda$ is a ring monomorphism and therefore an embedding of $\mathbb{A}$ into the ring $Q(\mathbb{A}, M)$. In this sense, we see $Q(\mathbb{A}, M)$ as a ring extension of $\mathbb{A}$.

Furthermore, if $\mathbb{A}$ is an integral domain, $Q(\mathbb{A}, \mathbb{A}^*)$ is nothing else as the quotient field $Q(\mathbb{A})$.

**Lemma 2.4.5.** *Let $(\mathbb{A}, \sigma)$ be a difference ring, let $M \subseteq \mathbb{A}$ be a multiplicative monoid consisting only of units in $\mathbb{A}$, and let $\sigma : M \to M$ restricted on $M$ be a monoid automorphism. If $Q(\mathbb{A}, M)$ denotes the ring of fractions $\mathbb{A}$ by $M$ then there is a unique ring automorphism $\tau : Q(\mathbb{A}, M) \to Q(\mathbb{A}, M)$ such that $(Q(\mathbb{A}, M), \tau)$ is a difference ring extension of $(\mathbb{A}, \sigma)$.*

*Proof.* Consider the map $\tau : Q(\mathbb{A}, M) \to Q(\mathbb{A}, M)$ defined by

$$\tau(\frac{a}{b}) = \frac{\sigma(a)}{\sigma(b)}$$

for $a \in \mathbb{A}, b \in M$. Since $\sigma(a) \in \mathbb{A}$ and $\sigma(b) \in M$, the map is well defined and one can easily check that $\tau$ is a ring automorphism. Therefore $(Q(\mathbb{A}, M), \tau)$ is a difference ring extension of $(\mathbb{A}, \sigma)$. Now let $\lambda : Q(\mathbb{A}, M) \to Q(\mathbb{A}, M)$ be any ring automorphism with $\lambda(r) = \sigma(r)$ for all $r \in \mathbb{A}$ and let $f = \frac{a}{b}$ with $a \in \mathbb{A}, b \in M$. Then

$$\lambda(f)\,\sigma(b) = \lambda(f)\,\lambda(b) = \lambda(f\,b) = \lambda(\frac{a}{b}\,b) = \lambda(a\,\frac{b}{b}) = \lambda(a) = \sigma(a)$$

[13] See for instance [Lan97, Part I, Chapter 2, Localization 2] or [Coh89, 9.3 Localization].

and since $\sigma(b) \in M$ we have

$$\lambda(f) = \lambda(f)\,\frac{\sigma(b)}{\sigma(b)} = \lambda(f)\,\sigma(b)\,\frac{1}{\sigma(b)} = \sigma(a)\,\frac{1}{\sigma(b)} = \frac{\sigma(a)}{\sigma(b)}.$$

Consequently

$$\lambda(f) = \frac{\sigma(a)}{\sigma(b)}$$

and thus $\lambda = \tau$. □

Let $\mathbb{A}$ be a ring, $\mathbb{D} \subseteq \mathbb{A}$ be an integral domain, $Q(\mathbb{D})$ the quotient field of $\mathbb{D}$ and $Q(\mathbb{D})[x]$ the corresponding polynomial ring, $x$ transcendental over $Q(\mathbb{D})$. Let $r \in \mathbb{A}$ such that $\mathbb{D}^*$ is a subset of the units in $\mathbb{D}[r]$. Then clearly

$$h : \left\{ \begin{array}{lcl} \mathbb{D}[x] & \to & \mathbb{A} \\ \sum_{i=0}^{n} f_i\,x^i & \mapsto & \sum_{i=0}^{n} f_i\,r^i \end{array} \right.$$

is a ring homomorphism. This motivates the notation

$$\mathbb{D}[r] := h(\mathbb{D}[x])$$

as

$$h(\mathbb{D}[x]) = \left\{ \sum_{i=0}^{n} f_i\,r^i \mid f_i \in \mathbb{D},\ n \geq 0 \right\}.$$

Now consider the ring of fractions $Q(\mathbb{D}[r], \mathbb{D}^*)$ with the multiplicative monoid $\mathbb{D}^*$. Since $\mathbb{D}^*$ is contained in the set of all units in $\mathbb{D}[r]$, by Lemma 2.4.5 $Q(\mathbb{D}[r], \mathbb{D}^*)$ is a ring extension of $\mathbb{D}[r]$. An element of this ring can be described by

$$\frac{\sum_{i=0}^{n} f_i\,r^i}{d} = \sum_{i=0}^{n} \frac{f_i\,r^i}{d}$$

where $f_i \in \mathbb{D}, d \in \mathbb{D}^*$. By the notation

$$\frac{f}{d}\,r := \frac{f\,r}{d}$$

we can represent the elements by

$$\sum_{i=0}^{n} \frac{f_i}{d_i}\,r^i$$

where $f_i \in \mathbb{D}, d_i \in \mathbb{D}^*$. Finally, this motivates the notation

$$Q(\mathbb{D})[r] := Q(\mathbb{D}[r], \mathbb{D}^*).$$

Besides the fact that $Q(\mathbb{D})[r]$ is a ring extension of $\mathbb{D}[r]$ one can see $Q(\mathbb{D})[r]$ also as a ring extension of $Q(\mathbb{D})$.

**Proposition 2.4.7.** *Let $(\mathbb{A}, \sigma)$ be a difference ring extension of $(\mathbb{D}, \sigma)$ where $\mathbb{D}$ is an integral domain. Let $r \in \mathbb{A}$ be such that $\mathbb{D}^*$ is contained in the set of all units of $\mathbb{D}[r]$. Then $(Q(\mathbb{D})[r], \sigma)$ is a difference ring extension of $(\mathbb{D}[r], \sigma)$. Furthermore $(Q(\mathbb{D})[r], \sigma)$ is a difference ring extension of $(Q(\mathbb{D}), \sigma)$.*

### 2.4.8 Lifting of Difference Ring Extensions to $\Pi\Sigma$-Fields

Let $\mathbb{F}[t]$ be a polynomial ring with coefficients in the field $\mathbb{F}$, i.e. $t$ is transcendental over the field $\mathbb{F}$. Let $I$ be a ideal of $\mathbb{F}[t]$ and consider the set of cosets

$$S := \{h + I \,|\, h \in \mathbb{F}[t]\}$$

where we define the cosets by

$$h + I := \{h + p \,|\, p \in \mathbb{F}[t]\}.$$

Then we can define multiplication and addition by the rules

$$(h_1 + I)\,(h_2 + I) = h_1\,h_2 + I, \qquad (h_1 + I) + (h_2 + I) = (h_1 + h_2) + I. \tag{2.9}$$

One can verify that these operations are well defined and that the set of cosets $S$ forms a ring under the operations (2.9) with $0 + I$ as zero and $1 + I$ as unit element. This ring is called *factor ring* modulo the ideal $I$ and is denoted by $\mathbb{F}[t]/I$.

**Lemma 2.4.6.** *Let $\mathbb{F}[t]$ be the polynomial ring with coefficients in the field $\mathbb{F}$, $g \in \mathbb{F}[t]$ and $\mathcal{R} := \mathbb{F}[t]/I$ be the factor ring modulo the ideal $I := \langle g \rangle$. Let $r \in \mathcal{R}^*$. Then there is a $u \in \mathbb{F}[t]^*$ with $g \nmid u$ and*

$$r = u + I.$$

*Furthermore $r$ is a zero divisor in $\mathcal{R}$ if and only if*

$$\gcd(u, g)_{\mathbb{F}[t]} \neq 1.$$

*Proof.* Since $r \in \mathcal{R}^*$, it follows that $r \neq 0 + I$ which means that there is a $u \in \mathbb{F}[t]^*$ such that

$$r = u + I$$

with $g \nmid u$ and consequently the first statement of the lemma is proven. Furthermore we have

$$\begin{aligned} r \text{ is a zero divisor} &\Leftrightarrow \exists s \in \mathcal{R}^* : \; r\,s = 0 \\ &\Leftrightarrow \exists v \in \mathbb{F}[t]^* : \; g \nmid v \;\&\; g \mid u\,v \\ &\Leftrightarrow \gcd(u, g) \neq 1. \end{aligned}$$

The first equivalence follows by the first statement of the lemma. Let us consider the second equivalence. Assume $g \nmid v$, $g \mid u\,v$ and $\gcd(u, g) = 1$. Then it follows that $g \mid v$ by $g \mid u\,v$ and $\gcd(u, g) = 1$, a contradiction. Conversely, assume $\gcd(u, g) \neq 1$. Then take

$$v := \frac{g}{\gcd(u, g)} \in \mathbb{F}[t]^*.$$

Since $\gcd(u, g) \neq 1$, it follows that $\deg(v) < \deg(g)$ and therefore $g \nmid v$. Additionally, we have

$$u\,v = u\,\frac{g}{\gcd(u, g)} = \frac{u}{\gcd(u, g)}\,g.$$

As $\frac{u}{\gcd(u,g)} \in \mathbb{F}[t]$, it follows that $g \mid u\,v$; therefore the second equivalence and thus the second statement of the lemma is proven. $\square$

**Lemma 2.4.7.** *Let $(\mathbb{F}[x], \sigma)$ be a difference ring and let $\mathbb{F}[t]$ be a ring for which there is a ring isomorphism $\tau : \mathbb{F}[x] \to \mathbb{F}[t]$. Then*

$$\sigma' : \begin{cases} \mathbb{F}[t] & \to \quad \mathbb{F}[t] \\ f & \mapsto \quad \tau(\sigma(\tau^{-1}(f))) \end{cases}$$

*is a ring isomorphism. Furthermore we have*

$$(\mathbb{F}[x], \sigma) \overset{\tau}{\simeq} (\mathbb{F}[t], \sigma').$$

*Proof.* Since $\sigma$ and $\tau$ are ring isomorphisms, it follows immediately that $\sigma'$ is a ring isomorphism. Additionally we have

$$\sigma'(\tau(f)) = \tau(\sigma(\tau(\tau^{-1}(f)))) = \tau(\sigma(f))$$

for all $f \in \mathbb{F}[x]$ and therefore $\tau$ is a difference ring isomorphism. □

**Proposition 2.4.8.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t + \beta$$

*where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Furthermore, let $(\mathbb{F}[x], \sigma)$ be a difference ring extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(x) = \alpha\, x + \beta$$

*where $x$ is not a zero divisor in case of $\beta = 0$. Then $\mathbb{F}[x]$ is an integral domain.*

*Proof.* Assume $\mathbb{F}[x]$ has zero divisors. As $(\mathbb{F}(t), \sigma)$ is a $\Pi\Sigma$-extension, it follows that $t$ is transcendental over $\mathbb{F}$. Then clearly,

$$\phi : \begin{cases} \mathbb{F}[t] & \to \quad \mathbb{F}[x] \\ \sum_{i=0}^{n} f_i\, t^i & \mapsto \quad \sum_{i=0}^{n} f_i\, x^i \end{cases}$$

is a surjective ring homomorphism[14] and therefore $\phi$ is a difference ring epimorphism. Furthermore there is an isomorphism between $\mathbb{F}[x]$ and the factor ring $\mathbb{F}[t]/\ker\phi$, i.e.

$$\mathbb{F}[t]/\ker\phi \overset{\tau}{\simeq} \mathbb{F}[x].$$

Additionally, we have that the following diagram

$$\begin{tikzcd} \mathbb{F}[t] \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \mathbb{F}[x] \arrow[d, "\tau"] \\ & \mathbb{F}[t]/\ker\phi \end{tikzcd}$$

commutes for

$$\pi : \begin{cases} \mathbb{F}[t] & \to \quad \mathbb{F}[t]/\ker\phi \\ f & \mapsto \quad f + \ker\phi. \end{cases}$$

[14] $\phi$ is also called evaluation homomorphism [Lan97].

By Lemma 2.4.7 there is a difference ring extension $(\mathbb{F}[t]/\ker\phi, \sigma)$ of $(\mathbb{F}, \sigma)$ such that $\tau$ is a difference ring isomorphism, i.e.

$$(\mathbb{F}[t]/\ker\phi, \sigma) \overset{\tau}{\simeq} (\mathbb{F}[x], \sigma)$$

As $\phi$ is a difference ring epimorphism,

$$\rho : \begin{cases} \mathbb{F}[t] & \rightarrow & \mathbb{F}[t]/\ker\phi \\ f & \mapsto & \phi(\tau(f)) \end{cases}$$

is a difference ring epimorphism and since the above diagram commutes it follows by $\rho = \pi$ that $\pi$ is a difference ring epimorphism from $(\mathbb{F}[t], \sigma)$ onto $(\mathbb{F}[t]/\ker\phi, \sigma)$.

As $\mathbb{F}[t]$ is a principle ideal domain, there is a $g \in \mathbb{F}[t]$ with

$$\ker\phi = \langle g\rangle.$$

If $g$ is irreducible then $\mathbb{F}[t]/\langle g\rangle$ is a field and therefore also $\mathbb{F}[x]$ is a field which contradicts to the assumption that $\mathbb{F}[x]$ has zero divisors. If $g$ is reducible, we may write $g = p\,q$ with $p, q, \in \mathbb{F}[t] \setminus \mathbb{F}$ and $p$ is irreducible. We have

$$0 + \langle g\rangle = \pi(g) = \pi(p\,q) = \pi(p)\,\pi(q)$$

and thus

$$0 + \langle g\rangle = \sigma^i(\pi(p)\,\pi(q)) = \pi(\sigma^i(p))\,\pi(\sigma^i(q)) \tag{2.10}$$

for all $i \in \mathbb{Z}$. As $\deg(\sigma^i(p)), \deg(\sigma^i(q)) < \deg(g)$, it follows that

$$\pi(\sigma^i(p)) \neq 0 + \langle g\rangle \neq \pi(\sigma^i(q))).$$

For any $h \in \mathbb{F}[t]^*$ with $\deg(h) < \deg(p)$ it follows by Lemma 2.4.6 that

$$\pi(h) \text{ is a zero divisor} \Leftrightarrow \gcd(h, g) \neq 1.$$

Hence it follows by (2.10) that

$$\forall i \in \mathbb{Z} : \gcd(\sigma^i(p), g) \neq 1. \tag{2.11}$$

Therefore $\mathrm{spread}_\sigma(g, p)$ is an infinite set and thus by Theorem 2.2.5 $(\mathbb{F}(t), \sigma)$ must be a $\Pi$-extension of $(\mathbb{F}, \sigma)$ and $t \mid \gcd(p, g)$. Consequently

$$p = u\,t$$

for some $u \in \mathbb{F}$ and thus $\pi(t)$ and therefore also $\phi(t) = x$ must be a zero divisor, a contradiction. Consequently $\mathbb{F}[x]$ cannot have any zero divisors and is therefore an integral domain. □

**Corollary 2.4.5.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\,t + \beta$$

*for some $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$ and $\mathrm{const}_\sigma\mathbb{F} = \mathbb{K}$. Furthermore, let $(\mathbb{F}[x], \sigma)$ be a difference ring extension of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(x) = \alpha\,x + \beta$$

*where $x$ is not a zero divisor in case of $\beta = 0$. Then $\mathbb{F}[x]$ is an integral domain. Furthermore there is a uniquely defined difference ring extension $(Q(\mathbb{F}[x]), \sigma) = (\mathbb{F}(x), \sigma)$ of $(\mathbb{F}[x], \sigma)$. $(\mathbb{F}(x), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and*

$$(\mathbb{F}(t), \sigma) \simeq (\mathbb{F}(x), \sigma).$$

*Proof.* By Proposition 2.4.8 it follows that $\mathbb{F}[x]$ is an integral domain. Thus we can build the field of fractions $\mathbb{F}(x)$ and get by Lemma 2.4.5 the uniquely defined difference ring extension $(\mathbb{F}(x), \sigma)$ of $(\mathbb{F}[x], \sigma)$. Furthermore $(\mathbb{F}(x), \sigma)$ is a difference field extension of $(\mathbb{F}, \sigma)$. If $(\mathbb{F}(t), \sigma)$ is a $\Sigma$-extension of $(\mathbb{F}, \sigma)$, the corollary follows by Proposition 2.3.3. If $(\mathbb{F}(t), \sigma)$ is a $\Pi$-extension then there does not exist an $n > 0$ with

$$\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$$

and thus $(\mathbb{F}(x), \sigma)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma)$. Clearly, $\tau : \mathbb{F}(t) \to \mathbb{F}(x)$ canonically defined by $\tau(t) = x$ is a difference field isomorphism. $\square$

## 2.5 ΠΣ-Fields and the Ring of Sequences

Let $\mathbb{K}$ be a field with characteristic zero. By $\mathbb{K}^{\mathbb{N}}$ we denote the set of all sequences

$$(a_n)_{n=0}^{\infty} = \langle a_0, a_1, a_2, \dots \rangle$$

with $a_i \in \mathbb{K}$. By component-wise addition and multiplication $\mathbb{K}^{\mathbb{N}}$ forms a commutative ring in which the field $\mathbb{K}$ can be naturally embedded by identifying $k \in \mathbb{K}$ with the sequence

$$\langle k, k, k, \dots \rangle.$$

One can define a shift-operation via the following ring epimorphism

$$S : \begin{cases} \mathbb{K}^{\mathbb{N}} & \rightarrow \quad \mathbb{K}^{\mathbb{N}} \\ \langle a_0, a_1, a_2, \dots \rangle & \mapsto \quad \langle a_1, a_2, a_3, \dots, \rangle \end{cases}$$

but one can immediately see that $S$ is not a ring automorphism. We define an equivalence relation on $\mathbb{K}^{\mathbb{N}}$ where two sequences $a$ and $b$ are equivalent if there exists a $\delta \geq 0$ such that

$$\forall k \geq \delta : \; a_k = b_k.$$

The equivalence classes form a ring which is denoted by $\mathcal{S}(\mathbb{K})$. The elements of $\mathcal{S}(\mathbb{K})$ will be denoted by sequence notation. We can now define a ring epimorphism

$$S : \begin{cases} \mathcal{S}(\mathbb{K}) & \rightarrow \quad \mathcal{S}(\mathbb{K}) \\ \langle a_0, a_1, a_2, \dots \rangle & \mapsto \quad \langle a_1, a_2, a_3, \dots, \rangle \end{cases}$$

like for the ring $\mathbb{K}^{\mathbb{N}}$ but this time $S$ is even a ring monomorphism and thus a ring automorphism. Therefore $(\mathcal{S}(\mathbb{K}), S)$ forms a difference ring.

**Definition 2.5.1.** The difference ring $(\mathcal{S}(\mathbb{K}), S)$ introduced above is called the *ring of* $\mathbb{K}$-*sequences.* ◊

The ring of $\mathbb{K}$-sequences $(\mathcal{S}(\mathbb{K}), \sigma)$ has zero divisors, so for example $a := \langle 1, 0, 1, 0, 1, 0, \dots \rangle$ and $b := \langle 0, 1, 0, 1, 0, 1, \dots \rangle$ are not the zero element, but we have

$$a\, b = \mathbf{0} := \langle 0, 0, 0, \dots \rangle.$$

If one looks closer at the ring of sequences then one notices that nonzero divisors of $\mathcal{S}(\mathbb{K})$ are exactly those elements $a$ with the property that there exists a $\delta \geq 0$ with

$$\forall k \geq \delta : \; a_k \neq 0.$$

On the other side, these are exactly those elements which are invertible, i.e. units.

The following lemma is inspired by [NP97] where P. Paule and I. Nemes introduced an evaluation map in order to link a difference ring $(\mathbb{A}, \sigma)$ with constant field $\mathbb{K}$ with the ring of $\mathbb{K}$-sequences. I extended this concept by using this evaluation map in order to define a difference ring homomorphism from the difference ring $(\mathbb{A}, \sigma)$ to the ring of $\mathbb{K}$-sequences.

**Lemma 2.5.1.** *Let* $(\mathbb{A}, \sigma)$ *be a difference ring with constant field* $\mathbb{K}$ *together with a map* $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \rightarrow \mathbb{K}$. *The map*

$$h : \begin{cases} (\mathbb{A}, \sigma) & \rightarrow \quad (\mathcal{S}(\mathbb{K}), S) \\ f & \mapsto \quad \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$

*is a difference ring homomorphism if and only if the map* ev *has the following properties: for all* $f, g \in \mathbb{A}$ *there exists a* $\delta \geq 0$ *such that*

$$\forall i \geq \delta: \ \mathrm{ev}(f\,g, i) = \mathrm{ev}(f, i)\,\mathrm{ev}(g, i),$$
$$\forall i \geq \delta: \ \mathrm{ev}(f+g, i) = \mathrm{ev}(f, i) + \mathrm{ev}(g, i)$$

*and for all* $f \in \mathbb{A}$ *there exists a* $\delta \geq 0$ *with*

$$\forall i \geq \delta: \ \mathrm{ev}(\sigma(f), i) = \mathrm{ev}(f, i+1).$$

*Proof.* **"⇒"** Let $f, g \in \mathbb{A}$. We have

$$h(f+g) = h(f) + h(g), \qquad h(f\,g) = h(f)\,h(g), \qquad h(\sigma(f)) = S(h(f)),$$

and thus

$$h(f+g)_i = h(f)_i + h(g)_i, \qquad h(f\,g)_i = h(f)_i\,h(g)_i, \qquad h(\sigma(f))_i = h(f)_{i+1}$$

for all $i \geq \delta$ for some $\delta \geq 0$; i.e.

$$\mathrm{ev}(f+g, i) = \mathrm{ev}(f, i) + \mathrm{ev}(g, i), \quad \mathrm{ev}(f\,g, i) = \mathrm{ev}(f, i)\,\mathrm{ev}(g, i), \quad \mathrm{ev}(\sigma(f), i) = \mathrm{ev}(f, i+1).$$

**"⇐"** Assume for all $f, g \in \mathbb{A}$ there exists a $\delta \geq 0$ such that $\mathrm{ev}(f+g, i) = \mathrm{ev}(f, i) + \mathrm{ev}(g, i)$ and $\mathrm{ev}(f\,g, i) = \mathrm{ev}(f, i)\,\mathrm{ev}(g, i)$ for all $i \geq \delta$. Then it follows immediately that $h$ is a ring homomorphism. Now let $f \in \mathbb{A}$ and let $\delta \geq 0$ with $\mathrm{ev}(\sigma(f), i) = \mathrm{ev}(f, i+1)$ for all $i \geq \delta$. Then $h$ is a difference ring homomorphism, since

$$\begin{aligned} h(\sigma(f)) &= \langle \mathrm{ev}(\sigma(f), 0), \dots, \mathrm{ev}(\sigma(f), \delta-1), \mathrm{ev}(\sigma(f), \delta), \mathrm{ev}(\sigma(f), \delta+1) \dots \rangle \\ &= \langle \mathrm{ev}(\sigma(f), 0), \dots, \mathrm{ev}(\sigma(f), \delta-1), \mathrm{ev}(f, \delta+1), \mathrm{ev}(f, \delta+2) \dots \rangle \\ &= \langle \mathrm{ev}(f, 1), \dots, \mathrm{ev}(f, \delta), \mathrm{ev}(f, \delta+1), \mathrm{ev}(f, \delta+2) \dots \rangle \\ &= S(h(f)). \end{aligned}$$

Here we make use of our convention that $\langle \dots \rangle$ denotes equivalence classes of sequences. □

**Definition 2.5.2.** Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$ together with a map $L : \mathbb{A} \to \mathbb{N}_0$. The function $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ is called *L-homomorphic* if for all $f, g \in \mathbb{A}$ and for all $i \geq \max(L(f), L(g))$ we have

$$\mathrm{ev}(f+g, i) = \mathrm{ev}(f, i) + \mathrm{ev}(g, i),$$
$$\mathrm{ev}(f\,g, i) = \mathrm{ev}(f, i)\,\mathrm{ev}(g, i)$$

and it follows that

$$\mathrm{ev}(\sigma(f), i) = \mathrm{ev}(f, i+1)$$

for all $i \geq L(f)$. ◊

**Proposition 2.5.1.** *Let* $(\mathbb{A}, \sigma)$ *be a difference ring together with a map* $L : \mathbb{A} \to \mathbb{N}_0$. *If the map* $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ *is L-homomorphic then*

$$h : \begin{cases} (\mathbb{A}, \sigma) & \to \quad (\mathcal{S}(\mathbb{K}), S) \\ f & \mapsto \quad \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$

*is a difference ring homomorphism.*

*Proof.* This follows immediately by Lemma 2.5.1. □

**Example 2.5.1.** In this example we will construct a difference ring monomorphism from the $\Pi\Sigma$-field $(\mathbb{Q}(x), \sigma)$ canonically defined by

$$\sigma(x) = x + 1$$

into $\mathcal{S}(\mathbb{Q})$. We start with the constant difference field $(\mathbb{Q}, \sigma)$. In order to turn

$$h : \begin{cases} \mathbb{Q} & \rightarrow & \mathcal{S}(\mathbb{Q}) \\ q & \mapsto & \langle \mathrm{ev}(q, 0), \mathrm{ev}(q, 1), \dots \rangle \end{cases}$$

into a difference ring monomorphism we define the $L$-homomorphic map

$$\mathrm{ev} : \begin{cases} \mathbb{Q} \times \mathbb{N}_0 & \rightarrow & \mathbb{Q} \\ (q, i) & \mapsto & q \end{cases}$$

where $L$ is defined by

$$L : \begin{cases} \mathbb{Q} & \rightarrow & \mathbb{N}_0 \\ q & \mapsto & 0. \end{cases}$$

Now we extend the function ev from $\mathrm{ev} : \mathbb{Q} \times \mathbb{N}_0 \rightarrow \mathbb{Q}$ to $\mathrm{ev} : \mathbb{Q}[x] \times \mathbb{N}_0 \rightarrow \mathbb{Q}$ by defining

$$\mathrm{ev}(\sum_i p_i\, x^i, k) := \sum_i \mathrm{ev}(p_i, k)\, \mathrm{ev}(x, k)^i$$

and

$$\mathrm{ev}(x, i) := i.$$

Furthermore one can easily check that ev is an $L$-homomorphic map where $L$ is extended from $L : \mathbb{Q} \rightarrow \mathbb{N}_0$, as follows, to

$$L : \begin{cases} \mathbb{Q}[x] & \rightarrow & \mathbb{N}_0 \\ p & \mapsto & 0. \end{cases}$$

Then by Proposition 2.5.1 the map

$$h : \begin{cases} \mathbb{Q}[x] & \rightarrow & \mathcal{S}(\mathbb{Q}) \\ p & \mapsto & \langle \mathrm{ev}(p, 0), \mathrm{ev}(p, 1), \dots \rangle \end{cases}$$

is a difference ring homomorphism. As will be seen later by Theorem 2.5.1, $h$ is even a difference ring monomorphism.

Finally, we define $Z : \mathbb{Q}[x] \rightarrow \mathbb{N}_0$ by

$$Z(p) := z + 1$$

where $z$ is the greatest positive integer which is a root of $p \in \mathbb{Q}[x]$. If there does not exist such a root then $Z(p) := 0$. Clearly, this function is computable. Now we extend the function $L$ from $L : \mathbb{Q}[x] \rightarrow \mathbb{N}_0$ to $L : \mathbb{Q}(x) \rightarrow \mathbb{N}_0$ by[15]

$$L(f) := Z(\mathrm{den}(f))$$

[15]If $f \in \mathbb{Q}[x]$ then $\mathrm{den}(f) = 1$, therefore $Z(\mathrm{den}(f)) = Z(1) = 0$ and thus $L(f) = 0$ as it was defined for $L : \mathbb{Q}[x] \rightarrow \mathbb{N}_0$.

and ev from $\mathrm{ev} : \mathbb{Q}[x] \times \mathbb{N}_0 \to \mathbb{N}_0$ to $\mathrm{ev} : \mathbb{Q}(x) \times \mathbb{N}_0 \to \mathbb{N}_0$ by[16]

$$\mathrm{ev}(\frac{a}{b}, k) := \begin{cases} 0 & \text{if } k < L(a/b) \\ \frac{\mathrm{ev}(a,k)}{\mathrm{ev}(b,k)} & \text{if } k \geq L(a/b). \end{cases}$$

Then one can check that ev is an $L$-homomorphic map and hence by Proposition 2.5.1 the map

$$h : \begin{cases} \mathbb{Q}(x) & \to & \mathcal{S}(\mathbb{Q}) \\ p & \mapsto & \langle \mathrm{ev}(p, 0), \mathrm{ev}(p, 1), \dots \rangle \end{cases}$$

is a difference ring homomorphism. Moreover, one can easily see that $h$ is even a difference ring monomorphism. $\Diamond$

**Example 2.5.2.** Now consider the $\Sigma$-extension $(\mathbb{Q}(x)(y), \sigma)$ of $(\mathbb{Q}(x), \sigma)$ canonically defined by

$$\sigma(y) = y + \frac{1}{x+1}.$$

Then the function $L : \mathbb{Q}(x) \to \mathbb{N}_0$ from Example 2.5.1 can be extended to $L : \mathbb{Q}(x)[y] \to \mathbb{N}_0$ by

$$L(\sum_{i=0}^{m} p_i\, y^i) := \max(L(p_0), \dots, L(p_m)).$$

Additionally, ev can be extended from $\mathrm{ev} : \mathbb{Q}(x) \times \mathbb{N}_0 \to \mathbb{K}$ to $\mathrm{ev} : \mathbb{Q}(x)[y] \times \mathbb{N}_0 \to \mathbb{K}$ by

$$\mathrm{ev}(\underbrace{\sum_{j=0}^{d} r_j\, y^j}_{=f}, k) := \begin{cases} 0 & \text{if } i < L(f) \\ \sum_{j=0}^{d} \mathrm{ev}(r_j, k)\, \mathrm{ev}(y, k)^j & \text{if } i \geq L(f) \end{cases}$$

and

$$\mathrm{ev}(y, k) = \sum_{i=1}^{k} \mathrm{ev}(\frac{1}{x}, i).$$

Since ev is an $L$-homomorphic map, by Proposition 2.5.1 the map

$$h : \begin{cases} \mathbb{Q}(x)[y] & \to & \mathcal{S}(\mathbb{Q}) \\ f & \mapsto & \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$

is a difference ring homomorphism. As will be shown later by Theorem 2.5.1, $h$ is even a difference ring monomorphism. $\Diamond$

In the next step one should try to lift this monomorphism to the quotient difference field $(\mathbb{Q}(x, h), \sigma)$. When we lifted the difference ring monomorphism form $\mathbb{Q}[x]$ to $\mathbb{Q}(x)$, it was a crucial step that we could define a function $Z$ which tells us form which index value on a sequence has nonzero entries. Is it guaranteed that for all elements $p \in \mathbb{Q}(x)[y]$ there exists a $\delta \in \mathbb{N}_0$ such that

$$\forall k \geq \delta : \ \mathrm{ev}(p, k) \neq 0?$$

More precisely, can one always construct a monomorphism from a $\Pi\Sigma$-field with constant field $\mathbb{K}$ into $\mathcal{S}(\mathbb{K})$? I could not find an answer to this question. The following example illustrates how I try to deal with this problem.

[16]For all $f \in \mathbb{Q}[x]$ we have $L(f) = 0$ and therefore ev is properly extended.

**Example 2.5.3.** Given the difference ring monomorphism $h : \mathbb{Q}(x)[h] \to \mathcal{S}(\mathbb{Q})$, we consider the multiplicative monoid

$$M = \{f \in \mathbb{Q}[x, y] \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{Q})\}.$$

As will be shown later by Proposition 2.5.2 $(Q(\mathbb{Q}(x)[y], M), \sigma)$ is a uniquely defined difference ring extension of $(\mathbb{Q}(x)[y], \sigma)$. Furthermore, for this difference ring $(Q(\mathbb{Q}(x)[y], M), \sigma)$ we can define a function $Z : M \to \mathbb{N}_0$ such that for all $f \in M$ we have

$$\forall k \geq Z(f) : \ \mathrm{ev}(f, k) \neq 0.$$

As we will shown later, we are now able to define a difference ring monomorphism from $(Q(\mathbb{Q}(x)[y], M), \sigma)$ into $\mathcal{S}(\mathbb{Q})$. But please note, that this function $Z$ is in general not constructive. $\diamondsuit$

**Remark 2.5.1.** Let $\mathbb{K}[t_1, \dots, t_n]$ a polynomial ring with coefficients in the field $\mathbb{K}$ and let $\mathbb{K}(t_1, \dots, t_n)$ be the *field of rational functions* over $\mathbb{K}$, this means $\mathbb{K}(t_1, \dots, t_n)$ is the quotient field of $\mathbb{K}[t_1, \dots, t_n]$. Let $<$ be any admissible ordering[17] on the monoid $[t_1, \dots, t_n]$ of power products $t_1^{i_1} \dots t_n^{i_n}$. Then for $f \in \mathbb{K}[t_1, \dots, t_n]$ we denote by $\mathrm{lc}(f) \in \mathbb{K}$ the coefficient of the greatest monomial in $f$ with respect to the admissible ordering $<$. If $f \in \mathbb{K}(t_1, \dots, t_n)$ then there are uniquely determined $f_1, f_2 \in \mathbb{K}[t_1, \dots, t_n]$ such that

$$f = \frac{f_1}{f_2}$$

and $\mathrm{lc}(f_2) = 1$. In the following we write

$$\mathrm{num}(f) = f_1, \qquad \qquad \mathrm{den}(f) = f_2$$

as the *numerator* and *denominator* of $f$.

If we consider the field of rational functions $\mathbb{K}[t]$ over $\mathbb{K}$ then by convention we choose the admissible ordering

$$t^0 < t^1 < t^2 < t^3 < \dots$$

and we obtain for $f \in \mathbb{F}(t)$ that

$$f = \frac{\mathrm{num}(f)}{\mathrm{den}(f)}$$

is just the reduced representation introduced in Definition 2.2.2. $\diamondsuit$

Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$. In the following we will consider difference rings $(\mathbb{A}, \sigma) = (Q(\mathbb{K}[t_1, \dots, t_n], B), \sigma)$ where $B$ is a multiplicative sub-monoid of $\mathbb{K}[t_1, \dots, t_n]$, together with difference ring extensions $(\mathbb{F}, \sigma)$ of $(\mathbb{A}, \sigma)$.

In this concrete situation we will always have the following properties for the function $L : \mathbb{A} \to \mathbb{N}_0$ in Definition 2.5.2.

1. For all $k \in \mathbb{K}$ we have $L(k) = 0$,

2. for all $f, g \in \mathbb{A}$ we obtain

$$L(f\,g) \leq \max(L(f), L(g)), \tag{2.12}$$
$$L(f + g) \leq \max(L(f), L(g)), \tag{2.13}$$
$$L(\sigma(f)) \leq L(f) \tag{2.14}$$

and

[17] See for instance [Win96, Chapter 8.2]

3. for all for all $f, g \in \mathbb{K}[t_1, \ldots, t_n]$ it follows that

$$L(f) \leq L(f\,g) = \max(L(f), L(g)), \tag{2.15}$$
$$L(f) \leq L(f + g) = \max(L(f), L(g)). \tag{2.16}$$

4. Furthermore for all $f \in Q(\mathbb{A}, M)$ we have

$$\max(L(\mathrm{num}(f)), L(\mathrm{den}(d))) \leq L(f). \tag{2.17}$$

This motivates the following definition:

**Definition 2.5.3.** Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \ldots, t_n), \sigma)$ be a $\Pi\Sigma$-field, $B \subseteq \mathbb{K}[t_1, \ldots, t_n]^*$ a multiplicative monoid and $(\mathbb{F}, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma) = (Q(\mathbb{K}[t_1, \ldots, t_n], B), \sigma)$. Let $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ and $L : \mathbb{A} \to \mathbb{N}_0$ be maps. ev is called *homomorphic map bounded by $L$*, if ev is $L$-homomorphic and $L$ possesses the properties 1. to 4. from above. $\diamondsuit$

**Example 2.5.4.** In the Examples 2.5.1 and 2.5.2 we always have that ev is a homomorphic map bounded by $L$. $\diamondsuit$

**Corollary 2.5.1.** *Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \ldots, t_n), \sigma)$ be a $\Pi\Sigma$-field, $B \subseteq \mathbb{K}[t_1, \ldots, t_n]^*$ a multiplicative monoid and $(\mathbb{F}, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma) = (Q(\mathbb{K}[t_1, \ldots, t_n], B), \sigma)$. If there exists a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : \mathbb{A} \to \mathbb{N}_0$ then*

$$h : \begin{cases} (\mathbb{A}, \sigma) & \to & (\mathcal{S}(\mathbb{K}), S) \\ f & \mapsto & \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \ldots \rangle \end{cases}$$

*is a difference ring homomorphism.*

*Proof.* Assume there exists a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : \mathbb{A} \to \mathbb{N}_0$. Then by definition ev is an $L$-homomorphic map. But then by Proposition 2.5.1 it follows immediately that

$$h : \begin{cases} (\mathbb{A}, \sigma) & \to & (\mathcal{S}(\mathbb{K}), S) \\ f & \mapsto & \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \ldots \rangle \end{cases}$$

is a difference ring homomorphism. $\square$

### 2.5.1 Lifting of Polynomial Extensions

**Lemma 2.5.2.** *Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ be a $\Pi\Sigma$-field, $B \subseteq \mathbb{K}[t_1, \dots, t_n]^*$ a multiplicative monoid and $(\mathbb{F}, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma) = (Q(\mathbb{K}[t_1, \dots, t_n], B), \sigma)$. Assume there is a difference ring homomorphism*

$$h : \begin{cases} \mathbb{A} & \to & \mathcal{S}(\mathbb{K}) \\ f & \mapsto & \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$

*for a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : \mathbb{A} \to \mathbb{N}_0$. Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$ with*

$$\sigma(t) = \alpha\, t, \quad \alpha \in \mathbb{A}^*$$

*and where $h(\alpha)$ is a unit in $\mathcal{S}(\mathbb{K})$. Then there are maps $\tilde{\mathrm{ev}} : \mathbb{A}[t] \times \mathbb{N}_0 \to \mathbb{K}$ and $\tilde{L} : \mathbb{A}[t] \to \mathbb{N}_0$ such that $\tilde{\mathrm{ev}}$ is a homomorphic map bounded by $\tilde{L}$. If $h$ is even a difference ring monomorphism then*

$$\langle \mathrm{ev}(t, 0), \mathrm{ev}(t, 1), \dots \rangle \notin \mathrm{const}_{S}\mathcal{S}(\mathbb{K}).$$

*Proof.* As $h(\alpha)$ is a unit in $\mathcal{S}(\mathbb{K})$, there is an $\epsilon \geq L(\sigma^{-1}(\alpha))$ such that

$$\forall i \geq \epsilon : \ \mathrm{ev}(\sigma^{-1}(\alpha), i) \neq 0. \tag{2.18}$$

Let us fix such an $\epsilon$. We define $\tilde{L} : \mathbb{A}[t] \to \mathbb{N}_0$ by[18]

$$\tilde{L}(f) := \begin{cases} L(f) & \text{if } f \in \mathbb{A} \\ \max(L(r_0), \dots, L(r_d), \epsilon) & \text{if } f = \sum_{i=0}^{d} r_i\, t^i \notin \mathbb{A} \end{cases}$$

and $\tilde{\mathrm{ev}} : \mathbb{A}[t] \times \mathbb{N}_0 \to \mathbb{K}$ by

$$\tilde{\mathrm{ev}}(t, k) := \begin{cases} 0 & \text{if } i < \tilde{L}(t) (= \epsilon) \\ c \prod_{j=\epsilon}^{k} \mathrm{ev}(\sigma^{-1}(\alpha), j) & \text{if } i \geq \tilde{L}(t) \end{cases} \tag{2.19}$$

for some[19] $c \in \mathbb{K}^*$ and

$$\tilde{\mathrm{ev}}(\underbrace{\sum_{j=0}^{d} r_j\, t^j}_{=f}, k) := \begin{cases} 0 & \text{if } i < \tilde{L}(f) \\ \sum_{j=0}^{d} \mathrm{ev}(r_j, k)\, \tilde{\mathrm{ev}}(t, k)^j & \text{if } i \geq \tilde{L}(f). \end{cases}$$

For $f = \sum_{j=0}^{m} f_i\, t^i, g = \sum_{j=0}^{m} g_i\, t^i \in \mathbb{A}[t]$ we have

$$\begin{aligned} \tilde{L}(f + g) &= \max(L(f_0 + g_0), \dots, L(f_m + g_m), \epsilon) \\ &\overset{(2.13)}{\leq} \max(L(f_0), \dots, L(f_m), L(g_0), \dots, L(g_m), \epsilon) \\ &= \max(\max(L(f_0), \dots, L(f_m), \epsilon), \max(L(g_0), \dots, L(g_m), \epsilon)) = \max(\tilde{L}(f), \tilde{L}(g), \\ \tilde{L}(f\, g) &= \tilde{L}(\sum_{i=0}^{2m} t^i \sum_{j=0}^{i} f_j\, g_{i-j}) = \max(L(f_0\, g_0), \dots, L(\sum_{j=0}^{2m} f_j\, g_{2m-j}), \epsilon) \\ &\overset{(2.12),(2.13)}{\leq} \max(L(f_0), \dots, L(f_m), L(g_0), \dots, L(g_m), \epsilon) = \max(\tilde{L}(f), \tilde{L}(g)) \end{aligned}$$

[18] Note that $\tilde{L}(t) = \epsilon$.

[19] Please note that the choice of $c \in \mathbb{K}^*$ is free. In order to deal later with indefinite summation, it will be essential to specify an appropriate value for this $c$ with respect to the given summation problem.

and one can easily see that for all $f, g \in \mathbb{K}[t_1, \dots, t_n][t]$ with (2.15) and (2.16) even equality holds. Furthermore we obtain for $f \in \mathbb{A}[t]$ as above that

$$\begin{aligned} L(\sigma(f)) &= L(\sum_{j=0}^{m} \sigma(f_j)\,\alpha^j\,t^j) = \max(L(\sigma(f_0)), L(\sigma(f_1)\,\alpha), \dots, L(\sigma(f_m)\,\alpha^m), \epsilon) \\ &\overset{(2.12)}{\leq} \max(L(\sigma(f_0)), \dots, L(\sigma(f_m)), L(\alpha), \epsilon) \overset{(2.14)}{\leq} \max(L(f_0), \dots, L(f_m), \epsilon) = L(f). \end{aligned}$$

Now let $f = \sum_{i=0}^{m} \frac{f_i}{d_i} t^i$ with $f_i \in \mathbb{K}[t_1, \dots, t_n]$, $d_i \in M$ and $\gcd(f_i, d_i) = 1$. Furthermore let $f = \frac{p}{d}$ with $p \in \mathbb{K}[t_1, \dots, t_n]$, $d \in M$ and $\gcd(p, d) = 1$. Then we have $d = \operatorname{lcm}(d_0, \dots, d_m)$ and there are $q_i \in \mathbb{K}[t_1, \dots, t_n]$ with $q_i\,f_i = p_i$ and $q_i \mid d$. It follows that

$$\begin{aligned} \max(L(f)) &= \max(L(f_0/d_0), \dots, L(f_m/d_m), \epsilon) \\ &\overset{(2.17)}{\geq} \max(L(f_0), \dots, L(f_m), L(d_0), \dots, L(d_m), \epsilon) \\ &\overset{(2.15)}{=} \max(L(f_0), \dots, L(f_m), L(q_0), \dots, L(q_m), L(d), \epsilon) \\ &\overset{(2.15)}{=} \max(L(f_0\,q_0), \dots, L(f_m\,q_m), L(d), \epsilon) \\ &= \max(L(p_0), \dots, L(p_m), L(d), \epsilon) = \max(L(p), L(d)). \end{aligned}$$

By definition we have for all $f, g \in \mathbb{A}[t]$ and all $k \geq \max(\tilde{L}(f), \tilde{L}(g))$ that

$$\begin{aligned} \tilde{\text{ev}}(f\,g, k) &= \tilde{\text{ev}}(f, k)\,\tilde{\text{ev}}(g, k), \\ \tilde{\text{ev}}(f + g, k) &= \tilde{\text{ev}}(f, k) + \tilde{\text{ev}}(g, k). \end{aligned}$$

Since for $k \geq \tilde{L}(t) = \epsilon$ we have

$$\begin{aligned} \tilde{\text{ev}}(t, k+1) &= c \prod_{j=\epsilon}^{k+1} \text{ev}(\sigma^{-1}(\alpha), j) = c \left( \prod_{j=\epsilon}^{k} \text{ev}(\sigma^{-1}(\alpha), j) \right) \text{ev}(\sigma^{-1}(\alpha), k+1) \\ &= \tilde{\text{ev}}(t, k)\,\text{ev}(\alpha, k) = \tilde{\text{ev}}(\alpha\,t, k) = \tilde{\text{ev}}(\sigma(t)), \end{aligned}$$

it follows with $\tilde{L}(\sigma(f)) \leq \tilde{L}(f)$ that

$$\begin{aligned} \text{ev}(\sigma(f), k) &= \text{ev}(\sigma(\sum_{j=0}^{m} f_j\,t^j, k)) = \text{ev}(\sum_{j=0}^{m} \sigma(f_j)\sigma(t)^j, k) = \sum_{j=0}^{m} \text{ev}(\sigma(f_j), k)\,\text{ev}(\sigma(t), k)^j \\ &= \sum_{j=0}^{m} \text{ev}(f_j, k+1)\,\text{ev}(t, k+1)^j = \text{ev}(\sum_{j=0}^{m} f_j\,t^j, k+1). \end{aligned}$$

Consequently for all $f \in \mathbb{A}[t]$ we have

$$\forall k \geq \tilde{L}(f):\ \text{ev}(f, k+1) = \text{ev}(\sigma(f), k)$$

and thus $\tilde{\text{ev}}$ is a homomorphic map bounded by $\tilde{L}$.

Now assume that $h$ is a difference ring monomorphism and that

$$h(t) \in \text{const}_S \mathcal{S}(\mathbb{K}),$$

i.e. there is a $\delta \geq \epsilon$ such that

$$\forall i \geq 0: \ \tilde{ev}(t, \delta + i) = \tilde{\mathrm{ev}}(t, \delta + i + 1).$$

Then

$$\begin{aligned} 0 &= \tilde{\mathrm{ev}}(t, \delta + i + 1) - \tilde{\mathrm{ev}}(t, \delta + i) = \tilde{\mathrm{ev}}(\sigma(t) - t, \delta + i) \\ &= \tilde{\mathrm{ev}}((\alpha - 1)\, t, \delta + i) = \tilde{\mathrm{ev}}(\alpha - 1, \delta + i)\, \tilde{\mathrm{ev}}(t, \delta + i). \end{aligned}$$

As

$$\tilde{\mathrm{ev}}(t, \delta + i) = c \prod_{j=\epsilon}^{\delta+i} \mathrm{ev}(\sigma^{-1}(\alpha), j) \neq 0$$

by (2.18), we must have $\tilde{\mathrm{ev}}(\alpha - 1, \delta + i) = 0$ and thus

$$\tilde{\mathrm{ev}}(\alpha, \delta + i) = 1$$

for all $i \geq 0$. Consequently $\alpha = 1$, a contradiction to the assumption that $(\mathbb{F}(t), \sigma)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma)$. □

**Lemma 2.5.3.** *Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \ldots, t_n), \sigma)$ be a $\Pi\Sigma$-field, $B \subseteq \mathbb{K}[t_1, \ldots, t_n]^*$ a multiplicative monoid and $(\mathbb{F}, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma) = (Q(\mathbb{K}[t_1, \ldots, t_n], B), \sigma)$. Assume there is a difference ring monomorphism*

$$h : \left\{ \begin{array}{lcl} \mathbb{A} & \rightarrow & \mathcal{S}(\mathbb{K}) \\ f & \mapsto & \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \ldots \rangle \end{array} \right.$$

*for a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \rightarrow \mathbb{K}$ bounded by $L : \mathbb{A} \rightarrow \mathbb{N}_0$. Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ with*

$$\sigma(t) = t + \beta, \quad \beta \in \mathbb{A}.$$

*Then there are maps $\tilde{\mathrm{ev}} : \mathbb{A}[t] \times \mathbb{N}_0 \rightarrow \mathbb{K}$ and $\tilde{L} : \mathbb{A}[t] \rightarrow \mathbb{N}_0$ such that $\tilde{\mathrm{ev}}$ is a homomorphic map bounded by $\tilde{L}$. If $h$ is even a difference ring monomorphism then*

$$\langle \mathrm{ev}(t, 0), \mathrm{ev}(t, 1), \ldots \rangle \notin \mathrm{const}_S \mathcal{S}(\mathbb{K}).$$

*Proof.* Let $\epsilon := L(\sigma^{-1}(\beta))$. We define $\tilde{L} : \mathbb{A}[t] \rightarrow \mathbb{N}_0$ by[20]

$$\tilde{L}(f) := \left\{ \begin{array}{ll} L(f) & \text{if } f \in \mathbb{A} \\ \max(L(r_0), \ldots, L(r_d), \epsilon) & \text{if } f = \sum_{i=0}^{d} r_i\, t^i \notin \mathbb{A} \end{array} \right.$$

and $\tilde{\mathrm{ev}} : \mathbb{A}[t] \times \mathbb{N}_0 \rightarrow \mathbb{K}$ by

$$\tilde{\mathrm{ev}}(t, k) := \left\{ \begin{array}{ll} 0 & \text{if } i < \tilde{L}(t) (= \epsilon) \\ c + \sum_{j=\epsilon}^{k} \mathrm{ev}(\sigma^{-1}(\beta), j) & \text{if } i \geq \tilde{L}(t) \end{array} \right. \tag{2.20}$$

for some[21] $c \in \mathbb{K}$ and

$$\tilde{\mathrm{ev}}(\underbrace{\sum_{j=0}^{d} r_j\, t^j}_{=f}, k) := \left\{ \begin{array}{ll} 0 & \text{if } i < \tilde{L}(f) \\ \sum_{j=0}^{d} \mathrm{ev}(r_j, k)\, \tilde{\mathrm{ev}}(t, k)^j & \text{if } i \geq \tilde{L}(f). \end{array} \right.$$

[20] Note that $\tilde{L}(t) = \epsilon$.

[21] Please note that the choice of $c \in \mathbb{K}^*$ is free. In order to deal later with indefinite summation, it will be essential to specify an appropriate value for this $c$ with respect to the given summation problem.

Properties (2.12), (2.13), (2.15), (2.16) and (2.17) can be shown similar as in the proof of Lemma 2.5.2. Additionally we have for $f = \sum_{j=0}^{m} f_j\, t^j \in \mathbb{A}[t]$ that

$$\begin{aligned}
\tilde{L}(\sigma(f)) = \tilde{L}(\sum_{j=0}^{m} \sigma(f_j)\,(t+\beta)^j) &= L(\sum_{j=0}^{m} t^j \sum_{i=j}^{m} \binom{i}{j} \sigma(f_i)\,\beta^{m-i}) \\
&= \max(L(\sum_{i=0}^{m} \binom{i}{0} \sigma(f_i)\,\beta^{m-i})), \dots, L(\sigma(f_m)), \epsilon) \\
&\overset{(2.12),(2.13)}{\leq} \max(L(\sigma(f_0)), \dots, L(\sigma(f_m)), L(\beta), \epsilon) \\
&\overset{(2.12)}{\leq} \max(L(f_0), \dots, L(f_m), \epsilon) = \tilde{L}(f).
\end{aligned}$$

By definition we have for all $f, g \in \mathbb{A}[t]$ and all $k \geq \max(\tilde{L}(f), \tilde{L}(g))$ that

$$\begin{aligned}
\tilde{\text{ev}}(f\,g, k) &= \tilde{\text{ev}}(f, k)\,\tilde{\text{ev}}(g, k), \\
\tilde{\text{ev}}(f + g, k) &= \tilde{\text{ev}}(f, k) + \tilde{\text{ev}}(g, k).
\end{aligned}$$

Since for $k \geq \tilde{L}(t) = \epsilon$ we have

$$\begin{aligned}
\tilde{\text{ev}}(t, k+1) = c + \sum_{j=\epsilon}^{k+1} \text{ev}(\sigma^{-1}(\beta), j) &= c + \sum_{j=\epsilon}^{k} \text{ev}(\sigma^{-1}(\beta), j) + \text{ev}(\sigma^{-1}(\beta), k+1) \\
&= \tilde{\text{ev}}(t, k) + \text{ev}(\beta, k) = \tilde{\text{ev}}(t + \beta, k) = \tilde{\text{ev}}(\sigma(t)),
\end{aligned}$$

it follows by $\tilde{L}(\sigma(f)) \leq \tilde{L}(f)$ as in the proof of Lemma 2.5.2, that for all $f \in \mathbb{A}[t]$ we have

$$\forall k \geq \tilde{L}(f) : \ \text{ev}(f, k+1) = \text{ev}(\sigma(f), k).$$

Consequently $\tilde{\text{ev}}$ is a homomorphic map bounded by $\tilde{L}$.

Now assume that $h$ is a difference ring monomorphism and that

$$h(t) \in \text{const}_S \mathcal{S}(\mathbb{K}),$$

i.e. there is a $\delta \geq \epsilon$ such that

$$\forall i \geq 0 : \ \tilde{ev}(t, \delta + i) = \tilde{\text{ev}}(t, \delta + i + 1).$$

Then

$$0 = \tilde{\text{ev}}(t, \delta + i + 1) - \tilde{\text{ev}}(t, \delta + i) = \tilde{\text{ev}}(\sigma(t) - t, \delta + i) = \tilde{\text{ev}}(\beta, \delta + i).$$

Consequently $\beta = 0$, a contradiction to the assumption that $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$. □

**Theorem 2.5.1.** *Let* $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ *be a ΠΣ-field,* $B \subseteq \mathbb{K}[t_1, \dots, t_n]^*$ *a multiplicative monoid and* $(\mathbb{F}, \sigma)$ *be a difference ring extension of* $(\mathbb{A}, \sigma) = (Q(\mathbb{K}[t_1, \dots, t_n], B), \sigma)$. *Assume there is a difference ring homomorphism*

$$h : \begin{cases} \mathbb{A} & \to \quad \mathcal{S}(\mathbb{K}) \\ f & \mapsto \quad \langle \text{ev}(f, 0), \text{ev}(f, 1), \dots \rangle \end{cases}$$

*for a homomorphic map* $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ *bounded by* $L : \mathbb{A} \to \mathbb{N}_0$*. Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F}, \sigma)$ *with*

$$\sigma(t) = \alpha\, t, \quad \alpha \in \mathbb{A}^*$$

*and where* $h(\alpha)$ *is a unit in* $\mathcal{S}(\mathbb{K})$*, or*

$$\sigma(t) = t + \beta, \quad \beta \in \mathbb{A}.$$

*Then there is a difference ring homomorphism*

$$h' : \begin{cases} \mathbb{A}[t] & \to \quad \mathcal{S}(\mathbb{K}) \\ f & \mapsto \quad \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$

*for a homomorphic map* $\mathrm{ev} : \mathbb{A}[t] \times \mathbb{N}_0 \to \mathbb{K}$ *bounded by* $L : \mathbb{A}[t] \to \mathbb{N}_0$*. If* $h$ *is even a difference ring monomorphism and all elements of* $h(\mathbb{A})$ *are units in* $h(\mathbb{A}[t])$ *then there is a difference ring monomorphism* $h'$ *with the above properties.*

*Proof.* By Lemmas 2.5.2 and 2.5.3 we get a homomorphic map $\mathrm{ev} : \mathbb{A}[t] \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : \mathbb{A}[t] \to \mathbb{N}_0$. Define the map

$$h : \begin{cases} \mathbb{A}[t] & \to \quad \mathcal{S}(\mathbb{K}) \\ f & \mapsto \quad \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}.$$

By Corollary 2.5.1 we get that $h$ is a difference ring homomorphism and therefore the first statement is proven.

Now assume that $h$ is a difference ring monomorphism and all elements in $\mathbb{A}$ are units in $\mathbb{A}[t]$. Clearly, $(h(\mathbb{A}[t]), S)$ is a difference ring extension of $(h(\mathbb{A}), S)$ and we have

$$(h(\mathbb{A}[t]), S) = (h(\mathbb{A})[x], S).$$

Since all elements of $h(\mathbb{A})$ are units in $h(\mathbb{A}[t])$ we may apply Proposition 2.4.7 and it follows that $(Q(h(\mathbb{A}))[x], S)$ is a difference ring extension of $(h(\mathbb{A})[x], S)$ with

$$S(x) - x = S(h(t)) - h(t) = h(\sigma(t)) - h(t) = h(\sigma(t) - t) = h(\beta), \text{ or}$$
$$S(x) = S(h(t)) = h(\sigma(t)) = h(\alpha\, t) = h(\alpha)\, h(t) = h(\alpha)\, x.$$

Furthermore $(Q(h(\mathbb{A}))[x], S)$ can be seen as a difference ring extension of $(Q(h(\mathbb{A})), \sigma)$. As $(\mathbb{A}, \sigma) \simeq (h(\mathbb{A}), \sigma)$ and the difference extension $(Q(\mathbb{A}), \sigma)$ of $(\mathbb{A}, \sigma)$ is uniquely defined by Lemma 2.4.5, it follows that

$$(Q(\mathbb{A}), \sigma) \simeq (Q(h(\mathbb{A})), S). \tag{2.21}$$

Since $(Q(\mathbb{A})(t), \sigma)$ is a $\Pi\Sigma$-extension of $(Q(\mathbb{A}), \sigma)$ and (2.21), we get by Lemma 2.4.2 that there is a $\Pi\Sigma$-extension $(Q(h(\mathbb{A}))(\tilde{t}), S)$ of $(Q(h(\mathbb{A})), S)$ with

$$S(\tilde{t}) = h(\alpha)\, \tilde{t} \qquad \textit{or} \qquad S(\tilde{t}) = \tilde{t} + h(\beta).$$

So we may apply Corollary 2.4.5 to the $\Pi\Sigma$-extension $(Q(h(\mathbb{A}))(\tilde{t}), S)$ of $(Q(h(\mathbb{A})), S)$ and the difference ring extension $(Q(h(\mathbb{A}))[x], S)$ of $(Q(h(\mathbb{A})), S)$. Therefore $x$ is transcendental over $Q(h(\mathbb{A}))$ and thus also transcendental over $h(\mathbb{A})$. Thus

$$h : \mathbb{A}[t] \to h(\mathbb{A})[x] = h(\mathbb{A}[t])$$

is a difference ring isomorphism and therefore

$$h : \mathbb{A}[t] \to \mathcal{S}(\mathbb{K})$$

is a difference ring monomorphism. □

### 2.5.2 Lifting to a Quotient Ring

**Proposition 2.5.2.** *Let $(\mathbb{F},\sigma)=(\mathbb{K}(t_1,\dots,t_n),\sigma)$ be a ΠΣ-field, $\mathbb{A}=\mathbb{K}[t_1,\dots,t_n]$ and $B\subseteq\mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F},\sigma)$ is a difference ring extension of $(Q(\mathbb{A},B),\sigma)$ for which there is a difference ring homomorphism $h:Q(\mathbb{A},B)\to\mathcal{S}(\mathbb{K})$. Let*

$$M=\{f\in\mathbb{A}\,|\,h(f)\text{ is a unit in }\mathcal{S}(\mathbb{K})\}.$$

*Then there is a uniquely defined difference ring $(Q(\mathbb{A},M),\sigma)$ such that*

$$(Q(\mathbb{A},B),\sigma)\leq(Q(\mathbb{A},M),\sigma)\leq(\mathbb{F},\sigma).$$

*Proof.* For any $f,g\in M$ we have that $h(f\,g)=h(f)\,h(g)$ is a unit in $\mathcal{S}(\mathbb{K})$ and thus $f\,g\in M$. Therefore $M$ is a multiplicative sub-monoid of $\mathbb{A}^*$ and thus we can construct the quotient ring $Q(\mathbb{A},M)$. As $h$ is a difference ring homomorphism, it follows that for all $f\in B$ we have $h(f),h(1/f)\in\mathcal{S}(\mathbb{K})$ with

$$h(f)\,h(\frac{1}{f})=h(1)=(1,1,1,\dots),$$

and thus $h(f)\in M$. Therefore

$$B\subseteq M$$

and consequently

$$Q(\mathbb{A},B)\leq Q(\mathbb{A},M)\leq\mathbb{F}.$$

Now consider

$$\tilde{M}:=\{f\in Q(\mathbb{A},B)\,|\,h(f)\text{ is a unit in }\mathcal{S}(\mathbb{K})\}.$$

As for $M$, it follows immediately that $\tilde{M}$ is a multiplicative sub-monoid of $Q(\mathbb{A},B)^*$. Since for all $f\in\tilde{M}$ we have that $h(\sigma(f))=S(h(f))$ and $h(\sigma^{-1}(f))=S^{-1}(h(f))$ are units in $\mathcal{S}(\mathbb{K})$, it follows immediately that $\sigma:\tilde{M}\to\tilde{M}$ is a monoid automorphism. Thus by Lemma 2.4.5, $(Q(Q(\mathbb{A},B),\tilde{M}),\sigma)$ is a uniquely defined difference ring extension of $(Q(\mathbb{A},B),\sigma)$ and it follows immediately that

$$(Q(\mathbb{A},B),\sigma)\leq(Q(Q(\mathbb{A},B),\tilde{M}),\sigma)\leq(\mathbb{F},\sigma).$$

Now let $q\in Q(Q(\mathbb{A},B),\tilde{M})$, i.e. $q=f/g$ where

$$\begin{aligned}f&=\frac{f_1}{f_2}\in Q(\mathbb{A},B), && f_1\in\mathbb{A},f_2\in B\\ g&=\frac{g_1}{g_2}\in\tilde{M}\subseteq Q(\mathbb{A},B), && g_1\in\mathbb{A},g_2\in B.\end{aligned}$$

As $h(g_2)$ and $h(g)$ are units in $\mathcal{S}(\mathbb{K})$, also $h(g_1)$ is a unit in $\mathcal{S}(\mathbb{K})$ and thus $g_1\in M$. Therefore $\tilde{g}=\frac{g_2}{g_1}\in Q(\mathbb{A},M)$ where

$$g\,\tilde{g}=1.$$

Thus $\frac{1}{g}\in Q(\mathbb{A},M)$, therefore $\frac{f}{g}\in Q(\mathbb{A},M)$ and consequently

$$Q(Q(\mathbb{A},B),\tilde{M})\subseteq Q(\mathbb{A},M).$$

On the other side we have

$$\mathbb{A}\subseteq Q(\mathbb{A},B),\quad M\subseteq\tilde{M}$$

and thus

$$Q(Q(\mathbb{A},B),\tilde{M})\supseteq Q(\mathbb{A},M).$$

Altogether we get $Q(Q(\mathbb{A},B),\tilde{M})=Q(\mathbb{A},M)$ and thus the proposition is proven. □

Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ be a $\Pi\Sigma$-field, $\mathbb{A} := \mathbb{K}[t_1, \dots, t_n]$ and $B \subseteq \mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F}, \sigma)$ is a difference ring extension of $(Q(\mathbb{A}, B), \sigma)$ for which there is a difference ring homomorphism

$$h : \begin{cases} Q(\mathbb{A}, B) & \to & \mathcal{S}(\mathbb{K}) \\ f & \mapsto & \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$

with $\mathrm{ev} : Q(\mathbb{A}, B) \times \mathbb{N}_0 \to \mathbb{K}$ being a homomorphic map bounded by $L : Q(\mathbb{A}, B) \to \mathbb{N}_0$. Let

$$M = \{f \in \mathbb{A} \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{K})\}.$$

In the following we will construct a difference ring homomorphism $\tilde{h} : Q(\mathbb{A}, M) \to \mathcal{S}(\mathbb{K})$ from a given difference ring monomorphism $h : Q(\mathbb{A}, B) \to \mathcal{S}(\mathbb{K})$ defined by an homomorphic map ev bounded by $L$. In order to achieve this construction we need to define a function $\tilde{Z} : M \to \mathbb{N}_0$ with

$$\tilde{Z}(f) := \begin{cases} 0 & \text{if } \nexists k \in \mathbb{N}_0 : \ \mathrm{ev}(f, k) = 0 \\ \max\{k + 1 \,|\, \mathrm{ev}(f, k) = 0\} & \text{otherwise} \end{cases}$$

and thereby the function $Z : M \to \mathbb{N}_0$ by

$$Z(f) := \max(\tilde{Z}(f), L(f)). \tag{2.22}$$

For this function we have the following properties.

**Lemma 2.5.4.** *Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ be a $\Pi\Sigma$-field, $\mathbb{A} := \mathbb{K}[t_1, \dots, t_n]$ and $B \subseteq \mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F}, \sigma)$ is a difference ring extension of $(Q(\mathbb{A}, B), \sigma)$ for which there is a difference ring homomorphism*

$$h : \begin{cases} Q(\mathbb{A}, B) & \to & \mathcal{S}(\mathbb{K}) \\ f & \mapsto & \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$

*with $\mathrm{ev} : Q(\mathbb{A}, B) \times \mathbb{N}_0 \to \mathbb{K}$ being a homomorphic map bounded by $L : Q(\mathbb{A}, B) \to \mathbb{N}_0$. Let*

$$M = \{f \in \mathbb{A} \,|\, h(f) \textit{ is a unit in } \mathcal{S}(\mathbb{K})\}.$$

*Then we have for the function $Z : M \to \mathbb{N}_0$ as defined in (2.22) the following properties:*

$$\forall f, g \in M : \ Z(f\, g) = \max(Z(f), Z(g)) \geq Z(f), \tag{2.23}$$

$$\forall f \in Q(\mathbb{A}, B) : \ L(f) \geq Z(\mathrm{den}(\sigma(f))). \tag{2.24}$$

*Proof.* Let $f, g \in M \subseteq \mathbb{K}[t_1, \dots, t_n]$. If $z := \tilde{Z}(f\, g) \geq \max(L(f), L(g))$ then

$$\forall k \geq z : \ \mathrm{ev}(f\, g, k) = \mathrm{ev}(f, k)\, \mathrm{ev}(g, k)$$

and thus

$$\tilde{Z}(f\, g) = \max(\tilde{Z}(f), \tilde{Z}(g));$$

so by $L(f\, g) = \max(L(f), L(g))$ it follows that

$$Z(f\, g) = Z(f)\, Z(g).$$

Otherwise, if $\tilde{Z}(f\, g) < \max(L(f), L(g)) =: l$ then

$$\forall k \geq l : \ \mathrm{ev}(f, k)\, \mathrm{ev}(g, k) = \mathrm{ev}(f\, g, k) \neq 0$$

and thus
$$\max(\tilde{Z}(f), \tilde{Z}(g)) \leq l;$$
therefore
$$\max(Z(f), Z(g)) = \max(\tilde{Z}(f), \tilde{Z}(f), L(f), L(g)) = \max(L(f), L(g)) = L(f\,g) = Z(f\,g).$$
Thus the first statement is proven. Now assume $f \in \mathbb{A}$. Then we have for all $k \geq L(f) \geq L(\sigma(f))$ that
$$\mathrm{ev}(\sigma(f), k) = \mathrm{ev}(\frac{\mathrm{num}(\sigma(f))}{\mathrm{den}(\sigma(f))}, k) = \frac{\mathrm{ev}(\mathrm{num}(\sigma(f)), k)}{\mathrm{ev}(\mathrm{den}(\sigma(f)), k)},$$
thus $\mathrm{ev}(\mathrm{den}(\sigma(f)), k) \neq 0$ for all $k \geq L(f)$, and consequently
$$L(f) \geq \tilde{Z}(\mathrm{den}(\sigma(f))).$$
Since
$$L(f) \overset{(2.14)}{\geq} L(\sigma(f)) \overset{(2.17)}{\geq} \max(L(\mathrm{num}(\sigma(f))), L(\mathrm{den}(\sigma(f)))), \tag{2.25}$$
it follows that
$$\forall f \in \mathbb{A}:\; L(f) \geq Z(\mathrm{den}(\sigma(f))). \tag{2.26}$$
Let $f \in B$ and $k \geq Z(f)$. Since
$$k \geq Z(f) \geq L(f) \geq L(\sigma(f)),$$
it follows that
$$0 \neq \mathrm{ev}(f, k+1) = \mathrm{ev}(\sigma(f), k) = \mathrm{ev}(\frac{\mathrm{num}(\sigma(f))}{\mathrm{den}(\sigma(f))}, k) = \frac{\mathrm{ev}(\mathrm{num}(\sigma(f)), k)}{\mathrm{ev}(\mathrm{den}(\sigma(f)), k)},$$
and hence
$$\mathrm{ev}(\mathrm{num}(\sigma(f)), k) \neq 0.$$
Since (2.25), it follows that
$$\forall f \in B:\; Z(f) \geq Z(\mathrm{num}(\sigma(f))). \tag{2.27}$$
Now let $f = \frac{f_1}{f_2} \in Q(\mathbb{A}, B)$ with $\gcd(f_1, f_2) = 1$. Since $\mathrm{den}(\sigma(f_1)), \mathrm{num}(\sigma(f_2)) \in B$, we have
$$\begin{aligned} Z(\mathrm{den}(\sigma(f))) &\overset{(2.23)}{\leq} Z(\mathrm{num}(\sigma(f_2))\,\mathrm{den}(\sigma(f_1))) \overset{(2.23)}{=} \max(Z(\mathrm{num}(\sigma(f_2))), Z(\mathrm{den}(\sigma(f_1)))) \\ &\overset{(2.27),(2.26)}{\leq} \max(Z(f_2), L(f_1)) \leq \max(L(f_1), L(f_2), Z(f_2)) = L(f) \end{aligned}$$
and thus the second statement is proven. □

**Lemma 2.5.5.** *Let* $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ *be a* ΠΣ*-field,* $\mathbb{A} := \mathbb{K}[t_1, \dots, t_n]$ *and* $B \subseteq \mathbb{A}^*$ *be a multiplicative monoid. Assume that* $(\mathbb{F}, \sigma)$ *is a difference ring extension of* $(Q(\mathbb{A}, B), \sigma)$ *for which there is a difference ring homomorphism*
$$h : \begin{cases} Q(\mathbb{A}, B) & \to \quad \mathcal{S}(\mathbb{K}) \\ f & \mapsto \quad \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle \end{cases}$$
*with* $\mathrm{ev} : Q(\mathbb{A}, B) \times \mathbb{N}_0 \to \mathbb{K}$ *being a homomorphic map bounded by* $L : Q(\mathbb{A}, B) \to \mathbb{N}_0$*. Let*
$$M = \{f \in \mathbb{A} \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{K})\}.$$
*Then there are maps* $\mathrm{ev} : Q(\mathbb{A}, M) \times \mathbb{N}_0 \to \mathbb{K}$ *and* $L : Q(\mathbb{A}, M) \to \mathbb{N}_0$ *such that* ev *is a homomorphic map bounded by* $L$.

*Proof.* Define the function $Z : M \to \mathbb{N}_0$ as in (2.22), $\tilde{L} : Q(\mathbb{A}, M) \to \mathbb{N}_0$ by

$$\tilde{L}(\frac{a}{b}) = \max(L(a), L(b), Z(b))$$

and $\tilde{\mathrm{ev}} : Q(\mathbb{A}, M) \to Q(\mathbb{A}, M)$ by

$$\tilde{\mathrm{ev}}(\frac{a}{b}, k) := \begin{cases} 0 & \text{if } k < \tilde{L}(a/b) \\ \frac{\mathrm{ev}(a,k)}{\mathrm{ev}(b,k)} & \text{if } k \geq \tilde{L}(a/b) \end{cases}$$

where $a \in \mathbb{A}$, $b \in M$ with $\gcd(a, b) = 1$. Clearly, $\tilde{L}$ and $\tilde{\mathrm{ev}}$ are well defined functions. If $b \in \mathbb{K}$ then $Z(b) = L(b) = 0$, and therefore $\tilde{L}(a/b) = L(a/b)$. Thus properties (2.15) and (2.16) hold. For $f = \frac{f_1}{f_2}, g = \frac{g_1}{g_2} \in Q(\mathbb{A}, M)$ with $f_i, g_i \in \mathbb{A}$ and $\gcd(f_1, f_2) = \gcd(g_1, g_2) = 1$ we have

$$\begin{aligned}
\tilde{L}(f + g) =& \max(L(\mathrm{num}(f + g)), L(\mathrm{den}(f + g)), Z(\mathrm{den}(f + g))) \\
&\overset{(2.15)}{\leq} \max(L(f_1\, g_2 + f_2\, g_1), L(f_2\, g_2), Z(f_2\, g_2)) \\
&\overset{(2.12),(2.13),(2.23)}{\leq} \max(L(f_1), L(f_2), Z(f_2), L(g_1), L(g_2), Z(g_2)) = \max(L(f), L(g)), \\
\tilde{L}(f\, g) =& \max(L(\mathrm{num}(f\, g)), L(\mathrm{den}(f\, g)), Z(\mathrm{den}(f\, g))) \\
&\overset{(2.15),(2.23)}{\leq} \max(L(f_1\, g_1), L(f_2\, g_2), Z(f_2\, g_2)) \\
&\overset{(2.12),(2.23)}{\leq} \max(L(f_1), L(f_2), Z(f_2), L(g_1), L(g_2), Z(g_2)) = \max(L(f), L(g)), \\
\tilde{L}(\sigma(f)) =& \max(L(\mathrm{num}(\sigma(f))), L(\mathrm{den}(\sigma(f))), Z(\mathrm{den}(\sigma(f)))) \\
&\overset{(2.15)}{\leq} \max(L(\mathrm{num}(\sigma(f_1))\, \mathrm{den}(\sigma(f_2))), L(\mathrm{den}(\sigma(f_1))\, \mathrm{num}(\sigma(f_2))), Z(\mathrm{den}(\sigma(f)))) \\
&\overset{2.15)}{=} \max( L(\mathrm{num}(\sigma(f_1))), L(\mathrm{den}(\sigma(f_2))), \\
&\qquad L(\mathrm{den}(\sigma(f_1))), L(\mathrm{num}(\sigma(f_2))), Z(\mathrm{den}(\sigma(f)))) \\
&\overset{(2.17),(2.24)}{\leq} \max(L(\sigma(f_1)), L(\sigma(f_2)), L(f)) \leq \max(L(f_1), L(f_2), Z(f_2), L(f)) \\
&= L(f).
\end{aligned}$$

Property (2.17) follows immediately by the definition of $L$. Again by definition, we have for all $f, g \in Q(\mathbb{A}, M)$ and all $k \geq \max(\tilde{L}(f), \tilde{L}(g))$ that

$$\begin{aligned}
\tilde{\mathrm{ev}}(f\, g, k) &= \tilde{\mathrm{ev}}(f, k)\, \tilde{\mathrm{ev}}(g, k), \\
\tilde{\mathrm{ev}}(f + g, k) &= \tilde{\mathrm{ev}}(f, k) + \tilde{\mathrm{ev}}(g, k).
\end{aligned}$$

Finally, let $f = \frac{f_1}{f_2} \in Q(\mathbb{A}, M)$ with $f_i \in \mathbb{K}[t_1, \ldots, t_n]$, $\gcd(f_1, f_2) = 1$ and let $k \geq \tilde{L}(f)$. Since

$$\begin{aligned}
k \geq& \tilde{L}(f) \geq \max(L(f_1), L(f_2)), \\
\forall f \in \mathbb{A} :\ & L(f) \overset{(2.14)}{\geq} L(\sigma(f)) \overset{(2.17)}{\geq} \max(L(\mathrm{num}(\sigma(f))), L(\mathrm{den}(\sigma(f)))),
\end{aligned} \tag{2.28}$$

we have

$$\begin{aligned}
\tilde{\mathrm{ev}}(f, k + 1) =& \frac{\mathrm{ev}(f_1, k + 1)}{\mathrm{ev}(f_2, k + 1)} = \frac{\mathrm{ev}(\sigma(f_1), k)}{\mathrm{ev}(\sigma(f_2), k)} \\
=& \frac{\mathrm{ev}(\mathrm{num}(\sigma(f_1)), k)/\mathrm{ev}(\mathrm{den}(\sigma(f_1)), k)}{\mathrm{ev}(\mathrm{num}(\sigma(f_2)), k)/\,\mathrm{ev}(\mathrm{den}(\sigma(f_2)), k)} = \frac{\mathrm{ev}(\mathrm{num}(\sigma(f_1)), k)\,\mathrm{ev}(\mathrm{den}(\sigma(f_2)), k)}{\mathrm{ev}(\mathrm{num}(\sigma(f_2)), k)\mathrm{ev}(\mathrm{den}(\sigma(f_1)), k)} \\
\overset{(2.28)}{=}& \frac{\mathrm{ev}(\mathrm{num}(\sigma(f_1))\,\mathrm{den}(\sigma(f_2)), k)}{\mathrm{ev}(\mathrm{num}(\sigma(f_2))\,\mathrm{ev}(\mathrm{den}(\sigma(f_1)), k)} = \mathrm{ev}(\frac{\sigma(f_1)}{\sigma(f_2)}, k) = \mathrm{ev}(\sigma(f), k).
\end{aligned}$$

Consequently $\tilde{\mathrm{ev}}$ is a homomorphic map bounded by $\tilde{L}$. □

**Theorem 2.5.2.** *Let $(\mathbb{F},\sigma) = (\mathbb{K}(t_1,\dots,t_n),\sigma)$ be a ΠΣ-field, $\mathbb{A} := \mathbb{K}[t_1,\dots,t_n]$ and $B \subseteq \mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F},\sigma)$ is a difference ring extension of $(Q(\mathbb{A},B),\sigma)$ for which there is a difference ring homomorphism*

$$h : \begin{cases} Q(\mathbb{A},B) & \rightarrow \quad \mathcal{S}(\mathbb{K}) \\ f & \mapsto \quad \langle \mathrm{ev}(f,0), \mathrm{ev}(f,1), \dots \rangle \end{cases}$$

*with* $\mathrm{ev} : Q(\mathbb{A},B) \times \mathbb{N}_0 \to \mathbb{K}$ *being a homomorphic map bounded by* $L : Q(\mathbb{A},B) \to \mathbb{N}_0$*. Let*

$$M = \{f \in \mathbb{A} \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{K})\}.$$

*Then there is a difference ring homomorphism*

$$h' : \begin{cases} Q(\mathbb{A},M) & \rightarrow \quad \mathcal{S}(\mathbb{K}) \\ f & \mapsto \quad \langle \mathrm{ev}(f,0), \mathrm{ev}(f,1), \dots \rangle \end{cases}$$

*for a homomorphic map* $\mathrm{ev} : Q(\mathbb{A},M) \times \mathbb{N}_0 \to \mathbb{K}$ *bounded by* $L : Q(\mathbb{A},M) \to \mathbb{N}_0$*. Furthermore, if $h$ is a difference ring monomorphism then there is a difference ring monomorphism $h'$ with the above properties.*

*Proof.* By Lemma 2.5.5 there is a homomorphic map $\tilde{\mathrm{ev}} : Q(\mathbb{A},M) \times \mathbb{N}_0 \to \mathbb{K}$ bounded by the map $\tilde{L} : Q(\mathbb{A},M) \to \mathbb{N}_0$ with

$$\forall f \in \mathbb{A} : \ \tilde{L}(f) = L(f)$$

and

$$\forall f \in \mathbb{A}\ \forall k \geq 0 : \ \tilde{\mathrm{ev}}(f,k) = \mathrm{ev}(f,k).$$

Therefore by Corollary 2.5.1 there is a difference ring homomorphism

$$h : \begin{cases} Q(\mathbb{A}[t],M) & \rightarrow \quad \mathcal{S}(\mathbb{K}) \\ f & \mapsto \quad \langle \mathrm{ev}(f,0), \mathrm{ev}(f,1), \dots \rangle. \end{cases}$$

If $h : \mathbb{A}[t] \to \mathcal{S}(\mathbb{K})$ is even a difference ring monomorphism, it follows directly that

$$h : Q(\mathbb{A}[t],M) \to \mathcal{S}(\mathbb{K})$$

is also a difference ring monomorphism. □

### 2.5.3 ΠΣ-Fields and Indefinite Summation

**Example 2.5.5.** In the following we indicate how one translate the indefinite summation problem

$$F_n^{(1)} := \sum_{m=1}^{n}\sum_{k=2}^{m}\frac{\sum_{i=1}^{k}\frac{1}{i}}{k^2-1}$$

into an expression from a ΠΣ-field $(\mathbb{Q}(t_1,t_2,t_3),\sigma)$ and how one defines the corresponding difference ring homomorphism $h:(\mathbb{Q}(t_1)[t_2,t_3],\sigma)\to(\mathcal{S}(\mathbb{K}),S)$.

| | |
|---|---|
| $S_nF_n^{(1)} = F_n^{(1)} + S_n\underbrace{\boxed{\sum_{k=2}^{n}\frac{\sum_{i=1}^{k}\frac{1}{i}}{k^2-1}}}_{:=F_n^{(2)}}$<br>$\downarrow$ | We have<br>$\boxed{\sigma(w) = w + \sigma(t_3)}$<br>and<br>$\forall n \geq 2:\ h(w)_n = F_n^{(1)}$<br>$\uparrow$ |
| $\downarrow$<br>$S_nF_n^{(2)} = F_n^{(2)} + \frac{S_n\overbrace{\boxed{\sum_{i=1}^{n}\frac{1}{i}}}^{:=F_n^{(3)}}}{(n+1)^2-1}$<br>$\downarrow$ | $\uparrow$<br>Define ΠΣ-field $(\mathbb{Q}(t_1,t_2,t_3),\sigma)$ with<br>$\boxed{\sigma(t_3) = t_3 + \frac{\sigma(t_2)}{(t_1+1)^2-1}}$<br>and homomorphism $h:(\mathbb{F}(t_1)[t_2,t_3],\sigma)\to(\mathcal{S}(\mathbb{K}),S),$ s.t.<br>$\forall\, n \geq 2:\ h(t_3)_n = F_n^{(2)}$<br><br>We have<br>$\exists w \in \mathbb{Q}(t_1,t_2,t_3):\sigma(w)-w=\sigma(t_3),$<br>i.e. $w := -\frac{-1-t_1-t_2+t_1t_2^2+t_1^2t_2^2-2t_1^2t_3-2t_1^3t_3}{2(t_1(1+t_1))}$<br>$\uparrow$ |
| $\downarrow$<br>$S_nF_n^{(3)} = F_n^{(3)} + \frac{1}{\boxed{n}+1}$<br>$\downarrow$ | $\uparrow$<br>Define ΠΣ-field $(\mathbb{Q}(t_1,t_2),\sigma)$ with<br>$\boxed{\sigma(t_2) = t_2 + \frac{1}{t_1+1}}$<br>and monomorphism $h:(\mathbb{F}(t_1)[t_2],\sigma)\to(\mathcal{S}(\mathbb{K}),S),$ s.t.<br>$\forall\, n \geq 1:\ h(t_2)_n = F_n^{(3)}$<br><br>We have<br>$\nexists w \in \mathbb{Q}(t_1,t_2):\sigma(w)-w=\frac{\sigma(t_2)}{(t_1+1)^2-1}$<br>$\uparrow$ |
| $\downarrow$<br>$S_nn = n+1$<br>$\downarrow$ | $\uparrow$<br>Define ΠΣ-field $(\mathbb{Q}(t_1),\sigma)$ with<br>$\boxed{\sigma(t_1) = t_1 + 1}$<br>and monomorphism $h:(\mathbb{F}(t_1),\sigma)\to(\mathcal{S}(\mathbb{K}),S),$ s.t.<br>$\forall\, n \geq 0:\ h(t_1)_n = n$<br><br>We have<br>$\nexists w \in \mathbb{Q}(t_1):\sigma(w)-w=\frac{1}{t_1+1}$<br>$\uparrow$ |
| $\downarrow$<br>$\rightarrow$ | $\uparrow$<br>Let $(\mathbb{Q},\sigma)$ with<br>$\forall k \in \mathbb{K}\ \sigma(k) = k$<br>$\rightarrow$<br>We have<br>$\nexists w \in \mathbb{Q}:\sigma(w)-w=1$ |

◊

**Implementation Note 2.5.1.** In functions like `SigmaReduce`, `GenerateRecurrence`, `Find-SumSolutions` or `SolveRecurrence`, introduced in Chapter 1, input expressions in terms of nested sums and products are tried to be transformed to expressions from a ΠΣ-field. Besides this, as sketched in the previous example, a difference ring homomorphism from a sub-difference ring of the ΠΣ-field to the ring of sequences is constructed. After solving a summation problem in terms of this ΠΣ-field, the result is translated back by this difference ring homomorphism to a corresponding solution in the ring of sequences. The following subsections describe in more details how this translation process works and actually is implemented.

**Lemma 2.5.6.** *Let a ΠΣ-field $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ and $\mathbb{F}=\mathbb{K}(t_1,\dots,t_n)$ be a difference ring extension of $(\mathbb{A},\sigma)$ where $\mathbb{K}[t_1,\dots,t_n]\subseteq\mathbb{A}$ and assume there is a difference ring homomorphism $h:\mathbb{A}\to\mathcal{S}(\mathbb{K})$. Let*

$$M=\{f\in\mathbb{K}[t_1,\dots,t_n]\,|\,h(f)\text{ is a unit in }\mathcal{S}(\mathbb{K})\}.$$

*Then for all $p,q\in M$ and $s,r,\in\mathbb{K}[t_1,\dots,t_n]$ we have*

$$\gcd(p,r)\in M,\tag{2.29}$$
$$\operatorname{lcm}(p,q)\in M,\tag{2.30}$$
$$\operatorname{lcm}(r,s)\in M\Rightarrow r\in M,\tag{2.31}$$
$$\gcd(r,s)\in M\Rightarrow\gcd(r\,p,s)\in M.\tag{2.32}$$

*Proof.* Let $\mathbb{A}':=\mathbb{K}[t_1,\dots,t_n]$, $p,q\in M$ and $r,s\in\mathbb{A}'$. There exists a $p'\in\mathbb{A}'$ with $p=\gcd(p,r)\,p'$ and thus

$$h(p)=h(\gcd(p,r))\,h(p').$$

Since $h(p)$ is a unit in $\mathcal{S}(\mathbb{K})$, it follows that also $h(\gcd(p,r))$ is a unit in $\mathcal{S}(\mathbb{K})$ and therefore $\gcd(p,r)\in M$. Consequently (2.29) is proven. In particular we have $\gcd(p,q)\in\mathbb{A}$ and thus by $\operatorname{lcm}(p,q)=p\,\gcd(p,q)$ it follows immediately that also $\operatorname{lcm}(p,q)\in M$ and therefore (2.30) is proven. If $\operatorname{lcm}(r,s)\in M$ then we may write

$$\operatorname{lcm}(r,s)=r\,r'$$

for some $r'\in\mathbb{A}'$ and thus

$$h(\operatorname{lcm}(r,s))=h(r)\,h(r').$$

As $h(\operatorname{lcm}(r,s))$ is a unit in $\mathcal{S}(\mathbb{K})$, it follows immediately that $h(r)$ is a unit in $\mathcal{S}(\mathbb{K})$ and thus $r\in M$ which proves (2.31). We have

$$\gcd(r\,p,s)=\gcd(\gcd(r,s)\ \gcd(p,s),s).$$

By (2.29) we get $\gcd(p,s)\in M$ and if we additionally assume $\gcd(r,s)\in M$, it follows directly that $\gcd(r\,p,s)\in M$ and consequently (2.32) is proven. □

#### 2.5.3.1 The Sum Case

**Proposition 2.5.3.** *Let $(\mathbb{F},\sigma)=(\mathbb{K}(t_1,\dots,t_n),\sigma)$ be a ΠΣ-field, $\mathbb{A}:=\mathbb{K}[t_1,\dots,t_n]$ and $B\subseteq\mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F},\sigma)$ is a difference ring extension of $(Q(\mathbb{A},B),\sigma)$ for which there is a difference ring homomorphism $h:Q(\mathbb{A},B)\to\mathcal{S}(\mathbb{K})$. Let*

$$M=\{f\in\mathbb{A}\,|\,h(f)\text{ is a unit in }\mathcal{S}(\mathbb{K})\}$$

*and assume $\beta \in Q(\mathbb{A}, M)$. If there exists a $g \in \mathbb{F}$ with*

$$\sigma(g) - g = \beta \quad \textit{and} \quad \gcd(\operatorname{den}(g), \operatorname{den}(\sigma(g))) \in M$$

*then $g \in Q(\mathbb{A}, M)$.*

*Proof.* Let $\beta = \frac{a}{b}$ with $a \in \mathbb{A}$, $b \in M$, $\gcd(a, b) = 1$ and $g = \frac{c}{d}$ with $c, d \in \mathbb{A}$ and $\gcd(c, d) = 1$. Let

$$\begin{aligned} c' &= \sigma(c)\, d/p + c\, \sigma(d)/p \\ d' &= \operatorname{lcm}(d, \sigma(d)) \end{aligned}$$

for $p = \gcd(d, \operatorname{num}(\sigma(d)))$. We have

$$\frac{a}{b} = \beta = \sigma(g) - g = \frac{c'}{d'}$$

By [Win96, Theorem 2.3.1] it follows that $\gcd(c', d') = \gcd(c', p)$ and thus

$$b = \frac{d'}{\gcd(c', p)}\, k$$

for some $k \in \mathbb{K}$. By assumption we have $p \in M$ and therefore by Lemma 2.5.6,(2.29), $\gcd(c', p) \in M$. As

$$h(b)\, h(\gcd(c', p)) = h(d')\, h(k),$$

$h(d')$ is a unit in $\mathcal{S}(\mathbb{K})$, i.e. $d' = \operatorname{lcm}(\sigma(d), d) \in M$. By Lemma 2.5.6,(2.31), it follows that $d \in M$ and thus $g = \frac{c}{d} \in Q(\mathbb{A}, M)$. $\square$

### The Sum Case

Consider the sequence $\langle *, \dots, *, F_\delta, F_{\delta+1}, \dots \rangle \in \mathcal{S}(\mathbb{K})$ with

$$\boxed{F_{n+1} = F_n + \frac{C_n}{D_n}}$$

for some sequences $C, D \in \mathcal{S}(\mathbb{K})$.

Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ be a ΠΣ-field, $\mathbb{A} := \mathbb{K}[t_1, \dots, t_n]$ and $B \subseteq \mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F}, \sigma)$ is a difference ring extension of $(Q(\mathbb{A}, B), \sigma)$ for which there is a difference ring homomorphism $h : Q(\mathbb{A}, B) \to \mathcal{S}(\mathbb{K})$ with $\mathrm{ev} : Q(\mathbb{A}, B) \times \mathbb{N}_0 \to \mathbb{K}$ being a homomorphic map bounded by $L : Q(\mathbb{A}, B) \to \mathbb{N}_0$ such that for suitable $c, d \in Q(\mathbb{A}, B)$ we have

$$\boxed{\forall n \geq L(c) : \quad h(c)_n = C_n} \qquad \boxed{\forall n \geq L(d) : \quad h(d)_n = D_n \neq 0.}$$

For

$$M = \{f \in \mathbb{A} \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{K})\}$$

Theorem 2.5.2 provides a ring homomorphism $h : Q(\mathbb{A}, M) \to \mathcal{S}(\mathbb{K})$ for a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : Q(\mathbb{A}, M) \to \mathbb{N}_0$. Especially, we get $d \in M$. Consequently for $\beta := \frac{c}{d} \in Q(\mathbb{A}, M)$ we have

$$\boxed{\forall n \geq L(\beta) : \quad h(\beta)_n = \frac{C_n}{D_n}.}$$

**Case 1:** There does not exist a $g \in \mathbb{F}$ such that

$$\sigma(g) - g = \beta.$$

Then there is a unique proper sum extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$\sigma(t) = t + \beta$$

and $\mathrm{const}_\sigma \mathbb{F}(t) = \mathbb{K}$ (Proposition 2.4.1) and a ring homomorphism $h : Q(\mathbb{A}, M)[t] \to \mathcal{S}(\mathbb{K})$ for a homomorphic map $\mathrm{ev} : Q(\mathbb{A}, M)[t] \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : Q(\mathbb{A}, M)[t] \to \mathbb{N}_0$ (Theorem 2.5.1). In addition, as already indicated in the proof of Lemma 2.5.3, the $c$ in (2.20) can be adjusted such that

$$\boxed{\forall n \geq \max(L(t), \delta) : \quad F_n = h(t)_n.}$$

**Case 2:** There exists a $g \in \mathbb{F}$ such that

$$\sigma(g) - g = \beta.$$

If $\gcd(\mathrm{den}(g), \sigma(\mathrm{den}(g))) \notin M$, STOP. Otherwise it follows by Proposition 2.5.3 that $g \in Q(\mathbb{A}, M)$ and thus there is a $k \in \mathbb{K}$ such that

$$\boxed{\forall n \geq \max(L(g), \delta) : \quad F_n = h(g + k)_n.}$$

#### 2.5.3.2 The Product Case

**Proposition 2.5.4.** *Let $(\mathbb{F},\sigma) = (\mathbb{K}(t_1,\dots,t_n),\sigma)$ be a $\Pi\Sigma$-field, $\mathbb{A} := \mathbb{K}[t_1,\dots,t_n]$ and $B \subseteq \mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F},\sigma)$ is a difference ring extension of $(Q(\mathbb{A},B),\sigma)$ for which there is a difference ring homomorphism $h: Q(\mathbb{A},B) \to \mathcal{S}(\mathbb{K})$. Let*

$$M = \{f \in \mathbb{A} \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{K})\}$$

*and let $\alpha \in Q(\mathbb{A},M)$ with $\operatorname{num}(\alpha) \in M$. If there exists a $g \in \mathbb{F}$ with*

$$\sigma(g) = \alpha\, g \quad \text{and} \quad \gcd(\operatorname{den}(g), \operatorname{num}(\sigma(\operatorname{den}(g)))) \in M$$

*then $g \in Q(\mathbb{A},M)$.*

*Proof.* Let $g = \frac{g_1}{g_2}$ where $g_1, g_2 \in \mathbb{A}$ and $\gcd(g_1,g_2) = 1$. Furthermore let $\alpha = \frac{\alpha_1}{\alpha_2}$ with $\alpha_1,\alpha_2 \in M$ and $\gcd(\alpha_1,\alpha_2) = 1$. We have

$$\frac{\alpha_1\, g_1}{\alpha_2\, g_2} = \alpha\, g = \sigma(g) = \sigma(\frac{g_1}{g_2}) = \frac{\operatorname{num}(\sigma(g_1))\operatorname{den}(\sigma(g_2))}{\operatorname{num}(\sigma(g_2))\operatorname{den}(\sigma(g_1))}$$

where by Lemma 2.5.6.(2.32) we get

$$\gcd(\alpha_1\, g_1, \alpha_2\, g_2) \in M.$$

Thus there are a $u \in M$ and a $v \in \mathbb{A}$ such that

$$\alpha_2\, g_2\, v = \operatorname{num}(\sigma(g_2))\operatorname{den}(\sigma(g_1))\, u.$$

Since

$$\gcd(g_2, \operatorname{num}(\sigma(g_2))) = \gcd(\operatorname{den}(g), \operatorname{num}(\sigma(\operatorname{den}(g)))) \in M,$$

$\alpha_2 \in M$, $u \in M$ and $\operatorname{den}(\sigma(g_1)) \in M$ it follows again by (2.32) that

$$\gcd(g_2\, \alpha_2, \operatorname{num}(\sigma(g_2)\operatorname{den}(\sigma(g_1))\, u) \in M.$$

And as

$$g_2\, \alpha_2 \mid \operatorname{num}(\sigma(g_2)\operatorname{den}(\sigma(g_1))\, u$$

we get $g_2\, \alpha \in M$, thus $g_2 \in M$ and consequently $g \in Q(\mathbb{A},M)$. □

**Proposition 2.5.5.** *Let $(\mathbb{F},\sigma) = (\mathbb{K}(s_1,\dots,s_m,t_1,\dots,t_n),\sigma)$ be a $\Pi\Sigma$-field where for all $\Pi$-extensions $t_i$ we have*

$$\sigma(t_i) = \alpha_i\, t_i$$

*for $\alpha_i \in \mathbb{K}(s_1,\dots,s_m)$. Let $\mathbb{A} := \mathbb{K}[s_1,\dots,s_m,t_1,\dots,t_n]$ and $B \subseteq \mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F},\sigma)$ is a difference ring extension of $(Q(\mathbb{A},B),\sigma)$ for which there is a difference ring homomorphism $h: Q(\mathbb{A},B) \to \mathcal{S}(\mathbb{K})$ with $h(t_i)$ and $h(\alpha_i)$ are units in $\mathcal{S}(\mathbb{K})$ for all $\Pi$-extensions $t_i$. Let*

$$M = \{f \in \mathbb{A} \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{K})\}$$

*and let $\alpha \in Q(\mathbb{A},M)$ with $\operatorname{num}(\alpha) \in M$. If there exists a $g \in \mathbb{F}$ with*

$$\sigma(g) = \alpha\, g$$

*then $g = w\, t_1^{k_1} \cdots t_n^{k_n}$ for some $w \in \mathbb{K}(s_1,\dots,s_m)$ and $k_i \in \mathbb{Z}$. Furthermore if*

$$\gcd(\operatorname{num}(\sigma(\operatorname{den}(w))), \operatorname{den}(w)) \in M$$

*then $g \in Q(\mathbb{A},M)$.*

*Proof.* Assume
$$\sigma(g) = \alpha\, g$$
for some $g \in \mathbb{F}$. By Corollary 2.2.6 it follows that
$$g = w\, t_1^{k_1} \cdots t_n^{k_n}$$
where $w \in \mathbb{K}(s_1, \dots, s_m)$ and $k_i = 0$ if $t_i$ is a Σ-extension and $k_i \in \mathbb{Z}$ if $t_i$ is a Π-extension. By assumption it follows that $t_i \in M$ and $\alpha_i \in Q(\mathbb{A}, M)$ with $\mathrm{num}(\alpha_i) \in M$ if $t_i$ is a Π-extension. Therefore we get
$$\frac{\sigma(g)}{g} = \frac{\sigma(w)}{w}\, u = \alpha$$
for some $u \in Q(\mathbb{A}, M)$ with $\mathrm{num}(u) \in M$. Thus we may apply Proposition 2.5.4 and obtain that $g \in Q(\mathbb{A}, M)$, if $\gcd(\mathrm{den}(w), \mathrm{den}(\sigma(w))) \in M$. □

### The Product Case

Consider the sequence $\langle *, \dots, *, F_\delta, F_{\delta+1}, \dots\rangle \in \mathcal{S}(\mathbb{K}) \setminus \{\mathbf{0}\}$ with

$$\boxed{F_{n+1} = \frac{C_n}{D_n} F_n}$$

for some sequences $C, D \in \mathcal{S}(\mathbb{K})$.

Let $(\mathbb{F}, \sigma) = (\mathbb{K}(t_1, \dots, t_n), \sigma)$ be a $\Pi\Sigma$-field, $\mathbb{A} := \mathbb{K}[t_1, \dots, t_n]$ and $B \subseteq \mathbb{A}^*$ be a multiplicative monoid. Assume that $(\mathbb{F}, \sigma)$ is a difference ring extension of $(Q(\mathbb{A}, B), \sigma)$ for which there is a difference ring homomorphism $h : Q(\mathbb{A}, B) \to \mathcal{S}(\mathbb{K})$ with $\mathrm{ev} : Q(\mathbb{A}, B) \times \mathbb{N}_0 \to \mathbb{K}$ being a homomorphic map bounded by $L : Q(\mathbb{A}, B) \to \mathbb{N}_0$ such that for suitable $c, d \in Q(\mathbb{A}, B)$ we have

$$\boxed{\forall n \geq L(c) : \quad h(c)_n = C_n \neq 0} \qquad \boxed{\forall n \geq L(d) : \quad h(d)_n = D_n \neq 0.}$$

For

$$M = \{f \in \mathbb{A} \,|\, h(f) \text{ is a unit in } \mathcal{S}(\mathbb{K})\}$$

Theorem 2.5.2 provides a ring homomorphism $h : Q(\mathbb{A}, M) \to \mathcal{S}(\mathbb{K})$ for a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : Q(\mathbb{A}, M) \to \mathbb{N}_0$. In particular, we have $c, d \in M$. Thus for $\alpha := \frac{c}{d}$ we get

$$\boxed{\forall n \geq L(\alpha) : \quad h(\alpha)_n = \frac{C_n}{D_n}.}$$

**Case 1:** There does not exist an $n > 0$:

$$\alpha^n := \left(\frac{c}{d}\right)^n \in \mathrm{H}_{(Q(\mathbb{A}),\sigma)}.$$

Then there is a unique $\Pi$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$\sigma(t) = \alpha\, t$$

and $\mathrm{const}_\sigma \mathbb{F}(t) = \mathbb{K}$ (Proposition 2.4.2) and a ring homomorphism $h : Q(\mathbb{A}, M)[t] \to \mathcal{S}(\mathbb{K})$ for a homomorphic map $\mathrm{ev} : Q(\mathbb{A}, M)[t] \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : Q(\mathbb{A}, M)[t] \to \mathbb{N}_0$ (Theorem 2.5.1). In addition, as already indicated in the proof of Lemma 2.5.2, the $c$ in (2.19) can be adjusted such that

$$\boxed{\forall n \geq \max(L(t), \delta) : \quad F_n = h(t)_n.}$$

**Case 2:** There exists a $g \in \mathbb{F}$ such that

$$\sigma(g) = \alpha\, g.$$

If[22] $\gcd(\mathrm{den}(g), \mathrm{num}(\sigma(\mathrm{den}(g)))) \notin M$, STOP. Otherwise it follows that $g \in Q(\mathbb{A}, M)$ by Proposition 2.5.4 and thus there is a $k \in \mathbb{K}$ with

$$\boxed{\forall n \geq \max(L(g), \delta) : \quad F_n = h(k\, g)_n.}$$

**Case 3:** OTHERWISE STOP

[22]See Corollary 2.5.5 for a simplification of the test.

# Chapter 3

# Solving Difference Equations

As already indicated in Section 1.4 we are mainly interested in solving linear difference equations in $\Pi\Sigma$-fields in order to handle definite and indefinite summation problems. In this chapter the main ideas and main algorithm will be described how one can solve difference equations. For some cases we even try to solve difference equations in difference rings although there are still a lot of open problems. But even in the $\Pi\Sigma$-field case not all problems are completely solved yet. If one runs in such problematic cases we provide heuristic methods to find all solutions.

Whereas in the first section the main ideas of the reduction strategy will be presented, in the remaining sections of this chapter the different parts of the reduction process will be considered in details.

All these steps and ideas in this chapter leads to an "algorithm" which I have implemented in the computer algebra system Mathematica.

## 3.1 The Reduction Strategy

### 3.1.1 The Solution Space

Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$. We are interested in the following problem:

- GIVEN $a_1, \dots, a_m \in \mathbb{A}$ with $(a_1 \dots a_m) \neq (0, \dots, 0) =: \mathbf{0}$ and $f_1, \dots, f_n \in \mathbb{A}$.
- FIND ALL $g \in \mathbb{A}$, $c_1, \dots, c_n \in \mathbb{K}$ such that

$$\boxed{a_1\,\sigma^{m-1}(g) + \cdots + a_m\,g = c_1\,f_1 + \cdots + c_n\,f_n.}$$

**Remark 3.1.1.** $\mathbb{A}$ is a vector space over $\mathbb{K}$. $\Diamond$

**Definition 3.1.1.** Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$ and consider a subspace $\mathbb{V}$ of $\mathbb{A}$ as a vector space over $\mathbb{K}$. Let

$$\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{A}^m,$$
$$\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{A}^n.$$

We define the *solution space* for $\mathbf{a}, \mathbf{f}$ in $\mathbb{V}$ by

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V}) = \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} : a_1\,\sigma^{m-1}(g) + \cdots + a_m\,g = c_1\,f_1 + \cdots + c_n\,f_n\}.$$

$\Diamond$

**Remark 3.1.2.** $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{V})$ is a vector space over $\mathbb{K}$. $\Diamond$

**Definition 3.1.2.** Let $(\mathbb{A},\sigma)$ be a difference ring with constant field $\mathbb{K}$. $\mathbf{0}\neq\mathbf{a}\in\mathbb{A}^m$ is called V-*finite*, if $\mathrm{V}(\mathbf{a},(0),\mathbb{A})$ is a finite dimensional vector space over $\mathbb{K}$. $\Diamond$

**Proposition 3.1.1.** *Let $(\mathbb{A},\sigma)$ be a difference ring with constant field $\mathbb{K}$ and assume $\mathbf{f}\in\mathbb{A}^n$ and $\mathbf{0}\neq\mathbf{a}\in\mathbb{A}^m$. Let $\mathbb{V}$ be a subspace of $\mathbb{A}$ as a vector space over $\mathbb{K}$. If $\mathbf{a}$ is* V*-finite then $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{V})$ is a finite dimensional vector space over $\mathbb{K}$, in particular,*

$$\dim\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{V})\leq\dim\mathrm{V}(\mathbf{a},(0),\mathbb{V})+n.$$

*Proof.* Let $d:=\dim\mathrm{V}(\mathbf{a},(0),\mathbb{V})$ and assume that

$$\dim\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{V})>n+d,$$

say there are $(c_{1i},\dots,c_{ni},g_i)\in\mathbb{K}^n\times\mathbb{V}$ for $1\leq i\leq n+d+1$ which are linearly independent over $\mathbb{K}$ and solutions of $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{V})$. Then one can transform the matrix

$$M:=\begin{pmatrix} c_{11} & \dots & c_{n1} & g_1\\ c_{12} & \dots & c_{n2} & g_2\\ \vdots & \vdots & \vdots & \vdots\\ c_{1,n+d+1} & \dots & c_{n,n+d+1} & g_{n+d+1}\end{pmatrix}$$

by row operations over $\mathbb{K}$ to a matrix

$$M':=\begin{pmatrix} c'_{11} & \dots & c'_{n1} & g'_1\\ c'_{12} & \dots & c'_{n2} & g'_2\\ \vdots & \vdots & \vdots & \vdots\\ c'_{1,n+d+1} & \dots & c'_{n,n+d+1} & g'_{n+d+1}\end{pmatrix}$$

where the submatrix

$$C':=\begin{pmatrix} c'_{11} & \dots & c'_{n1}\\ c'_{12} & \dots & c'_{n2}\\ \vdots & \vdots & \vdots\\ c'_{1,n+d+1} & \dots & c'_{n,n+d+1}\end{pmatrix}$$

is in row reduced form and the rows in $M'$ and the rows in $M$ are a basis of the same vector space $\mathbb{W}$. Since we assumed that the $(c_{1i},\dots,c_{ni},g_i)$ are linearly independent over $\mathbb{K}$, it follows that all rows in $M'$ have a nonzero entry and are linearly independent over $\mathbb{K}$. On the other side, only the first $n$ rows in $C'$ can have nonzero entries and therefore the last $d+1$ columns in $M'$ must be of the form

$$\left(0,\dots,0,g'_i\right)$$

where $g'_i\neq 0$. Therefore we find $d+1$ linearly independent solutions over $\mathbb{K}$ with

$$\sigma_{\mathbf{a}}g_i=0$$

which contradicts to the assumption. $\square$

**Proposition 3.1.2.** *Let $(\mathbb{F},\sigma)$ be a difference field. Then $\mathbf{0}\neq\mathbf{a}\in\mathbb{F}^m$ is* V*-finite and we have*

$$\dim\mathrm{V}(\mathbf{a},(0),\mathbb{V})\leq m-1.$$

*Proof.* This follows immediately by Lemma A.5 in [HS99]. □

**Corollary 3.1.1.** *Let* $(\mathbb{F}, \sigma)$ *be a difference field,* $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^m$ *and* $\mathbf{f} \in \mathbb{F}^n$*. Then* $\mathbf{a}$ *is* V*-finite and*

$$\dim \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V}) \leq m + n - 1.$$

*Proof.* This follows by Propositions 3.1.1 and 3.1.2. □

**Notation 3.1.1.** Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$ and consider a subspace $\mathbb{V}$ of $\mathbb{A}$ as a vector space over $\mathbb{K}$. Let $\mathbf{0} \neq \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{A}^m$ and $g \in \mathbb{A}$. Then we introduce the following notation

$$\sigma_{\mathbf{a}} g := a_1 \, \sigma^{m-1}(g) + \cdots + a_m \, g.$$

Furthermore, given $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{A}^n$ and $g \in \mathbb{A}$ we write

$$\mathbf{f} \wedge g = (f_1, \ldots, f_n, g)$$

for the *concatenation* of $\mathbf{f}$ with the element $g$. Additionally, for $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{F}^n$ and $\mathbf{g} = (g_1, \ldots, g_n) \in \mathbb{F}^n$ we write

$$\mathbf{f} \, \mathbf{g} = f_1 \, g_1 + \cdots + f_n \, g_n$$

for the *inner product*. By these notations we obtain the following compact description of the solution space:

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V}) = \{\overbrace{(c_1, \ldots, c_n, g)}^{\mathbf{c} \wedge g} \in \mathbb{K}^n \times \mathbb{V} : \underbrace{a_1 \, \sigma^{m-1}(g) + \cdots + a_m \, g}_{\sigma_{\mathbf{a}} g} = \underbrace{c_1 \, f_1 + \cdots + c_n \, f_n}_{\mathbf{c} \, \mathbf{f}}\}$$

$$\downarrow$$

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V}) = \{\mathbf{c} \wedge g \in \mathbb{K}^n \times \mathbb{V} \,|\, \sigma_{\mathbf{a}} g = \mathbf{c} \, \mathbf{f}\}.$$

◊

### 3.1.2 Decomposition of the Solution Range $\mathbb{F}(t)$

Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$. In the following we will decompose the solution range $\mathbb{F}(t)$ into a direct sum of subspaces of $\mathbb{F}(t)$ as a vector space over $\mathbb{K}$. Having this decomposition we can introduce in the next section the main reduction idea, namely to eliminate step by step the different parts of the direct sum components of the solutions range $\mathbb{F}(t)$.

#### 3.1.2.1 Sums and Direct Sums of Vector Spaces

**Definition 3.1.3.** Let $\mathbb{V}$ be a vector space over $\mathbb{K}$ and $\mathbb{V}_1, \dots, \mathbb{V}_n$ be subspaces of $\mathbb{V}$ over $\mathbb{K}$. $\mathbb{V}$ is called a *sum* of $\mathbb{V}_1, \dots, \mathbb{V}_n$, in symbols

$$V = \mathbb{V}_1 + \cdots + \mathbb{V}_n,$$

if each element $x \in \mathbb{V}$ can be represented in the form

$$x = x_1 + \cdots + x_n$$

where $x_i \in \mathbb{V}_i$. $\mathbb{V}$ is called a *direct sum* of $\mathbb{V}_1, \dots, \mathbb{V}_n$, in symbols

$$V = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_n,$$

if each element $x \in \mathbb{V}$ can be *uniquely* represented in the form

$$x = x_1 + \cdots + x_n$$

where $x_i \in \mathbb{V}_i$. $\diamondsuit$

#### 3.1.2.2 Partial Fraction Decomposition

Let $t$ be transcendental over a field $\mathbb{F}$, let $\mathbb{K}$ be a subfield of $\mathbb{F}$ and consider $\mathbb{F}(t)$ as a vector space over $\mathbb{K}$.

**Theorem 3.1.1.** *Any $f \in \mathbb{F}(t)$ can be uniquely represented in the form*

$$f = g + \frac{p}{q}$$

*where $g, p, q \in \mathbb{F}[t]$ such that* $\gcd(p, q) = 1$ *and* $\deg(p) < \deg(q)$ *and $q$ is monic.*

**Example 3.1.1.** By the polynomial division algorithm we obtain

$$\frac{1 - t - t^2 + t^3 + t^4 + t^5 - 2t^6 + t^7}{(-1 + t)^2 t^3} = t^2 + \frac{1 - t - t^2 + t^3 + t^4}{(-1 + t)^2 t^3}.$$

$\diamondsuit$

**Definition 3.1.4.** We define

$$\mathbb{F}(t)^{(frac)} := \{\frac{p}{q} \in \mathbb{F}(t) \mid \tfrac{p}{q} \text{ is in reduced representation and} \deg(p) < deg(q)\}.$$

$\diamondsuit$

**Corollary 3.1.2.** *Consider* $\mathbb{F}[t]$ *and* $\mathbb{F}(t)^{(frac)}$ *as subspaces of* $\mathbb{F}(t)$ *over* $\mathbb{K}$*. Then we have*

$$\mathbb{F}(t) = \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}.$$

$\boxed{\text{Let } P \subset \mathbb{F}[t] \text{ be the set of all monic polynomials being irreducible over } \mathbb{F}.}$

**Theorem 3.1.2.** *Any* $f \in \mathbb{F}(t)^{(frac)}$ *can be uniquely represented in the form*

$$f = \sum_{p \in P} \frac{f_p}{p^{d(p)}}$$

*where* $f_p \in \mathbb{F}[t]$ *and* $d(p) > 0$ *such that*

- $\deg(f_p) < \deg(p^{d(p)})$,
- $p \nmid f_p$.

**Example 3.1.2.** By the extended Euclidean algorithm we obtain the decomposition

$$\frac{1 - t - t^2 + t^3 + t^4}{(-1 + t)^2 t^3} = \frac{t + 1}{t^3} + \frac{t}{(t - 1)^2}.$$

$\diamondsuit$

**Corollary 3.1.3.** *Any* $f \in \mathbb{F}(t)^{(frac)}$ *can be uniquely represented in the form*

$$f = \frac{g}{t^k} + \frac{p}{q}$$

*where* $g \in \mathbb{F}[t]$, $\frac{p}{q} \in \mathbb{F}(t)$ *is in reduced representation and* $k > 0$ *such that*

- $\deg(g) < k$, $t \nmid g$,
- $\deg(p) < \deg(q)$, $t \nmid q$.

**Theorem 3.1.3.** *Let* $p \in P$, $f \in \mathbb{F}[t]$ *and* $d \geq 0$ *such that*

- $\deg(f) < \deg(p^d)$,
- $p \nmid f$.

*Then* $\frac{f}{p^d}$ *can be uniquely represented in the form*

$$\frac{f}{p^d} = \sum_{i=1}^{d} \frac{f_i}{p^i}$$

*where* $f_i \in \mathbb{F}[t]$ *with* $\deg(f_i) < \deg(p)$.

**Definition 3.1.5.** We define

$$\mathbb{F}(t)^{(fracpart)} := \{\frac{p}{q} \in \mathbb{F}(t)^{(frac)};\ t \nmid q\}.$$

$\diamondsuit$

**Corollary 3.1.4.** *Consider $\mathbb{F}[t]$, $\mathbb{F}[1/t]\setminus\mathbb{F}^*$ and $\mathbb{F}(t)^{(fracpart)}$ as subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$. Then we have*

$$\mathbb{F}(t)=\mathbb{F}[t]\oplus\underbrace{(\mathbb{F}[1/t]\setminus\mathbb{F}^*)\oplus\mathbb{F}(t)^{(fracpart)}}_{=\mathbb{F}(t)^{(frac)}}.$$

*Proof.* Define

$$F:=\Big\{\frac{p}{t^m}\,|\,m\geq 1\ \&\ p\in\mathbb{F}[t]\ \&\ \deg(p)<m\ \&\ t\nmid p\Big\}.$$

By Corollary 3.1.3 we have

$$\mathbb{F}(t)^{(frac)}=\mathbb{F}(t)^{(fracpart)}\oplus F$$

and by Theorem 3.1.3 it follows that

$$F=\mathbb{F}[1/t]\setminus\mathbb{F}^*.$$

Therefore by Corollary 3.1.2 the statement is proven. □

#### 3.1.2.3 Decomposition of a ΠΣ-Extension

Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$ and let us recall that by Theorem 2.2.4 the period of $f\in\mathbb{F}(t)^*$ is given by

$$\mathrm{per}_{(\mathbb{F},\sigma)}(f):=\begin{cases}1 & \text{if } f=c\,t^i \text{ where } c\in\mathbb{F}^*,\ i\in\mathbb{Z} \text{ and } (\mathbb{F}(t),\sigma) \text{ is a }\Pi\text{-extension}\\ 0 & \text{otherwise.}\end{cases}$$

**Definition 3.1.6.** Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$. $h\in\mathbb{F}[t]^*$ has *pure period* $m\in\{0,1\}$, if

$$\forall f\in\mathbb{F}[t]\setminus\mathbb{F}: f|h\Rightarrow \mathrm{per}_{(\mathbb{F},\sigma)}(f)=m.$$

◊

**Remark 3.1.3.** Note that $f\in\mathbb{F}[t]^*$ has pure period 1 if and only if $f=h\,t^i$ for some $i\in\mathbb{N}_0$ and $h\in\mathbb{F}^*$. Therefore $f$ has pure period 1 if and only if it has period 1. ◊

**Definition 3.1.7.** Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$. We define

$$\mathbb{F}(t)^{(0)}:=\Big\{\frac{p}{q}\in\mathbb{F}(t)^{(frac)}\,|\,q \text{ has pure period } 0\Big\},$$
$$\mathbb{F}(t)^{(1)}:=\Big\{\frac{p}{q}\in\mathbb{F}(t)^{(frac)}\,|\,\mathrm{per}_{(\mathbb{F},\sigma)}(q)=1\Big\}.$$

◊

**Corollary 3.1.5.** *If $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ then*

$$\mathbb{F}(t)^{(1)}=\{0\},$$
$$\mathbb{F}(t)^{(0)}=\mathbb{F}(t)^{(frac)}.$$

*If $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ then*

$$\mathbb{F}(t)^{(1)}=\mathbb{F}[1/t]\setminus\mathbb{F}^*,$$
$$\mathbb{F}(t)^{(0)}=\mathbb{F}(t)^{(fracpart)}.$$

*Proof.* This follows immediately by Theorem 2.2.4. $\square$

**Corollary 3.1.6.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$. Then*

$$\mathbb{F}(t) = \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \oplus \mathbb{F}(t)^{(0)}$$

*where $\mathbb{F}[t]$, $\mathbb{F}(t)^{(1)}$ and $\mathbb{F}(t)^{(0)}$ are considered as subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$.*

*Proof.* This is a direct consequence of Corollary 3.1.4 and Corollary 3.1.5. $\square$

**Definition 3.1.8.** Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and

$$h = \underbrace{h_p}_{\in\mathbb{F}[t]} + \underbrace{h_1}_{\in\mathbb{F}(t)^{(1)}} + \underbrace{h_0}_{\in\mathbb{F}(t)^{(0)}} \in \mathbb{F}(t).$$

Then $h_p$ is called *polynomial part*, $h_1$ is called *fractional part with period 1* and $h_0$ is called *fractional part with pure period 0.* $\diamondsuit$

**Example 3.1.3.** Let $(\mathbb{Q}(t_1),\sigma)$ be the $\Pi\Sigma$-field over the constant field $\mathbb{Q}$ canonically defined by

$$\sigma(t_1) = t_1 + 1.$$

Consider the $\Pi$-extension $(\mathbb{Q}(t_1,t_2),\sigma)$ of $(\mathbb{Q}(t_1),\sigma)$ canonically defined by

$$\sigma(t_2) = (t_1+1)\,t_2.$$

Then the upper braces indicate how the following rational function splits into the polynomial part and the fractional parts with pure period 0 and period 1.

$$\frac{t_1 + t_2 + t_1t_2 + 2t_2^2 + t_1t_2^3 + t_1t_2^4}{t_2^2(t_1+t_2)} = \underbrace{\overbrace{t_2t_1}^{\mathbb{Q}(t_1)[t_2]}}_{\mathbb{Q}(t_1)[t_2]} + \underbrace{\overbrace{\frac{1}{t_2} + \frac{t_1}{t_2^2}}^{\mathbb{Q}(t_1)(t_2)^{(1)}} + \overbrace{\frac{1}{t_2+1}}^{\mathbb{Q}(t_1)(t_2)^{(0)}}}_{\mathbb{Q}(t_1)(t_2)^{(0)}} \in \mathbb{Q}(t_1,t_2)$$

Consider the $\Sigma$-extension $(\mathbb{Q}(t_1,t_2),\sigma)$ of $(\mathbb{Q}(t_1),\sigma)$ canonically defined by

$$\sigma(t_2) = t_2 + \frac{1}{t_1+1}.$$

Then the lower braces indicate how the rational function splits into the polynomial part and the fractional parts with pure period 0 only.

$\diamondsuit$

### 3.1.3 The Basic Reduction Strategy for $\Pi\Sigma$-Fields

In the following section we try to give a first description how linear difference equations are solved in a $\Pi\Sigma$-field $(\mathbb{F}(t), \sigma)$ over the constant field $\mathbb{K}$. Given $\mathbf{f} \in \mathbb{F}(t)^n$ and $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}(t)^m$ there is the following reduction process to find a basis for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$.

$$\text{Find a basis for} \left\{ \begin{tikzcd} \mathrm{V}(\mathbf{a},\mathbf{f},\overbrace{\boxed{\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}\oplus\mathbb{F}(t)^{(0)}}}^{\mathbb{F}(t)}) \arrow[d, shift right=3ex, "\substack{\text{period } 0\\ \text{elimination}}"'] \\ \mathrm{V}(\mathbf{a}',\mathbf{f}',\boxed{\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}}) \arrow[u, shift right=3ex, "\text{by denominator bounding}"'] \arrow[d, shift right=3ex, "\substack{\text{period } 1\\ \text{elimination}}"'] \\ \mathrm{V}(\mathbf{a}'',\mathbf{f}'',\boxed{\mathbb{F}[t]}) \arrow[u, shift right=3ex, "\text{by denominator bounding}"'] \arrow[d, shift right=3ex, "\substack{\text{polynomial}\\ \text{degree}\\ \text{elimination}}"'] \\ \mathrm{V}(\mathbf{a}''',\mathbf{f}''',\boxed{\{0\}}) \arrow[u, shift right=3ex, "\text{by incremental reduction}"'] \arrow[d, equal] \\ \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}''') \times \{0\} \end{tikzcd} \right.$$

In the following subsections I will explain in more details the methods for the different reduction steps.

#### 3.1.3.1 The Denominator Bounding Method

**The Period 0 Denominator Bounding** We will give the main idea how we can achieve the following reduction.

$$\text{Find a basis for} \left\{ \begin{tikzcd} \mathrm{V}(\mathbf{a},\mathbf{f},\overbrace{\boxed{\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}\oplus\mathbb{F}(t)^{(0)}}}^{\mathbb{F}(t)}) \arrow[d, shift right=3ex, "\substack{\text{period } 0\\ \text{elimination}}"'] \\ \mathrm{V}(\mathbf{a}',\mathbf{f}',\boxed{\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}}) \arrow[u, shift right=3ex] \end{tikzcd} \right.$$

In this reduction the following simple Lemma 3.1.1 gives the main idea.

**Lemma 3.1.1.** *Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and let $\mathbb{F}(t) = \mathbb{W} \oplus \mathbb{V}$ be a direct sum of subspaces $\mathbb{V}$ and $\mathbb{W}$ of $\mathbb{F}(t)$ as vector spaces over $\mathbb{K}$. Let $\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ and $\mathbf{f} \in \mathbb{F}[t]^n$. Assume there is a $d \in \mathbb{F}(t)^*$ such that for all*

$$\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$$

*we have $d\,g \in \mathbb{W}$. Then*

$$\begin{gathered}\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t)) \\ \Updownarrow \\ \mathbf{c} \wedge (gd) \in \mathrm{V}(\mathbf{a}', \mathbf{f}, \mathbb{W})\end{gathered}$$

*for*

$$\mathbf{a}' := \left( \frac{a_1}{\sigma^{m-1}(d)}, \dots, \frac{a_{m-1}}{\sigma(d)}, \frac{a_m}{d} \right).$$

*Proof.* We have

$$\begin{gathered}\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t)) \\ \Updownarrow \\ \sigma_{\mathbf{a}} g = \mathbf{c}\mathbf{f} \\ \Updownarrow \\ a_1 \sigma^{m-1}(g) + \dots + a_{m-1}\sigma(g) + a_m g = \mathbf{c}\,\mathbf{f} \\ || \\ a_1 \frac{\sigma^{m-1}(d)}{\sigma^{m-1}(d)} \sigma^{m-1}(g) + \dots + a_{m-1} \frac{\sigma(d)}{\sigma(d)} \sigma(g) + \frac{d}{d} a_m\, g \\ || \\ \frac{a_1}{\sigma^{m-1}(d)} \sigma^{m-1}(g\,d) + \dots + \frac{a_{m-1}}{\sigma(d)} \sigma(g\,d) + \frac{a_m}{d} d\,g \\ \Updownarrow \\ \sigma_{\mathbf{a}'}(g\,d) = \mathbf{c}\,\mathbf{f} \\ \Updownarrow \\ \mathbf{c} \wedge (g\,d) \in \mathrm{V}(\mathbf{a}', \mathbf{f}, \mathbb{W}).\end{gathered}$$

□

Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)^{(1)}$ as a vector space over $\mathbb{K}$. As will be shown in Section 3.5.3, more precisely in Theorem 3.5.6, we are able to compute a particular $d \in \mathbb{F}[t]^*$ such that for all

$$\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t] \oplus \mathbb{W} \oplus \mathbb{F}(t)^{(0)})$$

we have

$$d\,g \in \mathbb{F}[t] \oplus \mathbb{W}. \tag{3.1}$$

Giving this $d$ we may compute

$$\mathbf{a}' := \left( \frac{a_1}{\sigma^{m-1}(d)}, \dots, \frac{a_{m-1}}{\sigma(d)}, \frac{a_m}{d} \right).$$

If we can solve $\mathrm{V}(\mathbf{a}', \mathbf{f}, \mathbb{F}[t] \oplus \mathbb{W})$ then by Lemma 3.1.1 it follows that

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t] \oplus \mathbb{W} \oplus \mathbb{F}(t)^{(0)}) = \left\{\mathbf{c}\wedge\frac{g}{d} \mid \mathbf{c}\wedge g \in \mathrm{V}(\mathbf{a}', \mathbf{f}, \mathbb{F}[t] \oplus \mathbb{W})\right\}.$$

Therefore we achieve the following general reduction strategy

$$\text{Find a basis for } \begin{cases} \mathrm{V}(\mathbf{a}, \mathbf{f}, \boxed{\mathbb{F}[t] \oplus \mathbb{W} \oplus \mathbb{F}(t)^{(0)}}) \\ \big\downarrow \ \begin{array}{c}\text{period } 0\\ \text{elimination}\end{array} \ \big\uparrow \\ \mathrm{V}(\mathbf{a}', \mathbf{f}', \boxed{\mathbb{F}[t] \oplus \mathbb{W}}) \end{cases}$$

Please note that this reduction process is more general than what is actually needed. But solving this more general problem gives us more flexibility as will be indicated in Section 3.1.4. In particular, for $\mathbb{W} = \mathbb{F}(t)^{(1)}$ we get the reduction process described above.

One can immediately see that the particular $d \in \mathbb{F}[t]^*$ fulfilling (3.1) bounds the denominator of the fractional part with pure period 0. In this sense we call this reduction method *period* 0 *denominator bounding.*

**Example 3.1.4.** Let $(\mathbb{Q}(t_1), \sigma)$ be the difference field canonically defined by $\sigma(t_1) = t_1 + 1$ and consider the $\Sigma$-extension $(\mathbb{Q}(t_1, t_2), \sigma)$ of $(\mathbb{Q}(t_1), \sigma)$ canonically defined by

$$\sigma(t_2) = t_2 + \frac{1}{t_1 + 1}.$$

For the difference equation

$$\sigma^2(g) + \sigma(g) - g = f$$

with

$$f := \frac{-3 + 2\,t_2^2 + 4\,t_1^2\,t_2^2 + t_1^3\,t_2^2 + t_1\,(-2 + 5\,t_2^2)}{t_2\,(1 + t_2 + t_1\,t_2)(3 + 2\,t_2 + t_1^2\,t_2 + t_1\,(2 + 3\,t_2))} \in \mathbb{Q}(t_1, t_2)$$

we have the following solution space

$$\mathrm{V}((1, 1, -1), (f), \mathbb{Q}(t_1, t_2)) = \{c_1 \left(1, \frac{1}{t_2}\right) + c_2\,(0, 1) \mid c_1, c_2 \in \mathbb{Q}\}.$$

Thus $d := t_2$ has the property that for all

$$\mathbf{c}\wedge g \in \mathrm{V}((1, 1, -1), (f), \mathbb{Q}(t_1, t_2))$$

it follows that

$$d\,g \in \mathbb{Q}(t_1)[t_2]$$

and one can see that $d$ bounds the denominator of the fractional part with pure period 0.

Additionally, by Corollary 3.1.5 and Lemma 3.1.1 we get the following elimination of the fractional part of pure period 0:

$$\mathrm{V}((1, 1, -1), (f), \overbrace{\mathbb{Q}(t_1)[t_2] \oplus \mathbb{Q}(t_1)(t_2)^{(0)} \oplus \underbrace{\mathbb{Q}(t_1)(t_2)^{(1)}}_{=\{0\}}}^{=\mathbb{F}(t)})$$

$$||$$

$$\begin{array}{c}
\mathrm{V}((1,1,-1),(f)\,,\boxed{\mathbb{Q}(t_1)[t_2]\oplus\mathbb{Q}(t_1)(t_2)^{(0)}})\\
\downarrow \qquad\qquad\qquad \uparrow\\
\mathrm{V}(\left(\frac{1}{\sigma^2(t_2)},\frac{1}{\sigma(t_2)},-\frac{1}{t_2}\right),(f)\,,\boxed{\mathbb{Q}(t_1)[t_2]})
\end{array}$$

In other words, solving

$$\mathbb{V}:=\mathrm{V}(\left(\frac{1}{\sigma^2(t_2)},\frac{1}{\sigma(t_2)},-\frac{1}{t_2}\right),(f)\,,\mathbb{Q}(t_1)[t_2])$$

yields to

$$\mathrm{V}((1,1,-1),(f)\,,\mathbb{F}(t))=\left\{\left(c,\frac{g}{t_2}\right)\mid(c,g)\in\mathbb{V}\right\}.$$

$\Diamond$

**The Period 1 Denominator Bounding** Now we consider the second reduction step in the reduction process described in the beginning of Section 3.1.3.

$$\text{Find a basis for}\begin{cases}\mathrm{V}(\mathbf{a},\mathbf{f},\boxed{\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}})\\ \Big\downarrow\ \begin{array}{c}\text{period 1}\\ \text{elimination}\end{array}\ \Big\uparrow\\ \mathrm{V}(\mathbf{a}',\mathbf{f}',\boxed{\mathbb{F}[t]})\end{cases}$$

Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)^{(0)}$ as a vector space over $\mathbb{K}$. Similar to the period 0 denominator bounding we are interested in finding a particular $d\in\mathbb{F}[t]^*$ such that for all

$$\mathbf{c}\wedge g\in\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{F}[t]\oplus\mathbb{W}\oplus\mathbb{F}(t)^{(1)})$$

we have

$$d\,g\in\mathbb{F}[t]\oplus\mathbb{W}.$$

Given such a $d$ we may apply Lemma 3.1.1 in order to achieve the reduction process.

$$\text{Find a basis for}\begin{cases}\mathrm{V}(\mathbf{a},\mathbf{f},\boxed{\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}\oplus\mathbb{W}})\\ \Big\downarrow\ \begin{array}{c}\text{period 1}\\ \text{elimination}\end{array}\ \Big\uparrow\\ \mathrm{V}(\mathbf{a}',\mathbf{f}',\boxed{\mathbb{F}[t]\oplus\mathbb{W}})\end{cases}$$

In particular for $\mathbb{W}=\{0\}$, we get the reduction described above.

For first order linear difference equations this problem to find such a particular $d\in\mathbb{F}[t]^*$ is solved as one can see in Section 3.5.2.3. Unfortunately, for difference equations of higher order this $d$ can be only computed for some special cases (Section 3.5.2.4). If we run in a situation where we do not have an algorithm to compute such a $d$, we apply an heuristic method described in Section 3.5.2.1.

#### 3.1.3.2 The Incremental Reduction Method for Polynomial Degree Elimination

In this section we try to give a first "oversimplified" sketch how the incremental reduction method for the polynomial degree elimination works. In Section 3.2 we will finally consider this incremental reduction process in more details.

$$\text{Find a basis for} \begin{cases} \mathrm{V}(\mathbf{a}, \mathbf{f}, \boxed{\mathbb{F}[t]}) \\ \downarrow \begin{array}{c}\text{polynomial}\\ \text{degree}\\ \text{elimination}\end{array} \uparrow \\ \mathrm{V}(\mathbf{a}', \mathbf{f}', \boxed{\{0\}}) \end{cases}$$

In a first step one finds a bound[1] $b \in \mathbb{N}_0 \cup \{-1\}$ such that for all

$$\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$$

we have $\deg(g) \leq b$. Of course, for any $d \in \mathbb{N}_0 \cup \{-1\}$,

$$\mathbb{F}[t]_d := \{f \in \mathbb{F}[t] \mid \deg(f) \leq d\}$$

is a subspace of $\mathbb{F}[t]$ over $\mathbb{K}$. In other words, we try to find a $b \in \mathbb{N}_0 \cup \{-1\}$ such that

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]) = \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_b)$$

As will be shown in Sections 3.3.4.1 and 3.3.5.1 the polynomial degree bounding for the first order case is solved completely. In Sections 3.3.5.2 and 3.3.4.2 we extend these methods based on [Kar81] for some special cases to the $m$-th order case.

Finally in Section 3.4 we find further degree boundings for proper sum extensions. If we run in a situation where we do not have an algorithm to compute a bound, we apply an heuristic method described in Section 3.3.1.

If we find such a *polynomial degree bounding*, we can do in a second step the following reduction process:

$$\begin{array}{c} \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_b) \\ \downarrow \qquad \uparrow \\ \mathrm{V}(\mathbf{a_{b-1}}, \mathbf{f_{b-1}}, \mathbb{F}[t]_{b-1}) \\ \downarrow \qquad \uparrow \\ \vdots \qquad \vdots \\ \downarrow \qquad \uparrow \\ \mathrm{V}(\mathbf{a_0}, \mathbf{f_0}, \mathbb{F}[t]_0) \\ \downarrow \qquad \downarrow \\ \mathrm{V}(\underbrace{\mathbf{a_{-1}}}_{\mathbf{a}'}, \underbrace{\mathbf{f_{-1}}}_{\mathbf{f}'}, \mathbb{F}[t]_{-1}) \\ || \\ \mathrm{V}(\mathbf{a}', \mathbf{f}', \boxed{\{0\}}). \end{array}$$

How this reduction process works in details will be explained in Section 3.2.

[1] Since the zero-polynomial has degree $-\infty$ (see Section 3.2.1.1), we actually need only to consider $b \in \mathbb{N}_0 \cup \{-1\}$ to get a full filtration of $\mathbb{F}[t]$ (see Section 3.2.3).

**Example 3.1.5.** Consider the $\Pi\Sigma$-field $(\mathbb{Q}(t_1,t_2),\sigma)$ over the constant field $\mathbb{Q}$ canonically defined by

$$\sigma(t_1) = t_1 + 1,$$
$$\sigma(t_2) = t_2 + \frac{1}{t_1+1}.$$

In order to find a solution $g$ for

$$\sigma(g) - g = t_1\,t_2,$$

we find a basis of $\mathrm{V}((1,-1),(t_2),\mathbb{Q}(t_1,t_2))$ by the following reduction process:

$$\begin{array}{c}
\text{Find a basis of}\\
\mathrm{V}((1,-1),(t_2),\mathbb{Q}(t_1,t_2))\\
|| \quad \text{period 0 denominator bounding}\\
\mathrm{V}((1,-1),(t_2),\mathbb{Q}(t_1)[t_2])\\
|| \quad \text{polynomial degree bounding}^2\\
\mathrm{V}((1,-1),(t_2),\mathbb{Q}(t_1)[t_2]_2)\\
\downarrow \qquad\qquad \uparrow\\
\text{Find a basis of}\\
\mathrm{V}\left((1,-1),\left(\frac{-1-2t_2-2t_1t_2}{(1+t_1)^2},t_2\right),\mathbb{Q}(t_1)[t_2]_1\right)\\
\downarrow \qquad\qquad \uparrow^3\\
\text{Find a basis of}\\
\mathrm{V}\left((1,-1),\left(-\frac{1}{t_1+1},-1\right),\mathbb{Q}(t_1)[t_2]_0\right)\\
\downarrow \qquad\qquad \uparrow\\
\text{Find a basis of}\\
\mathrm{V}((1,-1),(0,0),\mathbb{Q}(t_1)[t_2]_{-1})\\
||\\
\mathrm{V}((1,-1),(0,0),\{0\})\\
||\\
\left\{(c_1,c_2,g)\in\mathbb{Q}^2\times\{0\}\,|\,\sigma(g)-g=c_1\,0+c_2\,0\right\}\\
||\\
\{c_1\ (1,0,0)+c_2\ (0,1,0)\ |\ c_1,c_2\in\mathbb{Q}\}.
\end{array}$$

Finally, this reduction process yields to

$$\mathrm{V}((1,-1),(t_2),\mathbb{Q}(t_1,t_2)) = \{(0,c_1)+c_2\ (1,t_1\,(t_2-1))\ |\ c_1,c_2\in\mathbb{Q}\}$$

which gives us the specific solution $g := t_1\,(t_2-1)$ for

$$\sigma(g) - g = t_2.$$

$\Diamond$

[2]See Corollary 3.3.3.

[3]This reduction step will be considered in more details in Example 3.2.7.

**Example 3.1.6.** Consider the $\Pi\Sigma$-field $(\mathbb{Q}(t_1,t_2),\sigma)$ over the constant field $\mathbb{Q}$ canonically defined by

$$\sigma(t_1) = t_1 + 1,$$
$$\sigma(t_2) = (t_1 + 1)\, t_2.$$

In order to find the solution $g := t_2$ for

$$\sigma(g) - g = t_1\, t_2$$

in Section 1.2.2, the following reduction process is involved.

$$\begin{array}{c}
\text{Find basis of}\\[6pt]
\mathrm{V}((1,-1),(t_1t_2)\,,\mathbb{Q}(t_1)(t_2))\\[6pt]
\| \qquad \text{period 0 and 1 denominator bounding}\\[6pt]
\mathrm{V}((1,-1),(t_1t_2)\,,\mathbb{Q}(t_1)[t_2])\\[6pt]
\| \qquad \text{polynomial degree bounding}^4\\[6pt]
\mathrm{V}((1,-1),(t_1t_2)\,,\mathbb{Q}(t_1)[t_2]_1)\\[6pt]
\downarrow \qquad\qquad\qquad \uparrow\\[6pt]
\text{Find basis of}\\[6pt]
\mathrm{V}((1,-1),(0)\,,\mathbb{Q}(t_1)[t_2]_0)\\[6pt]
\downarrow \qquad\qquad\qquad \uparrow\\[6pt]
\text{Find basis of}\\[6pt]
\mathrm{V}((1,-1),(0)\,,\underbrace{\mathbb{Q}(t_1)[t_2]_{-1}}_{\{0\}})\\[6pt]
\|\\[6pt]
\{(c,g)\in\mathbb{Q}\times\{0\}\,|\,\sigma(g)-g=0\}\\[6pt]
\|\\[6pt]
\{(c,0)\ |\, c\in\mathbb{Q}\}.
\end{array}$$

The complete reduction process for all subproblems will be shown in Example 3.2.8.

◊

[4] See Corollary 3.3.2.

#### 3.1.3.3 The First Base Case

Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}^m$ and $\mathbf{f} \in \mathbb{A}^n$. As can be seen in Section 3.1.3.2, we have to deal with the following base case problem: find a basis of

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \{0\}).$$

The following Theorem 3.1.4 allows us to reduce this problem to a nullspace problem of $\mathbb{A}$ as a vector space over $\mathbb{K}$.

**Definition 3.1.9.** Let $\mathbb{A}$ be a vector space over $\mathbb{K}$ and consider $\mathbb{A}^n$ as a vector space over $\mathbb{K}$. Let $\mathbf{f} \in \mathbb{A}^n$. Then we define the nullspace of $\mathbf{f}$ over $\mathbb{K}$ by

$$\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}) = \{\mathbf{c} \in \mathbb{K}^n \,|\, \mathbf{c}\,\mathbf{f} = 0\}.$$

$\diamondsuit$

**Lemma 3.1.2.** *Let $\mathbb{A}$ be a vector space over $\mathbb{K}$ and $\mathbf{f} \in \mathbb{A}^n$. Then $\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f})$ is a subspace of $\mathbb{A}^n$ over $\mathbb{K}$.*

**Theorem 3.1.4.** *Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$ and assume $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}^m$ and $\mathbf{f} \in \mathbb{A}^n$. Then*

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \{0\}) = \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}) \times \{0\}.$$

*Proof.* We have

$$\begin{aligned}
\mathbf{c}\wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \{0\}) &\Leftrightarrow \sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f} \;\&\; g = 0\\
&\Leftrightarrow \mathbf{c}\,\mathbf{f} = \mathbf{0} \;\&\; g = 0\\
&\Leftrightarrow \mathbf{c} \in \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}) \;\&\; g = 0\\
&\Leftrightarrow \mathbf{c}\wedge g \in \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}) \times \{0\}.
\end{aligned}$$

$\square$

If we run into the base case $\mathrm{V}(\mathbf{a}, \mathbf{f}, \{0\})$, we will apply this Theorem 3.1.4; we only have to consider the problem to find a basis of the vector space $\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}) \times \{0\}$. In particular in Lemma 3.2.7 we will deal with this problem for a $\Pi\Sigma$-field $(\mathbb{F}(t), \sigma)$ and in Lemma 3.6.6 we will solve this problem if $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field and $(\mathbb{F}[t], \sigma)$ is a difference ring extension canonically defined by $\sigma(t) = -t$ with the relation $t^2 = 1$.

### 3.1.4 Variations of Strategies

Finally, I indicate that besides the reduction strategy presented in Section 3.1.3, other strategies might be possible.

$\boxed{\textbf{Our Reduction}}$

$$\begin{array}{c} \mathrm{V}(\mathbf{a}, \mathbf{f}, \overbrace{\mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}}^{\mathbb{F}(t)}) \\ \downarrow \quad \uparrow \text{den. bounding} \\ \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}[t]) \\ \downarrow \quad \uparrow \text{incr. reduction} \\ \mathrm{V}(\mathbf{a}'', \mathbf{f}'', \{0\}) \end{array}$$

*This is the strategy we focused on in the last sections and which will be the strategy under further discussion.*

**Denominator Bounding as Late as Possible**

$$\begin{array}{c} \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{incr reduction} \\ \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{period 0 den. bounding} \\ \mathrm{V}(\mathbf{a}'', \mathbf{f}'', \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{incr. reduction} \\ \mathrm{V}(\mathbf{a}''', \mathbf{f}''', \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{period 1 den. bounding} \\ \mathrm{V}(\mathbf{a}'''', \mathbf{f}'''', \mathbb{F}[t]) \\ \downarrow \quad \uparrow \text{incr. reduction} \\ \mathrm{V}(\mathbf{a}''''', \mathbf{f}''''', \{0\}) \end{array}$$

In this strategy we use the flexibility that the order of the period 0 and 1 denominator bounding can be exchanged.

**Direct Incremental Reduction for Period 1**

$$\begin{array}{c} \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{incr reduction} \\ \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{period 0 den. bounding} \\ \mathrm{V}(\mathbf{a}'', \mathbf{f}'', \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{incr. reduction} \\ \mathrm{V}(\mathbf{a}''', \mathbf{f}''', \mathbb{F}(t)^{(1)}) \\ \downarrow \quad \uparrow \text{incr. reduction} \\ \mathrm{V}(\mathbf{a}'''', \mathbf{f}'''', \{0\}) \end{array}$$

Karr introduced in [Kar81] an incremental reduction method which eliminates the period 1 factors in the denominator for the first order difference equation case. This reduction can be generalized for the higher order case.

Additionally, Karr developed in [Kar81] incremental reduction strategies, similar to the polynomial degree reduction, in order to eliminate the period 0 factors in the denominator for the first order difference equation case. These different strategies will not be considered further in this theses but for simplicity we will focus only on the first strategy.

## 3.2 The Incremental Reduction

In the following the incremental reduction method will be considered in details which allows us to eliminate the polynomial degrees of the possible solutions as it was indicated in Section 3.1.3.2.

### 3.2.1 Some Notations, Conventions and Definitions

#### 3.2.1.1 Polynomials

Let $\mathbb{A}[t]$ be a polynomial ring over a ring $\mathbb{A}$, this means we assume that $t$ is *transcendental* over $\mathbb{A}$. By convention the *zero-polynomial* 0 has degree $-\infty$, i.e.

$$\deg(0) = -\infty.$$

Furthermore, if $f = \sum_{i=0}^{n} f_i\, t_i \in \mathbb{A}[t]$ then the $i$-th *coefficient* $f_i$ of $f$ will be denoted by $[f]_i$, i.e.

$$[f]_i = f_i.$$

Additionally, for $\mathbf{g} = (g_1, \dots, g_n) \in \mathbb{A}[t]^n$ we introduce

$$[\mathbf{g}]_i = ([g_1]_i\,, \dots, [g_n]_i)\,.$$

#### 3.2.1.2 Vectors, Matrices and Basis Matrices

Let $\mathbb{A}$ be a vector space over $\mathbb{K}$ and, more generally, consider $\mathbb{A}^n$ as a vector space over $\mathbb{K}$.

**Vectors**

In the following we will consider a vector $\mathbf{f} \in \mathbb{A}^n$ either as a row **or** as a column vector. It will be convenient not to distinguish between these two types of presentations. This means the row vector

$$\mathbf{f} = (f_1, \dots, f_n)$$

may be also interpreted as the column vector

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

We will show that there cannot appear any ambiguous situations in the sequel. For the vector multiplication of the vectors $\mathbf{f}$ and $\mathbf{g} = (g_1, \dots, g_n)$ introduced in Notation 3.1.1 there cannot be confusion:

$$\begin{aligned}\sum_{i=1}^{n} f_i\, g_i = \mathbf{f}\,\mathbf{g} = (f_1, \dots, f_n)\ (g_1, \dots, g_n) &= \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \\ &= (f_1, \dots, f_n) \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (g_1, \dots, g_n)\,.\end{aligned}$$

### Matrices and Vectors

Whereas a vector will be denoted always with a small letter, matrices will be denoted by capital letters:

$$\mathbf{A} := \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in \mathbb{A}^{m \times n}, \qquad \mathbf{B} := \begin{pmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} \in \mathbb{A}^{n \times m}.$$

*Multiplying a matrix* $\mathbf{A}$ *with the vector* $\mathbf{f}$ *from the right* always means that the vector $\mathbf{f}$ is interpreted as a column vector:

$$\mathbf{A} \cdot \mathbf{f} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^n a_{1i} \, f_{1i} \\ \vdots \\ \sum_{i=0}^n a_{mi} \, f_{mi} \end{pmatrix}$$

whereas *multiplying a matrix* $\mathbf{B}$ *with the vector* $\mathbf{f}$ *from the left* means always that the vector $\mathbf{f}$ is interpreted as a row vector:

$$\mathbf{f} \cdot \mathbf{B} = (f_1, \dots, f_n) \cdot \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} = \left( \sum_{i=0}^n b_{i1} \, f_i, \dots, \sum_{i=0}^n b_{im} \, f_i \right).$$

In the following we will denote the multiplication of a matrix with a vector by the operation symbol $\cdot$. We will denote the usual matrix multiplication by

$$\mathbf{A}\,\mathbf{B} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^n a_{1i} \, b_{i1} & \dots & \sum_{i=0}^n a_{1i} \, b_{im} \\ \vdots & \vdots & \vdots \\ \sum_{i=0}^n a_{mi} \, b_{i1} & \dots & \sum_{i=0}^n a_{mi} \, b_{im} \end{pmatrix}.$$

Finally, given vectors $\mathbf{f_i} = (f_{i1}, \dots, f_{in}) \in \mathbb{A}^n$ for $1 \le i \le m$ we may obtain matrices $(\mathbf{f_1}, \dots, \mathbf{f_m})$ or $\begin{pmatrix} \mathbf{f_1} \\ \vdots \\ \mathbf{f_m} \end{pmatrix}$ by interpreting $\mathbf{f_i}$ as column vectors

$$(\mathbf{f_1}, \dots, \mathbf{f_m}) := \left( \begin{array}{c|c|c} f_{11} & \dots & f_{m1} \\ f_{12} & \dots & f_{m2} \\ \vdots & \vdots & \vdots \\ f_{1n} & \dots & f_{mn} \end{array} \right)$$

or by interpreting them as row vectors

$$\begin{pmatrix} \mathbf{f_1} \\ \vdots \\ \mathbf{f_m} \end{pmatrix} := \left( \begin{array}{cccc} f_{11} & f_{12} & \dots & f_{1n} \\ \hline \vdots & \vdots & \vdots & \\ \hline f_{m1} & f_{m2} & \dots & f_{mn} \end{array} \right).$$

### Concatenations

By the concatenation of the two matrices

$$\mathbf{A} := \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{d1} & \dots & a_{dm} \end{pmatrix} \in \mathbb{A}^{d\times m}, \qquad \mathbf{B} := \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{d1} & \dots & b_{dn} \end{pmatrix} \in \mathbb{A}^{d\times n}$$

we mean

$$\mathbf{C} = \begin{pmatrix} a_{11} & \dots & a_{1m} & b_{11} & \dots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{d1} & \dots & a_{dm} & b_{d1} & \dots & b_{dn} \end{pmatrix};$$

we also write

$$\mathbf{C} = \mathbf{A}\wedge\mathbf{B}$$

for the result. Similarly, by the concatenation of the two vectors

$$\mathbf{f} = (f_1, \dots, f_m)\,, \qquad \mathbf{g} = (g_1, \dots, g_n)$$

we mean

$$\mathbf{h} = (f_1, \dots, f_m, g_1, \dots, g_n)\,;$$

we also write

$$\mathbf{h} = \mathbf{f}\wedge\mathbf{g}.$$

As already introduced in Notation 3.1.1 we write

$$\mathbf{f}\wedge g = (f_1, \dots, f_d, g)$$

for $\mathbf{f} = (f_1, \dots, f_d) \in \mathbb{A}^d$ and $g \in \mathbb{A}$. Similarly, we will write

$$\mathbf{A}\wedge\mathbf{f} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{d1} & \dots & a_{dm} \end{pmatrix} \wedge \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1m} & f_1 \\ \vdots & \vdots & \vdots & \\ a_{d1} & \dots & a_{dm} & f_d \end{pmatrix}.$$

### Basis Matrices

Let $\mathbb{V}$ be a finite dimensional subspace of $\mathbb{A}^n$ as a vector space over $\mathbb{K}$. Let $B = \{\mathbf{b_1}, \dots, \mathbf{b_d}\}$ be a family of linearly independent vectors of $\mathbb{A}^n$ over $\mathbb{K}$ with

$$\mathbf{b_i} = \begin{pmatrix} b_{i1} \\ \vdots \\ b_{in} \end{pmatrix} \in \mathbb{A}^n$$

such that

$$\mathbb{V} = \{k_1\,\mathbf{b_1} + \dots + k_d\,\mathbf{b_d} \,|\, k_i \in \mathbb{K}\}.$$

Often we will represent the basis $B$ of $\mathbb{V}$ by the *basis matrix*

$$\mathbf{M_B} := \begin{pmatrix} \mathbf{b_1} \\ \vdots \\ \mathbf{b_d} \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{d1} & \dots & b_{dn} \end{pmatrix},$$

i.e. we have

$$\mathbb{V} = \left\{ \mathbf{k} \cdot \mathbf{M_B} \,|\, \mathbf{k} \in \mathbb{K}^d \right\}. \tag{3.2}$$

For the special situation $\mathbb{V} = \{(0, \ldots, 0)\} \subseteq \mathbb{A}^n$, the basis matrix is

$$\mathbf{M_B} = (0, \ldots, 0) \in \mathbb{A}^{1\times n}.$$

In order to indicate that $\mathbf{M_B}$ is the basis matrix of $\mathbb{V}$, we will write

$$\mathbb{V} \xleftrightarrow{\text{basis}} \mathbf{M_B}.$$

If $B$ is only a set of generators of $\mathbb{V}$ and not necessarily a basis, i.e. the elements $\mathbf{b_i}$ might be linearly dependent over $\mathbb{K}$, we write

$$\mathbb{V} \xleftrightarrow{\text{span}} \mathbf{M_B},$$

in order to indicate that the matrix $\mathbf{M_B}$ generates the vector space $\mathbb{V}$ by (3.2). In this situation we call $\mathbf{M_B}$ just a *generator matrix*.

**Lemma 3.2.1.** *Let $\mathbb{A}$ be a ring which is also a vector space over the field $\mathbb{K}$ and let $\mathbb{V} \subseteq \mathbb{W} \subseteq \mathbb{A}$ be finite dimensional vector spaces over $\mathbb{K}$ with*

$$\mathbf{M}_{\mathbb{V}} \xleftrightarrow{\text{span}} \mathbb{V},$$

$$\mathbf{M}_{\mathbb{W}} \xleftrightarrow{\text{span}} \mathbb{W}.$$

*Then there exists a matrix $\mathbf{K}$ with entries in $\mathbb{K}$ such that*

$$\mathbf{M}_{\mathbb{V}} = \mathbf{K}\,\mathbf{M}_{\mathbb{W}}.$$

*Proof.* Let

$$\mathbf{M}_{\mathbb{V}} = \begin{pmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_d} \end{pmatrix} \text{ and } \mathbf{M}_{\mathbb{W}} = \begin{pmatrix} \mathbf{w_1} \\ \vdots \\ \mathbf{w_e} \end{pmatrix}.$$

We have

$$\operatorname{span}_{\mathbb{K}}(\mathbf{v_1}, \ldots, \mathbf{v_d}) = \mathbb{V} \subset \mathbb{W} = \operatorname{span}_{\mathbb{K}}(\mathbf{w_1}, \ldots, \mathbf{w_e})$$

Thus there are vectors $\mathbf{k_i} = (k_{i1}, \ldots, k_{ie}) \in \mathbb{K}^e$ for $1 \le i \le d$ such that

$$\mathbf{v_i} = k_{i1}\mathbf{w_1} + \cdots + k_{ie}\mathbf{w_e} = \mathbf{k_i} \cdot \mathbf{M}_{\mathbb{W}}$$

and therefore

$$\mathbf{M}_{\mathbb{V}} = \mathbf{K}\,\mathbf{M}_{\mathbb{W}}$$

with

$$\mathbf{K} = \begin{pmatrix} \mathbf{k_1} \\ \vdots \\ \mathbf{k_d} \end{pmatrix}.$$

□

### 3.2.2 A Basis of the Solution Space

Starting from these notations and conventions we will describe the solution space in such a way that the incremental reduction method can be explained properly.

Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$ and consider $\mathbb{A}$ as a vector space over $\mathbb{K}$. Let $\mathbb{V}$ be a subspace of $\mathbb{A}$. Assume

$$\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{A}^m, \qquad \mathbf{f} = (f_1, \dots, f_n) \in \mathbb{A}^n$$

where $\mathbf{a}$ is $\mathbb{V}$-finite and let

$$B = \{ \begin{pmatrix} c_{11} \\ \vdots \\ c_{1n} \\ g_1 \end{pmatrix}, \dots, \begin{pmatrix} c_{d1} \\ \vdots \\ c_{dn} \\ g_d \end{pmatrix} \}$$

be a basis of the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V})$. We have

$$a_1\, \sigma^{m-1}(g_i) + \dots + a_m\, g_i = c_{i1}\, f_1 + \dots + c_{in}\, f_n$$

and

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V}) = \mathrm{span}_{\mathbb{K}}(B) = \{k_1 \begin{pmatrix} c_{11} \\ \vdots \\ c_{1n} \\ g_1 \end{pmatrix} + \dots + k_d \begin{pmatrix} c_{d1} \\ \vdots \\ c_{dn} \\ g_d \end{pmatrix} \mid k_i \in \mathbb{K}\}.$$

We represent the basis $B$ by the *basis matrix*

$$\mathbf{M_B} = \begin{pmatrix} c_{11} & \dots & c_{1n} & g_1 \\ c_{21} & \dots & c_{2n} & g_2 \\ \vdots & & \vdots & \vdots \\ c_{d1} & \dots & c_{dn} & g_d \end{pmatrix}$$

and thus get

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V}) = \{\mathbf{k} \cdot \mathbf{M_B} \mid \mathbf{k} \in \mathbb{K}^d\}.$$

In the following we compose the basis matrix by

$$\begin{tikzcd}
 & \mathbf{M_B} = \begin{pmatrix} c_{11} & \dots & c_{1n} & g_1 \\ c_{21} & \dots & c_{2n} & g_2 \\ \vdots & & \vdots & \vdots \\ c_{d1} & \dots & c_{dn} & g_d \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\mathbf{C} = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ c_{21} & \dots & c_{2n} \\ \vdots & & \vdots \\ c_{d1} & \dots & c_{dn} \end{pmatrix} \arrow[dr] & & \mathbf{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \arrow[dl] \\
 & \mathbf{M_B} = \mathbf{C} \wedge \mathbf{g} &
\end{tikzcd}$$

Since $\mathbf{C} \wedge \mathbf{g}$ is a basis matrix of $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V})$, we also write

$$\mathbf{C} \wedge \mathbf{g} \xleftrightarrow{\text{basis}} \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{V}).$$

### 3.2.3 Filtrations, Graduations and the Rank Function

As was already mentioned in Section 3.1.3.2, filtrations and graduations will play an essential role for the incremental reduction method. In the following we will have a closer look at these notions.

**Definition 3.2.1.** Let $\mathbb{V}$ be a vector space and $\langle \mathbb{V}_i \rangle_{i\in\mathbb{Z}}$ a sequence of subspaces of $\mathbb{V}$. $\mathbb{V}$ is a *direct sum* of $\langle \mathbb{V}_i \rangle_{i\in\mathbb{Z}}$, in symbols

$$V = \bigoplus_{i\in\mathbb{Z}} \mathbb{V}_i,$$

if every element $x \in \mathbb{V}$ has a *unique* representation

$$x = \sum_{i\in\mathbb{Z}} x_i, \quad x_i \in \mathbb{V}_i$$

where only *finitely* many $x_i$ are nonzero. $\diamondsuit$

**Example 3.2.1.** Let $\mathbb{K}$ be a subfield of $\mathbb{F}$, $t$ be transcendental over $\mathbb{F}$ and consider $\mathbb{F}[t]$ as a vector space over $\mathbb{K}$. Then for the subspaces

$$T_i := \begin{cases} t^i\,\mathbb{F} & \text{if } i \geq 0 \\ \{0\} & \text{if } i < 0 \end{cases}$$

of $\mathbb{F}[t]$ over $\mathbb{K}$ we have the direct sum

$$\mathbb{F}[t] = \bigoplus_{i\in\mathbb{Z}} T_i.$$

$\diamondsuit$

**Definition 3.2.2.** A direct sum $\langle \mathbb{G}_i \rangle_{i\in\mathbb{Z}}$ of $\mathbb{V}$ is called *graduation* of $\mathbb{V}$. $\diamondsuit$

**Definition 3.2.3.** A *filtration* of a vector space $\mathbb{V}$ is a sequence $\langle \mathbb{V}_i \rangle_{i\in\mathbb{Z}}$ of subspaces such that we have a chain

$$\cdots \subseteq \mathbb{V}_{d-1} \subseteq \mathbb{V}_d \subseteq \mathbb{V}_{d+1} \subseteq \cdots \to \mathbb{V}$$

whose limit is $\mathbb{V}$. $\diamondsuit$

**Definition 3.2.4.** Let $\mathbb{A}[t]$ be a polynomial ring over a ring $\mathbb{A}$, i.e. $t$ is transcendental over $\mathbb{A}$. We define the rank function $\|\ \|$ of $\mathbb{A}[t]$ by

$$\|f\| := \begin{cases} -1 & \text{if } f = 0 \\ \deg(f) & \text{otherwise.} \end{cases}$$

Furthermore, we define the rank function $\|\ \|$ of a rational function field $\mathbb{F}(t)$ by

$$\|f\| := \|f_p\|$$

where

$$f = f_p \oplus \tilde{f} \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}.$$

$\diamondsuit$

**Definition 3.2.5.** Let $\mathbb{K}$ be a subfield of $\mathbb{F}$ and let $\mathbb{F}(t)$ be a rational function field over $\mathbb{F}$. Consider $\mathbb{F}(t)$ as a vector space over $\mathbb{K}$ and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Then we define

$$\mathbb{W}_d := \{f \in \mathbb{W} \,|\, \|f\| \leq d\}.$$

Similarly, let $\mathbb{A}[t]$ be a polynomial ring with coefficients in the ring $\mathbb{A}$ and $\mathbb{K} \subseteq \mathbb{A}$ be a field. Consider $\mathbb{A}[t]$ as a vector space over $\mathbb{K}$ and let $\mathbb{W}$ be a subspace of $\mathbb{A}[t]$ over $\mathbb{K}$. Then we define

$$\mathbb{W}_d := \{f \in \mathbb{W} \,|\, \|f\| \leq d\}.$$

$\Diamond$

**Lemma 3.2.2.** *Let $\mathbb{K}$ be a subfield of $\mathbb{F}$, $\mathbb{F}(t)$ be a rational function field over $\mathbb{F}$ and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ as a vector space over $\mathbb{K}$. Then $\langle \mathbb{W}_i \rangle_{i \in \mathbb{Z}}$ is a filtration of $\mathbb{W}$.*

**Example 3.2.2.** Let $\mathbb{K}$ be a subfield of $\mathbb{F}$, $\mathbb{F}(t)$ be a rational function field over $\mathbb{F}$ an consider $\mathbb{F}[t]$ as a subspace of $\mathbb{F}(t)$ as a vector space over $\mathbb{K}$. We have the following filtration of $\mathbb{F}[t]$ and $\mathbb{F}(t)$:

$$\begin{aligned}
\mathbb{F}[t]_d &= \{f \in \mathbb{F}[t] \,|\, \|f\| \leq d\}, \\
\mathbb{F}(t)_d &= \{f \in \mathbb{F}(t) \,|\, \|f\| \leq d\} = \mathbb{F}[t]_d \oplus \mathbb{F}(t)^{(frac)},
\end{aligned}$$

i.e.

$$\begin{aligned}
\{0\} &= \mathbb{F}[t]_{-1} \subset \mathbb{F}[t]_0 \subset \mathbb{F}[t]_1 \subset \cdots \to \mathbb{F}[t], \\
\{0\} &= \mathbb{F}(t)_{-1} \subset \mathbb{F}(t)_0 \subset \mathbb{F}(t)_1 \subset \cdots \to \mathbb{F}(t).
\end{aligned}$$

$\Diamond$

**Lemma 3.2.3.** *Let $\mathbb{W}$ be a subspace of $\mathbb{V}$. Then*

$$\mathbb{V} \simeq \mathbb{V}/\mathbb{W} \oplus \mathbb{W}.$$

**Example 3.2.3.** Let $\mathbb{A}[t]$ be a polynomial ring over $\mathbb{A}$, $\mathbb{K} \subseteq \mathbb{A}$ be a field and consider $\mathbb{A}[t]$ as a vector space over $\mathbb{K}$. We have

$$\mathbb{A}[t]_d/\mathbb{A}[t]_{d-1} \simeq t^d \, \mathbb{A}$$

and thus

$$\mathbb{A}[t]_d = t^d \, \mathbb{A} \oplus \mathbb{A}[t]_{d-1}.$$

$\Diamond$

**Definition 3.2.6.** Let $\mathbb{A}[t]$ be a polynomial ring over $\mathbb{A}$ and let $\| \ \|$ be the rank function of $\mathbb{A}$. For $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{A}[t]^n$ we define

$$\|\mathbf{f}\| = \max_i \|f_i\|.$$

$\Diamond$

**Lemma 3.2.4.** *Let* $(\mathbb{A}[t], \sigma)$ *be a difference ring canonically defined by*

$$\sigma(t) = \alpha\, t + \beta$$

*with* $\alpha \in \mathbb{A}^*$, $\beta \in \mathbb{A}$ *and* $t$ *transcendental over* $\mathbb{A}$. *Let* $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}[t]^m$ *and* $f, g \in \mathbb{A}[t]$ *such that*

$$\sigma_{\mathbf{a}} g = f.$$

*Then*

$$\|f\| \leq \|\mathbf{a}\| + \|g\|.$$

*If* $\mathbb{A}$ *is an integral domain,* $g \neq 0$ *and if for* $\mathbf{a} = (a_1, \dots, a_m)$ *we have*

$$\begin{aligned} &\|a_r\| = \|\mathbf{a}\| \text{ for some } r \in \{1, \dots, m\}, \\ &\|a_i\| < \|\mathbf{a}\| \;\; \forall i \neq r \end{aligned}$$

*then*

$$\|f\| = \|\mathbf{a}\| + \|g\|.$$

*Proof.* If $g = 0$, we have $f = \sigma_{\mathbf{a}} g = 0$ and hence $-1 = \|f\| \leq \|\mathbf{a}\| + \|g\|$ with $\|g\| = -1$ and $\|\mathbf{a}\| \geq 0$ holds. Otherwise assume that $g \neq 0$. We have

$$\|f\| = \|\sigma_{\mathbf{a}} g\| = \|a_1\, \sigma^{m-1}(g) + \dots + a_m\, g\| \leq \max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|).$$

Please note that we have $\|a_i\, \sigma^{m-i}(g)\| \leq \|a_i\| + \|\sigma^{m-i}(g)\|$, if $a_i = 0$; otherwise, if $a_i \neq 0$, we even have equality. Moreover if $a_i = 0$ and $a_j \neq 0$ then $\|a_i\| + \|\sigma^{m-i}(g)\| < \|a_j\| + \|\sigma^{m-j}(g)\|$. Since there exists an $j$ with $a_j \neq 0$, it follows that

$$\max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|) = \max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|).$$

By Lemma 2.1.2 we have $\|\sigma^i(g)\| = \|g\|$ for all $i \in \mathbb{Z}$ and thus

$$\max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|) = \max(\|a_1\|, \dots, \|a_m\|) + \|g\| = \|\mathbf{a}\| + \|g\|$$

which proves the first statement of the lemma. If there exists additionally an $r$ with the above properties, we have

$$\|a_1\, \sigma^{m-1}(g) + \dots + a_m\, g\| = \|a_r\, \sigma^{m-r}(g)\| = \max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|)$$

and by the same argumentations as for the first statement the second second follows immediately. $\square$

### 3.2.4 The Incremental Solution Space

In order to deal with the incremental reduction technique, we finally have to introduce the incremental solution space.

**Definition 3.2.7.** Let $(\mathbb{A}[t], \sigma)$ be a difference ring with constant field $\mathbb{K}$ and $t$ transcendental over $\mathbb{A}$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}[t]^m$ with $l := \|\mathbf{a}\|$ and let $\mathbf{f} \in \mathbb{A}[t]_{d+l}$ for some $d \in \mathbb{N}_0$. We define the *incremental solution space* by

$$\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A}) := \{\mathbf{c} \wedge g \in \mathbb{K}^n \times t^d\,\mathbb{A} \,|\, \sigma_{\mathbf{a}} g - \mathbf{c}\mathbf{f} \in \mathbb{A}[t]_{d+l-1}\}.$$

$\diamondsuit$

Let $(\mathbb{A}[t], \sigma)$ be a difference ring canonically defined by

$$\sigma(t) = \alpha\, t + \beta$$

with $\alpha \in \mathbb{A}^*$, $\beta \in \mathbb{A}$, constant field $\mathbb{K}$ and $t$ transcendental over $\mathbb{A}$. If $g \in t^d\,\mathbb{A}$ and $f \in \mathbb{A}[t]_{d+l}$, by Lemma 3.2.4 we have

$$\begin{aligned} \|\sigma_{\mathbf{a}} g\| &\leq \|\mathbf{a}\| + \|g\| = l + d, \\ \|f\| &\leq l + d. \end{aligned}$$

In other words, the incremental solution space $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ delivers us all linear combinations $\mathbf{c} \in \mathbb{K}^n$ of $\mathbf{f}$ over the constant field $\mathbb{K}$, $\mathbf{c}\,\mathbf{f}$, and all elements $g \in t^d\,\mathbb{A}$ such that the $l + d$-th coefficient, the coefficient of highest possible degree, of the polynomial

$$\sigma_{\mathbf{a}} g - \mathbf{c}\mathbf{f} \in \mathbb{A}[t]$$

vanishes.

**Lemma 3.2.5.** *Let $(\mathbb{A}[t], \sigma)$ be a difference ring with constant field $\mathbb{K}$ and $t$ transcendental over $\mathbb{K}$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}[t]^m$ with $l := \|\mathbf{a}\|$ and let $\mathbf{f} \in \mathbb{A}[t]_{d+l}$ for some $d \in \mathbb{N}_0$. Then the incremental solution space $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ is a vector space over $\mathbb{K}$.*

In Section 3.1.3.2 we already indicated that we want to achieve the following reduction

$$\begin{array}{c} \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d) \\ \downarrow \qquad \uparrow \\ \mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_{d-1}). \end{array}$$

As will be shown in the next section we can do this reduction under some conditions in the following way:

$$\begin{tikzcd} \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d) \arrow[dr, shift right, "1."'] & \\ & \mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A}) \arrow[ul, shift right, "3."'] \arrow[dl, "2."] \\ \mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_{d-1}) \arrow[uu, "3."] & \end{tikzcd}$$

This diagram has to be read as follows:

1. If the incremental solution space $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ has a finite basis representation we can try to compute its basis matrix.
2. Given this basis matrix we can compute a specific $\tilde{\mathbf{f}} \in \mathbb{A}[t]^{\lambda}_{d-1}$ for some $\lambda \geq 1$ which will be specified later. Now we can try to find a basis matrix for the solution space $\mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_{d-1})$ - if it has a finite basis representation.
3. By using the basis matrices of $\mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_{d-1})$ and $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ we finally can compute the basis matrix of the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d)$.

As will be shown later, this reduction technique will always work, if $(\mathbb{A}(t), \sigma)$ is a $\Pi\Sigma$-extension of a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$.

In the sequel we will explore some properties of the incremental solution space $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ which tell us under which circumstances it is a finite vector space over the constant field $\mathbb{K}$, i.e. there exists a basis matrix of $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$, and how one can find this basis matrix.

**Example 3.2.4.** Consider the $\Pi\Sigma$-field $(\mathbb{Q}(t_1, t_2), \sigma)$ over the constant field $\mathbb{Q}$ canonically defined by

$$\sigma(t_1) = t_1 + 1,$$
$$\sigma(t_2) = t_2 + \frac{1}{t_1 + 1}.$$

Let $c_1, c_2 \in \mathbb{Q}$ and $w \in \mathbb{Q}(t_1)$. We have

$$(c_1, c_2, t_2\, w) \in \mathrm{I}((1, -1), \left(\frac{-1 - 2\,t_2 - 2\,t_1\,t_2}{(1 + t_1)^2}, t_2\right), t_2\,\mathbb{Q}(t_1))$$

$$\Updownarrow \quad \text{(d=1,l=0)}$$

$$c_1 \frac{-1 - 2\,t_2 - 2\,t_1\,t_2}{(1 + t_1)^2} + c_2\,t_2 - (\boxed{\sigma(t_2\, w)} - t_2\, w) \in \mathbb{Q}(t_1)$$

$$\Updownarrow$$

$$c_1 \left(-\frac{1}{(t_1 + 1)^2} - \frac{2}{t_1 + 1}\,t_2\right) + c_2\,t_2 - (\boxed{(t_2 + \frac{1}{t_1 + 1})\,\sigma(w)} - t_2\, w) \in \mathbb{Q}(t_1)$$

$$\Updownarrow$$

$$\frac{-2\,c_1}{t_1 + 1} + c_2 - (\sigma(w) - w) = 0$$

$$\Updownarrow$$

$$(c_1, c_2, w) \in \mathrm{V}((1, -1), \left(\frac{-2}{t_1 + 1}, 1\right), \mathbb{Q}(t_1)).$$

As will be shown later, we can compute a basis matrix of $\mathrm{V}((1, -1), \left(\frac{1}{(t_1+1)^2}, 1\right), \mathbb{Q}(t_1))$ in the $\Pi\Sigma$-field $(\mathbb{Q}(t_1), \sigma)$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & t_1 \end{pmatrix} \xleftrightarrow{\text{basis}} \mathrm{V}((1, -1), \left(\frac{-2}{(t_1 + 1)^2}, 1\right), \mathbb{Q}(t_1)),$$

and therefore we compute the basis matrix of the incremental solution space:

$$\begin{pmatrix} 0 & 0 & t_2 \\ 0 & 1 & t_2\,t_1 \end{pmatrix} \xleftrightarrow{\text{basis}} \mathrm{V}((1, -1), \left(\frac{-1 - 2\,t_2 - 2\,t_1\,t_2}{(1 + t_1)^2}, t_2\right), t_2\,\mathbb{Q}(t_1)).$$

$\diamondsuit$

**Example 3.2.5.** Consider the $\Pi\Sigma$-field $(\mathbb{Q}(t_1, t_2), \sigma)$ over the constant field $\mathbb{Q}$ canonically defined by

$$\sigma(t_1) = t_1 + 1,$$
$$\sigma(t_2) = (t_1 + 1)\, t_2$$

and let

$$\mathbf{f} := \big(1 + (3 + 3\, t_1 + t_1^2)\, t_2 + (6 + 10\, t_1 + 6\, t_1^2 + t_1^3)\, t_2^2\big) = (f)\,,$$
$$\mathbf{a} := (t_2, 1, -t_2, t_2)\,.$$

For $c \in \mathbb{Q}$ and $w \in \mathbb{Q}(t_1)$ we have

$$(c, t_2\, w) \in \mathrm{I}(\mathbf{a}, \mathbf{f}, t_2\, \mathbb{Q}(t_1))$$

$$\Updownarrow \qquad \text{(d=1,l=1)}$$

$$c\, f - \sigma_{\mathbf{a}}(t_2\, w) \in \mathbb{Q}(t_1)[t_2]_1$$

$$\Updownarrow$$

$$c\, f - (t_2 \boxed{\sigma^3(w\, t_2)} + \sigma^2(t_2\, w) - t_2\, \sigma(t_2\, w) + t_2^2\, w) \in \mathbb{Q}(t_1)[t_2]_1$$

$$\Updownarrow$$

$$c\, f - (t_2 \boxed{(t_1+3)(t_1+2)(t_1+1)\, t_2\, \sigma^3(w)} + (t_1+2)(t_1+1)\, t_2\, \sigma^2(w) - t_2\, (t_1+1)\, t_2 \sigma(w) + t_2^2\, w) \in \mathbb{Q}(t_1)[t_2]_1$$

$$\Updownarrow$$

$$c\, \underbrace{(6 + 10\, t_1 + 6\, t_1^2 + t_1^3)}_{=:\tilde{f}} - \big((t_1+3)(t_1+2)(t_1+1)\, \sigma^3(w) - (t_1+1)\, \sigma(w) + w\big) = 0$$

$$\Updownarrow$$

$$(c, w) \in \mathrm{V}(\tilde{\mathbf{a}}, \left(\tilde{f}\right), \mathbb{Q}(t_1))$$

where

$$\tilde{\mathbf{a}} := ((t_1+3)(t_1+2)(t_1+1), 0, -(t_1+1), 1)\,.$$

$\Diamond$

Let $(\mathbb{A}, \sigma)$ be a difference ring and let us recall the definition of the $\sigma$-factorial of $f \in \mathbb{A}$ from Definition 2.2.12 which has been defined by

$$(f)_k = \prod_{i=0}^{k-1} \sigma^i(f)$$

for $k \in \mathbb{Z}$.

**Example 3.2.6.** For the $\Pi\Sigma$-field $(\mathbb{Q}(t_1,t_2),\sigma)$ over the constant field $\mathbb{Q}$ canonically defined by

$$\begin{aligned}\sigma(t_1) &= t_1+1,\\ \sigma(t_2) &= (t_1+1)\,t_2\end{aligned}$$

we have for all $i \in \mathbb{N}_0$ that

$$\sigma^i(t_2) = (t_1+i)(t_1+i-1)\dots(t_1+1)\,t_2 = (t_1+1)_i\; t_2.$$

$\diamondsuit$

**Lemma 3.2.6.** *Let $(\mathbb{A}[t],\sigma)$ be a difference ring extension of $(\mathbb{A},\sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t+\beta, \quad \alpha \in \mathbb{A}^*,\, \beta \in \mathbb{A},$$

*with constant field $\mathbb{K}$ and $t$ is transcendental over $\mathbb{A}$. Let $\mathbf{0} \neq \mathbf{a} = (a_1,\dots,a_m) \in \mathbb{A}[t]^m$ with $l := \|\mathbf{a}\|$ and $\mathbf{f} \in \mathbb{A}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0$. Then*

$$\mathbf{c}\wedge(w\,t^d) \in \mathrm{I}(\mathbf{a},\mathbf{f},t^d\,\mathbb{A}) \Leftrightarrow \mathbf{c}\wedge w \in \mathrm{V}(\tilde{\mathbf{a}},\tilde{\mathbf{f}},\mathbb{A})$$

*where*

$$\begin{aligned}\mathbf{0} \neq \tilde{\mathbf{a}} &:= \Big([a_1]_l\;(\alpha)^d_{m-1},\dots,[a_{m-1}]_l\;(\alpha)^d_1,[a_m]_l\,(\alpha)^d_0\Big) \in \mathbb{A}^m,\\ \tilde{\mathbf{f}} &:= [\mathbf{f}]_{d+l} \in \mathbb{A}^n.\end{aligned}$$

*Proof.* We have

$$\begin{gathered}\mathbf{c}\wedge(w\,t^d) \in \mathrm{I}(\mathbf{a},\mathbf{f},t^d\,\mathbb{A})\\ \Updownarrow\\ \sigma_{\mathbf{a}}(w\,t^d) - \mathbf{c}\,\mathbf{f} \in \mathbb{A}[t]_{d+l-1}\\ \Updownarrow\\ a_1\,\sigma^{m-1}(w\,t^d)+\dots+a_m\,w\,t^d - \mathbf{c}\,\mathbf{f} \in \mathbb{A}[t]_{d+l-1}\\ \Updownarrow\\ a_1\,\sigma^{m-1}(w)\;(\alpha)^d_{m-1}\;t^d+\dots+a_{m-1}\,\sigma(w)\,\alpha^d\,t^d+a_m\,w\,t^d - \mathbf{c}\,\mathbf{f} \in \mathbb{A}[t]_{d+l-1}\\ \Updownarrow\\ \Big[a_1\,\sigma^{m-1}(w)\;(\alpha)^d_{m-1}\;t^d+\dots+a_{m-1}\,\sigma(w)\,\alpha^d\,t^d+a_m\,w\,t^d - \mathbf{c}\,\mathbf{f}\Big]_{d+l} = 0\\ \Updownarrow\\ [a_1]_l\;\sigma^{m-1}(w)\;(\alpha)^d_{m-1}+\dots+[a_{m-1}]_l\;\sigma(w)\,\alpha^d+[a_m]_l\;w - \mathbf{c}\;\underbrace{[\mathbf{f}]_{d+l}}_{\tilde{\mathbf{f}}} = 0\\ \Updownarrow\\ \sigma_{\tilde{\mathbf{a}}}w = \mathbf{c}\,\tilde{\mathbf{f}}\\ \Updownarrow\\ \mathbf{c}\wedge w \in \mathrm{V}(\tilde{\mathbf{a}},\tilde{\mathbf{f}},\mathbb{A})\end{gathered}$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{f}}$ as from above. $\square$

The following theorem is a generalization of Theorem 16 in [Kar81] which I have extended from the first order case to the $m$-th order case of linear difference equations. Furthermore I have extended the difference field domain $(\mathbb{F}[t], \sigma)$ to the more general domain $(\mathbb{A}[t], \sigma)$ where $(\mathbb{A}, \sigma)$ is a difference ring.

**Theorem 3.2.1.** *Let $(\mathbb{A}[t], \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ canonically defined by*

$$\sigma(t) = \alpha\, t + \beta, \quad \alpha \in \mathbb{A}^*, \, \beta \in \mathbb{A},$$

*with constant field $\mathbb{K}$ and $t$ is transcendental over $\mathbb{A}$. Let $\mathbf{0} \neq \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{A}[t]^m$ with $l := \|\mathbf{a}\|$, $\mathbf{f} \in \mathbb{A}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0$ and let*

$$\begin{aligned} \mathbf{0} \neq \tilde{\mathbf{a}} &:= \Big( [a_1]_l \; (\alpha)^d_{m-1}, \ldots, [a_{m-1}]_l \; (\alpha)^d_1, [a_m]_l \, (\alpha)^d_0 \Big) \in \mathbb{A}^m, \\ \tilde{\mathbf{f}} &:= [\mathbf{f}]_{d+l} \in \mathbb{A}^n. \end{aligned}$$

*Then $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\, \mathbb{A})$ is a finite dimensional vector space over $\mathbb{K}$ if and only if $\tilde{\mathbf{a}}$ is $\mathrm{V}$-finite. Furthermore, if $\tilde{\mathbf{a}}$ is $\mathrm{V}$-finite then we have*

$$\begin{array}{ccc} \mathbf{C}\wedge\mathbf{w} & \xleftrightarrow{\text{basis}} & \mathrm{V}(\tilde{\mathbf{a}}, \tilde{\mathbf{f}}, \mathbb{A}) \\ & \Updownarrow & \\ \mathbf{C}\wedge(\mathbf{w}\,\mathbf{t}^{\mathbf{d}}) & \xleftrightarrow{\text{basis}} & \mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\, \mathbb{A}). \end{array}$$

*Proof.* If $\tilde{\mathbf{a}}$ is V-finite, by Proposition 3.1.1 $\mathrm{V}(\tilde{\mathbf{a}}, \tilde{\mathbf{f}}, \mathbb{A})$ is a finite dimensional vector space over $\mathbb{K}$ and therefore there is a basis matrix $\mathbf{C}\wedge\mathbf{w}$ for $\mathrm{V}(\tilde{\mathbf{a}}, \tilde{\mathbf{f}}, \mathbb{A})$. But then by Lemma 3.2.6 $\mathbf{C}\wedge\mathbf{w}\,\mathbf{t}^{\mathbf{d}}$ is a basis matrix for $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\, \mathbb{A})$ and consequently $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\, \mathbb{A})$ is a finite dimensional vector space over $\mathbb{K}$. Conversely, if $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\, \mathbb{A})$ is a finite dimensional vector space over $\mathbb{K}$, there is a basis matrix $\mathbf{C}\wedge\mathbf{w}$ for $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\, \mathbb{A})$ and consequently by Lemma 3.2.6 $\mathbf{C}\wedge\frac{\mathbf{w}}{\mathbf{t}^{\mathbf{d}}}$ is a basis matrix for $\mathrm{V}(\tilde{\mathbf{a}}, \tilde{\mathbf{f}}, \mathbb{A})$ which means that $\mathrm{V}(\tilde{\mathbf{a}}, \tilde{\mathbf{f}}, \mathbb{A})$ is a finite dimensional vector space over $\mathbb{K}$.
The second statement follows directly from these argumentations. $\square$

This reduction given in Theorem 3.2.1 will be indicated by the diagram

$$\begin{array}{c} \mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\, \mathbb{A}) \\ \Big\downarrow 1. \quad 2. \Big\uparrow \\ \mathrm{V}(\tilde{\mathbf{a}}, \tilde{\mathbf{f}}, \mathbb{A}). \end{array}$$

### 3.2.5 The Incremental Reduction

If we are capable of computing a finite basis of the incremental solution space $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ then we can do the following reduction by Theorem 3.2.2 below.

$$\begin{tikzcd} \mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{A}[t]_d) \arrow[dd, shift left] \\ \\ \mathrm{V}(\mathbf{a},\tilde{\mathbf{f}},\mathbb{A}[t]_{d-1}) \arrow[uu, shift left] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{A}[t]_d) \arrow[dr, shift left, "1."'] & \\ & \mathrm{I}(\mathbf{a},\mathbf{f},t^d\,\mathbb{A}) \arrow[ul, shift left, "3."'] \arrow[dl, "2."] \\ \mathrm{V}(\mathbf{a},\tilde{\mathbf{f}},\mathbb{A}[t]_{d-1}) \arrow[uu, "3."'] & \end{tikzcd}$$

The following theorem is a generalization of Theorem 12 in [Kar81] which I have extended from the first order case to the $m$-th order case of linear difference equations. Furthermore I have extended the difference field domain $(\mathbb{F}[t], \sigma)$ to the more general domain $(\mathbb{A}[t], \sigma)$ where $(\mathbb{A}, \sigma)$ is a difference ring.

**Theorem 3.2.2 (Incremental Reduction Theorem).** *Let $(\mathbb{A}[t], \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ with constant field $\mathbb{K}$ and*

$$\sigma(t) = \alpha\, t + \beta, \quad \alpha \in \mathbb{A}^*,\, \beta \in \mathbb{A},$$

*where $t$ is transcendental over $\mathbb{A}$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}[t]^m$ with $l := \|\mathbf{a}\|$ and $\mathbf{f} \in \mathbb{A}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0$. If $\mathbf{a}$ is $\mathrm{V}$-finite and $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ is a finite dimensional vector space, we can carry out the following reduction process:*

1. *Let*
$$\mathbf{C}\wedge\mathbf{g} \xleftrightarrow{\text{basis}} \mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$$
*with $\mathbf{C} \in \mathbb{K}^{\lambda\times n}$ and $\mathbf{g} \in (t^d\,\mathbb{A})^\lambda$ for some $\lambda \geq 1$.*

2. *Take*[a]
$$\tilde{\mathbf{f}} := \mathbf{C}\cdot\mathbf{f} - \sigma_{\mathbf{a}}\mathbf{g} \in \mathbb{A}[t]^\lambda_{d+l-1}$$
*and let*
$$\mathbf{D}\wedge\mathbf{h} \xleftrightarrow{\text{basis}} \mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_{d-1})$$
*with $\mathbf{D} \in \mathbb{K}^{\mu\times\lambda}$ and $\mathbf{h} \in \mathbb{A}[t]^\mu_{d-1}$ for some $\mu \geq 1$.*

3. *Then*
$$(\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g}) \xleftrightarrow{\text{basis}} \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d)$$
*with $\mathbf{D}\,\mathbf{C} \in \mathbb{K}^{\mu\times n}$ and $\mathbf{h} + \mathbf{D}\cdot\mathbf{g} \in \mathbb{A}[t]^\mu_d$.*

[a] Note that $\tilde{\mathbf{f}}$ has entries in $\mathbb{A}_{d+l-1}$ by Definition 3.2.7.

#### 3.2.5.1 An Example of an Incremental Reduction Step

**Example 3.2.7.** Let $(\mathbb{Q}(t_1, t_2), \sigma)$ be the $\Pi\Sigma$-field over the constant field $\mathbb{Q}$ canonically defined by

$$\begin{aligned}\sigma(t_1) &= t_1 + 1,\\ \sigma(t_2) &= t_2 + \frac{1}{t_2 + 1}.\end{aligned}$$

We illustrate Theorem 3.2.2 by the following reduction:

$$\begin{array}{ccc}
\mathrm{V}((1,-1),\overbrace{\left(\frac{-1-2\,t_2-2\,t_1\,t_2}{(1+t_1)^2},t_2\right)}^{=:\mathbf{f}},\mathbb{Q}(t_1)[t_2]_1) & \underset{1.}{\searrow}\;\overset{3.}{\nwarrow} & \\
\uparrow & & \mathrm{I}((1,-1),\mathbf{f},t_2\,\mathbb{Q}(t_1)) \\
\mathrm{V}(1,-1,\underbrace{\left(\frac{-1}{t_1+1},-1\right)}_{=:\tilde{\mathbf{f}}},\underbrace{\mathbb{Q}(t_1)[t_2]_0}_{\mathbb{Q}(t_1)}) & \underset{2.}{\swarrow} &
\end{array}$$

1. Compute[5]

$$\mathrm{I}((1,-1),\mathbf{f},t_2\,\mathbb{Q}(t_1)) \xleftrightarrow{\text{basis}} \mathbf{C}\wedge\mathbf{g} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\wedge\begin{pmatrix} t_2 \\ t_2\,t_1 \end{pmatrix}.$$

2. Let

$$\tilde{\mathbf{f}} := \mathbf{C}\cdot\mathbf{f} - (\sigma(\mathbf{g}) - \mathbf{g}) = \begin{pmatrix} \frac{-1}{t_1+1} \\ -1 \end{pmatrix} \in \mathbb{Q}(t_1)[t_2]_0^2 = \mathbb{Q}(t_1)^2$$

and compute

$$\mathrm{V}((1,-1),\left(\frac{-1}{t_1+1},-1\right),\mathbb{Q}(t_1)[t_2]_0) \xleftrightarrow{\text{basis}} \mathbf{D}\wedge\mathbf{h} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\wedge\begin{pmatrix} t_1 \\ 1 \end{pmatrix}.$$

3. Compute

$$\mathbf{DC} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{h} + \mathbf{D}\cdot\mathbf{g} = \begin{pmatrix} t_1 - t_1\,t_2 \\ 1 \end{pmatrix}.$$

We get

$$\begin{gathered}(\mathbf{D\,C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g}) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\wedge\begin{pmatrix} t_1 - t_1\,t_2 \\ 1 \end{pmatrix}\\ \updownarrow \\ \mathrm{V}((1,-1),\left(\frac{-1-2\,t_2-2\,t_1\,t_2}{(1+t_1)^2},t_2\right),\mathbb{Q}(t_1)[t_2]_1).\end{gathered}$$

$\Diamond$

[5] See Example 3.2.4 for further details.

#### 3.2.5.2 The Proof of the Incremental Reduction Theorem

*Proof.* As $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ and $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t])$ are finite dimensional vector spaces, we may assume that $\mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_{d-1})$, $\mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_d)$ and $\mathrm{I}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ have basis matrices. Given the matrices $\mathbf{C}\wedge\mathbf{g}$ and $\mathbf{D}\wedge\mathbf{h}$ stated as above, we clearly have $\mathbf{D}\,\mathbf{C} \in \mathbb{K}^{\mu\times n}$ and $\mathbf{h} + \mathbf{D}\cdot\mathbf{g} \in \mathbb{A}[t]_d^{\mu}$.

**Step 1**

First we will show that $(\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\,\mathbf{g})$ generates a subspace of $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d)$. We have

$$\begin{aligned}\sigma_{\mathbf{a}}\mathbf{h} = \mathbf{D}\cdot\tilde{\mathbf{f}} = \mathbf{D}\cdot(\mathbf{C}\cdot\mathbf{f} - \sigma_{\mathbf{a}}\mathbf{g}) &\Leftrightarrow \sigma_{\mathbf{a}}\mathbf{h} = \mathbf{D}\cdot(\mathbf{C}\cdot\mathbf{f}) - \mathbf{D}\cdot\sigma_{\mathbf{a}}\mathbf{g}\\ &\Leftrightarrow \sigma_{\mathbf{a}}\mathbf{h} + \mathbf{D}\cdot\sigma_{\mathbf{a}}\mathbf{g} = (\mathbf{D}\,\mathbf{C})\cdot\mathbf{f}\\ &\Leftrightarrow \sigma_{\mathbf{a}}(\mathbf{h} + \mathbf{D}\cdot\mathbf{g}) = (\mathbf{D}\,\mathbf{C})\cdot\mathbf{f}\end{aligned}$$

and by $\mathbf{h}+\mathbf{D}\cdot\mathbf{g} \in \mathbb{A}[t]_d^{\mu}$ it follows that $(\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g})$ generates a subspace of $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d)$.

**Step 2**

Second we will show that $(\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g})$ is a basis matrix. If

$$(\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g}) = (0, \dots, 0) \tag{3.3}$$

then by convention it is a basis matrix and represents the vector space $\{\mathbf{0}\} \subseteq \mathbb{K}^n \times \mathbb{A}$.

Otherwise, assume that the basis matrix is not of the form (3.3). We will show that the rows in the matrix $(\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g})$ are linearly independent over $\mathbb{K}$ which proves that it is a basis matrix. Assume the rows are linearly dependent. Then there is a $\mathbf{0} \neq \mathbf{k} \in \mathbb{K}^{\mu}$ such that

$$\mathbf{k}\cdot((\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g})) = \mathbf{0}.$$

- Now assume that
$$\mathbf{k}\cdot\mathbf{D} = \mathbf{0}. \tag{3.4}$$
$\mathbf{D}\wedge\mathbf{h}$ is a basis matrix by assumption. If $\mathbf{D}\wedge\mathbf{h}$ consists of exactly one zero-row, we are in the case (3.3), a contradiction. Therefore we may assume that the rows are nonzero and linearly independent over $\mathbb{K}$. Hence by (3.4) it follows that
$$0 \neq \mathbf{k}\,\mathbf{h} \in \mathbb{A}[t]_{d-1}. \tag{3.5}$$
Since $\mathbf{g} \in (t^d\,\mathbb{A})^{\lambda}$, we can conclude that
$$\mathbf{k}\,(\mathbf{D}\cdot\mathbf{g}) \in t^d\,\mathbb{A}.$$
Therefore by (3.5) we have
$$\mathbf{0} \neq \mathbf{k}\,\mathbf{h} + \mathbf{k}\,(\mathbf{D}\cdot\mathbf{g}) = \mathbf{k}\,(\mathbf{h} + \mathbf{D}\cdot\mathbf{g})$$
and thus
$$\mathbf{k}\cdot((\mathbf{D}\,\mathbf{C})\wedge(\mathbf{h} + \mathbf{D}\cdot\mathbf{g})) \neq \mathbf{0},$$
a contradiction.

- Otherwise, assume that
$$\mathbf{v} := \mathbf{k} \cdot \mathbf{D} \neq \mathbf{0}.$$
Then we have
$$\begin{aligned}\mathbf{0} &= \mathbf{k} \cdot ((\mathbf{D}\,\mathbf{C}) \wedge (\mathbf{h} + \mathbf{D} \cdot \mathbf{g})) = (\mathbf{k} \cdot (\mathbf{D}\,\mathbf{C})) \wedge (\mathbf{k}\,(\mathbf{h} + \mathbf{D} \cdot \mathbf{g}))\\ &= (\mathbf{k} \cdot \mathbf{D}) \cdot \mathbf{C}) \wedge (\mathbf{k}\,\mathbf{h} + (\mathbf{k} \cdot \mathbf{D}) \cdot \mathbf{g})) = (\mathbf{v} \cdot \mathbf{C}) \wedge (\mathbf{k}\,\mathbf{h} + \mathbf{v}\,\mathbf{g})\end{aligned}$$
and thus
$$\mathbf{v} \cdot \mathbf{C} = \mathbf{0} \quad \text{and} \quad \mathbf{k}\,\mathbf{h} + \mathbf{v}\,\mathbf{g} = \mathbf{0}. \tag{3.6}$$
But $\mathbf{C} \wedge \mathbf{g}$ is a basis matrix.
  - Assume $\mathbf{C} \wedge \mathbf{g}$ does not consist of exactly one zero-row. Then by assumption the rows must be all nonzero and linearly independent over $\mathbb{K}$. Hence by (3.6) we have
  $$0 \neq \mathbf{v}\,\mathbf{g} \in t^d\,\mathbb{A}.$$
  As $\mathbf{k}\,\mathbf{h} \in \mathbb{A}[t]_{d-1}$, we finally get $\mathbf{k}\,\mathbf{h} + \mathbf{v}\,\mathbf{g} \neq 0$, a contradiction to (3.6).
  - Otherwise, $\mathbf{C} \wedge \mathbf{g}$ consists of exactly one zero-row. Then it follows that
  $$(\mathbf{D}\,\mathbf{C}) \wedge (\mathbf{h} + \mathbf{D} \cdot \mathbf{g}) = (0, \ldots, 0, h_1)\,.$$
  If $h_1 = 0$, we are in the case (3.3), a contradiction. Otherwise, the matrix $(0, \ldots, 0, h_1)$ has a nonzero row which is of course linearly independent over $\mathbb{K}$.

Altogether it follows that $(\mathbf{D}\,\mathbf{C}) \wedge (\mathbf{h} + \mathbf{D} \cdot \mathbf{g})$ is a basis matrix and there is a vector space $\mathbb{W}$ such that
$$\mathbb{W} \xleftrightarrow{\text{basis}} (\mathbf{D}\,\mathbf{C}) \wedge (\mathbf{h} + \mathbf{D} \cdot \mathbf{g})$$
where
$$\mathbb{W} \subseteq \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d). \tag{3.7}$$

**Step 3**

Finally we show equality for (3.7) which proves the theorem. Assume
$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d) \xleftrightarrow{\text{basis}} \tilde{\mathbf{E}} \wedge \tilde{\mathbf{h}}$$
with $\tilde{\mathbf{E}} \in \mathbb{K}^{\nu \times n}$, $\tilde{\mathbf{h}} \in \mathbb{A}[t]_d^\nu$ and write
$$\tilde{\mathbf{h}} = \underbrace{\mathbf{h_1}}_{(t^d\,\mathbb{A})^\nu} + \underbrace{\mathbf{h_2}}_{\mathbb{A}[t]_{d-1}^\nu}.$$
Let $\mathbb{V}$ be the vector space such that
$$\tilde{\mathbf{E}} \wedge \mathbf{h_1} \xleftrightarrow{\text{span}} \mathbb{V}.$$
Since
$$\mathbf{0} = \sigma_{\mathbf{a}} \tilde{\mathbf{h}} - \tilde{\mathbf{E}} \cdot \mathbf{f} = \sigma_{\mathbf{a}} \mathbf{h_1} + \sigma_{\mathbf{a}} \mathbf{h_2} - \tilde{\mathbf{E}} \cdot \mathbf{f} \tag{3.8}$$

by assumption and

$$\sigma_{\mathbf{a}}\mathbf{h_2} \in \mathbb{A}[t]^{\nu}_{d+l-1}$$

by Lemma 3.2.4, it follows that

$$\sigma_{\mathbf{a}}\mathbf{h_1} - \tilde{\mathbf{E}}\mathbf{f} \in \mathbb{A}[t]^{s}_{d+l-1}.$$

Therefore $\mathbb{V} \subset \mathrm{V}(\mathbf{a}, \mathbf{f}, t^d\,\mathbb{A})$ and thus by Lemma 3.2.1 we find a matrix $\tilde{\mathbf{D}} \in \mathbb{K}^{\lambda\times\nu}$ such that

$$\tilde{\mathbf{E}}\wedge\mathbf{h_1} = \tilde{\mathbf{D}}(\mathbf{C}\wedge\mathbf{g}) = (\tilde{\mathbf{D}}\,\mathbf{C})\wedge(\tilde{\mathbf{D}}\cdot\mathbf{g}),$$

this means

$$\tilde{\mathbf{E}} = \tilde{\mathbf{D}}\,\mathbf{C} \text{ and } \mathbf{h_1} = \tilde{\mathbf{D}}\cdot\mathbf{g}. \tag{3.9}$$

By (3.8) we have

$$\sigma_{\mathbf{a}}\mathbf{h_2} = \tilde{\mathbf{E}}\cdot\mathbf{f} - \sigma_{\mathbf{a}}\mathbf{h_1} \overset{(3.9)}{=} (\tilde{\mathbf{D}}\,\mathbf{C})\cdot\mathbf{f} - \sigma_{\mathbf{a}}(\tilde{\mathbf{D}}\cdot\mathbf{g}) = \tilde{\mathbf{D}}\cdot(\mathbf{C}\cdot\mathbf{f}) - \tilde{\mathbf{D}}\cdot\sigma_{\mathbf{a}}\mathbf{g} = \tilde{\mathbf{D}}\cdot(\mathbf{C}\cdot\mathbf{f} - \sigma_{\mathbf{a}}\mathbf{g})$$

and hence

$$\sigma_{\mathbf{a}}\mathbf{h_2} = \tilde{\mathbf{D}}\cdot\tilde{\mathbf{f}}. \tag{3.10}$$

Let $\mathbb{U}$ be the vector space such that

$$\mathbb{U} \xleftrightarrow{\text{span}} \tilde{\mathbf{D}}\wedge\mathbf{h_2}.$$

Then by (3.10) it follows that $\mathbb{U} \subseteq \mathrm{V}(\mathbf{a}, \tilde{\mathbf{f}}, \mathbb{A}[t]_{d-1})$ and thus by Lemma 3.2.1 we find a matrix $\mathbf{K} \in \mathbb{K}^{\nu\times\mu}$ such that

$$\tilde{\mathbf{D}}\wedge\mathbf{h_2} = \mathbf{K}\,(\mathbf{D}\wedge\mathbf{h}) = (\mathbf{K}\,\mathbf{D})\wedge(\mathbf{K}\cdot\mathbf{h}),$$

this means

$$\tilde{\mathbf{D}} = \mathbf{K}\,\mathbf{D} \text{ and } \mathbf{h_2} = \mathbf{K}\cdot\mathbf{h}. \tag{3.11}$$

Then

$$\begin{aligned}\tilde{\mathbf{E}}\wedge\tilde{\mathbf{h}} = \tilde{\mathbf{E}}\wedge(\mathbf{h_1}+\mathbf{h_2}) &\overset{(3.9)}{=} (\tilde{\mathbf{D}}\,\mathbf{C})\wedge(\tilde{\mathbf{D}}\cdot\mathbf{g}+\mathbf{h_2})\\ &\overset{(3.11)}{=} (\mathbf{K}\,\mathbf{D}\,\mathbf{C})\wedge((\mathbf{K}\,\mathbf{D})\cdot\mathbf{g}+\mathbf{K}\cdot\mathbf{h}) = \mathbf{K}\,((\mathbf{D}\,\mathbf{C})\wedge(\mathbf{D}\cdot\mathbf{g}+\mathbf{h})).\end{aligned}$$

and it follows[6] that

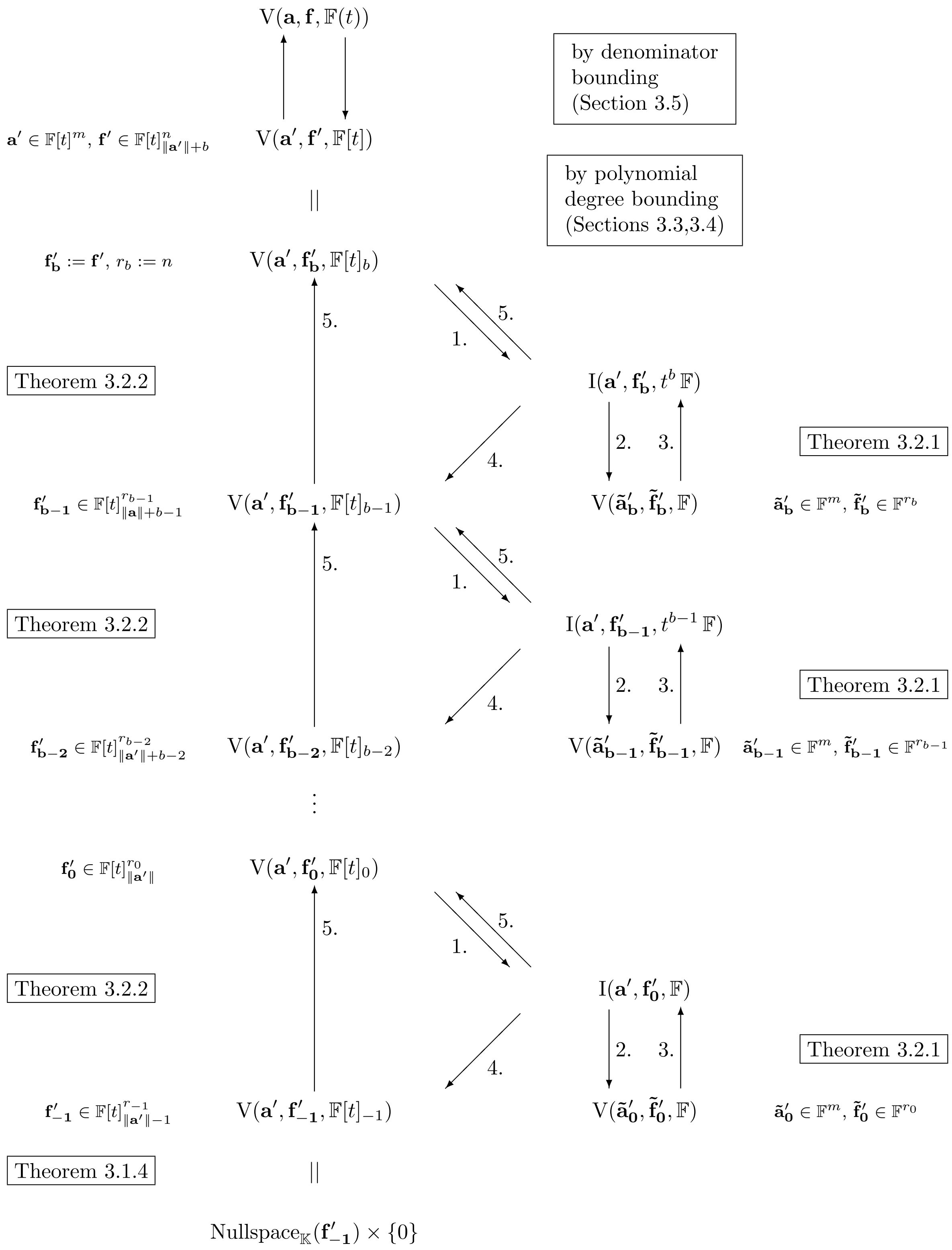
$$\mathbb{W} \supseteq \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_d)$$

which proves the theorem. □

[6]In particular, $\mathbf{K}$ is a basis transformation, i.e. $\nu = \mu$ and $\mathbf{K}$ is invertible.

### 3.2.6 The Complete Reduction Process in $\Pi\Sigma$-fields

Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-field over the constant field $\mathbb{K}$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$ and $\mathbf{f} \in \mathbb{F}[t]^n$. Since any vector $\mathbf{0} \neq \mathbf{b}$ from $\mathbb{F}(t)$ is V-finite by Proposition 3.1.2, we may apply Theorems 3.2.2 and 3.2.1 and obtain the following reduction process to compute a basis of $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$.



This has to be read as follows.

- Suppose we find $d \in \mathbb{F}[t]^*$, $\mathbf{0} \neq \mathbf{a}' \in \mathbb{F}[t]^m$ with $l := \|\mathbf{a}'\| \geq 0$ and $\mathbf{f}' \in \mathbb{F}[t]^n$ by the

  | denominator bounding method<br>(Sections 3.1.3.1 and 3.5) |
  |---|

  such that
  $$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t)) = \left\{ \mathbf{c} \wedge \frac{g}{d} \mid \mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}[t]) \right\}.$$
  Then given a basis matrix for $\mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}[t])$ we are able to compute a basis matrix for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$.

- Suppose we find a

  | polynomial degree bound<br>(Sections 3.3 and 3.4) |
  |---|

  $b \in \mathbb{N}_0 \cup \{-1\}$ such that
  $$\mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}[t]) = \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}[t]_b)$$
  where
  $$\mathbf{f}'_{\mathbf{b}} := \mathbf{f}' \in \mathbb{F}[t]^{r_b}_{l+b}, \quad r_b := n.$$

- In order to find a basis matrix for $\mathrm{V}(\mathbf{a}', \mathbf{f}'_{\mathbf{b}}, \mathbb{F}[t]_b)$,

  | we apply Theorem 3.2.2. |
  |---|

  1. We have to compute a basis matrix for the incremental solution space $\mathrm{I}(\mathbf{a}', \mathbf{f}'_{\mathbf{b}}, t^b\,\mathbb{F})$.
  2. In order to achieve this,

     | we apply Theorem 3.2.1. |
     |---|

     This tells us how we can compute $\mathbf{0} \neq \tilde{\mathbf{a}}'_{\mathbf{b}} \in \mathbb{F}^m$ and $\tilde{\mathbf{f}}'_{\mathbf{b}} \in \mathbb{F}^{r_b}$ and that we must find a basis matrix for $\mathrm{V}(\tilde{\mathbf{a}}'_{\mathbf{b}}, \tilde{\mathbf{f}}'_{\mathbf{b}}, \mathbb{F})$.

     | Now we have to start the reduction process again,<br>but this time in the $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$. |
     |---|

  3. Now we construct the basis matrix for $\mathrm{I}(\mathbf{a}', \mathbf{f}'_{\mathbf{b}}, t^b\,\mathbb{F})$ by using the basis matrix for $\mathrm{V}(\tilde{\mathbf{a}}'_{\mathbf{b}}, \tilde{\mathbf{f}}'_{\mathbf{b}}, \mathbb{F})$.
  4. Now Theorem 3.2.1 tells us how to compute $\mathbf{f}'_{\mathbf{b-1}} \in \mathbb{F}[t]^{r_{b-1}}_{l+b-1}$ for some $r_{b-1} \geq 1$ and that we have to find a basis matrix for $\mathrm{V}(\mathbf{a}', \mathbf{f}'_{\mathbf{b-1}}, \mathbb{F}[t]_{b-1})$.

     | For this we have again to apply Theorem 3.2.1. |
     |---|

  5. Finally we can obtain a basis matrix for $\mathrm{V}(\mathbf{a}', \mathbf{f}'_{\mathbf{b}}, \mathbb{F}[t]_b)$ by the basis matrices of $\mathrm{I}(\mathbf{a}', \mathbf{f}'_{\mathbf{b}}, t^b\,\mathbb{F})$ and $\mathrm{V}(\mathbf{a}', \mathbf{f}'_{\mathbf{b-1}}, \mathbb{F}[t]_{b-1})$.

#### 3.2.6.1 The First Base Case

Finally, by Theorem 3.1.4 we have to deal with the problem to find a basis of $\text{Nullspace}_{\mathbb{K}}(\mathbf{f}'_{-\mathbf{1}})$ where $\mathbf{f}'_{-\mathbf{1}} \in \mathbb{F}[t]^{r-1}_{\|\mathbf{a}\|-1}$. By the following lemma we obtain an algorithm which computes a basis of this vector space.

**Lemma 3.2.7.** *Let* $(\mathbb{F}, \sigma)$ *with* $\mathbb{F} := \mathbb{K}(t_1, \dots, t_e)$ *be a* $\Pi\Sigma$*-field over* $\mathbb{K}$ *and* $\mathbf{f} \in \mathbb{F}^n$*. Then* $\text{Nullspace}_{\mathbb{K}}(\mathbf{f})$ *is a finite dimensional subspace of* $\mathbb{K}^n$ *and a basis can be computed by linear algebra.*

*Proof.* Let $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. Since $\mathbb{F}$ is a $\Pi\Sigma$-field, it follows that $\mathbb{F}$ is the quotient field of the polynomial ring $\mathbb{K}[t_1, \dots, t_e]$. We can find a $d \in \mathbb{K}[t_1, \dots, t_e]^*$ such that

$$\mathbf{g} = (g_1, \dots, g_n) := (f_1\, d, \dots, f_n\, d) \in \mathbb{K}[t_1, \dots, t_e].$$

For $\mathbf{c} \in \mathbb{K}^n$ we have

$$\mathbf{c}\,\mathbf{f} = 0 \;\Leftrightarrow\; \mathbf{c}\,\mathbf{g} = 0$$

and therefore

$$\text{Nullspace}_{\mathbb{K}}(\mathbf{f}) = \text{Nullspace}_{\mathbb{K}}(\mathbf{g}).$$

Let $c_1, \dots, c_n$ be indeterminates and make the ansatz

$$c_1\, g_1 + \dots + c_n\, g_n = 0.$$

Then the coefficients of each monomial $t_1^{d_1} \dots t_e^{d_e}$ in $c_1\, g_1 + \dots + c_n\, g_n$ must vanish. Therefore we get a linear system of equations

$$\begin{array}{cccc} c_1\, p_{11}+ & \dots & +c_n\, p_{1n} & = 0 \\ & \vdots & & \\ c_r\, p_{r1}+ & \dots & +c_n\, p_{rn} & = 0 \end{array} \tag{3.12}$$

where each equation corresponds to a coefficient of a monomial which must vanish. Since $p_{ij} \in \mathbb{K}$, finding all $(c_1, \dots, c_n) \in \mathbb{K}^n$ which are a solution of (3.12) is a simple linear algebra problem. In particular applying Gaussian elimination we get immediately a basis for the vector space

$$\{\mathbf{c} \in \mathbb{K}^n \mid \mathbf{c} \text{ is a solution of (3.12)}\},$$

thus for $\text{Nullspace}_{\mathbb{K}}(\mathbf{g})$ and consequently also for $\text{Nullspace}_{\mathbb{K}}(\mathbf{f})$. $\square$

#### 3.2.6.2 An Example of the Complete Reduction Process

**Example 3.2.8.** Let $(\mathbb{Q}(t_1, t_2), \sigma)$ be the $\Pi\Sigma$-field over the constant field $\mathbb{Q}$ canonically defined by

$$t_1 = t_1 + 1,$$
$$t_2 = (t_1 + 1)\, t_2.$$

In order to find a $g \in \mathbb{Q}(t_1, t_2)$ such that

$$\sigma(g) - g = t_1\, t_2,$$

we compute a basis of the solution space $\mathrm{V}((1,-1),(t_1 t_2), \mathbb{Q}(t_1,t_2))$ by the following reduction.

$$\begin{tikzcd}
\mathrm{V}((1,-1),(t_1\, t_2), \mathbb{Q}(t_1,t_2))^{\dagger} \arrow[d, equal] & & \\
\mathrm{V}((1,-1),(t_1\, t_2), \mathbb{Q}(t_1)[t_2]) \arrow[d, equal] & & \\
\mathrm{V}((1,-1),(t_1\, t_2), \mathbb{Q}(t_1)[t_2]_1) \arrow[dr, shift left, "1."] & & \\
& \mathrm{I}((1,-1),(t_1\, t_2), t_2\, \mathbb{Q}(t_1)) \arrow[ul, shift left, "5."] \arrow[d, shift right, "2."'] \arrow[dl, "4."] & \\
\mathrm{V}((1,-1),(0), \mathbb{Q}(t_1)) \arrow[uu, "5."] \arrow[d, equal, "\boxed{\text{Short Cut}}"] & \mathrm{V}((t_1+1,-1),(t_1), \mathbb{Q}(t_1))^{\dagger} \arrow[u, shift right, "3."'] \arrow[d, equal] & \\
\mathbb{Q} \times \mathbb{Q} & \mathrm{V}((t_1+1,-1),(t_1), \mathbb{Q}[t_1]) \arrow[d, equal] & \\
& \mathrm{V}((t_1+1,-1),(t_1), \mathbb{Q}[t_1]_0) \arrow[dr, shift left, "1."] & \\
& & \mathrm{I}((t_1+1,-1),(t_1), t_1\, \mathbb{Q}) \arrow[ul, shift left, "5."] \arrow[d, shift right, "2."'] \arrow[dl, "4."] \\
& \mathrm{V}((t_1+1,-1),(0), \{0\}) \arrow[uu, "5."] \arrow[d, equal] & \mathrm{V}((1,0),(1), \mathbb{Q})^{\dagger} \arrow[u, shift right, "3."'] \arrow[d, equal, "\boxed{\text{New Base Case}}"] \\
& \mathrm{Nullspace}_{\mathbb{Q}}((0)) \times \{0\} & \mathrm{Nullspace}_{\mathbb{Q}}((-1,1))
\end{tikzcd}$$

$\Diamond$

What remains open in this reduction process is a shortcut and an additional base case indicated by framed boxes. These two situations will be considered in the following two sections.

#### 3.2.6.3 A Hidden Reduction Process and a New Base Case

Looking closer at the reduction process (look at the labels † in Example 3.2.8), one can notice a hidden reduction process, namely

$$\begin{array}{c} \mathrm{V}((1,-1),(t_1\,t_2)\,,\mathbb{Q}(t_1,t_2)) \\ \downarrow \qquad \uparrow \\ \mathrm{V}((t_1+1,-1),(t_1)\,,\mathbb{Q}(t_1)) \\ \downarrow \qquad \uparrow \\ \mathrm{V}((1,0),(1)\,,\mathbb{Q}). \end{array}$$

In the general case, for a ΠΣ-field $(\mathbb{F},\sigma)$ with $\mathbb{F} := \mathbb{K}(t_1,\dots,t_e)$ over the constant field $\mathbb{K}$ and $\mathbf{0} \neq \mathbf{a_e} \in \mathbb{F}^{m_e}$ and $\mathbf{f_e} \in \mathbb{F}^{n_e}$ the following **reduction process** pops up:

$$\begin{array}{c} \mathrm{V}(\mathbf{a_e},\mathbf{f_e},\mathbb{K}(t_1,\dots,t_e)) \\ \downarrow \qquad \uparrow \\ \mathrm{V}(\mathbf{a_{e-1}},\mathbf{f_{e-1}},\mathbb{K}(t_1,\dots,t_{e-1})) \\ \downarrow \qquad \uparrow \\ \mathrm{V}(\mathbf{a_{e-2}},\mathbf{f_{e-2}},\mathbb{K}(t_1,\dots,t_{e-2})) \\ \downarrow \qquad \uparrow \\ \vdots \qquad \vdots \\ \downarrow \qquad \uparrow \\ \mathrm{V}(\mathbf{a_1},\mathbf{f_1},\mathbb{K}(t_1)) \\ \downarrow \qquad \uparrow \\ \mathrm{V}(\mathbf{a_0},\mathbf{f_0},\mathbb{K}). \end{array}$$

Finally we have to deal with the problem to solve $\mathrm{V}(\mathbf{a_0},\mathbf{f_0},\mathbb{K})$ for some $\mathbf{0} \neq \mathbf{a_0} \in \mathbb{F}^{n_0}$ and $\mathbf{f_0} \in \mathbb{F}^{m_0}$. The following Theorem allows us to handle this second *base case*.

**Theorem 3.2.3.** *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$, $\mathbf{f} \in \mathbb{F}^n$ and $\mathbf{0} \neq \mathbf{a} = (a_1,\dots,a_m) \in \mathbb{F}^m$. Then*

$$\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{K}) = \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}\wedge u)$$

*where $u := -\sum_{i=1}^m a_i$.*

*Proof.* Let $\mathbf{c} \in \mathbb{K}^n$ and $g \in \mathbb{K}$. It follows that

$$\begin{aligned} \mathbf{c}\wedge g \in \mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{K}) &\Leftrightarrow \mathbf{c}\,\mathbf{f} - \sigma_{\mathbf{a}} g = 0 \\ &\Leftrightarrow \mathbf{c}\,\mathbf{f} - g\left(\sum_{i=0}^m a_i\right) = 0 \\ &\Leftrightarrow \mathbf{c}\wedge g \in \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}\wedge u). \end{aligned}$$

□

**Remark 3.2.1.** Given $\mathbf{f} \in \mathbb{K}^n$ in a field $\mathbb{K}$ a basis of $\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f})$ can be immediately computed by linear algebra. ◊

#### 3.2.6.4 A Shortcut

In Example 3.2.8 we used a shortcut [Kar81, Proposition 10] for the reduction process which is based on the following lemma.

**Lemma 3.2.8.** *Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$, $\mathbb{V}$ be a subspace of $\mathbb{A}$ over $\mathbb{K}$ and let $\mathbf{0_n} = (0, \dots, 0) \in \mathbb{A}^n$. Then*

$$\mathrm{V}((1, -1), \mathbf{0_n}, \mathbb{V}) := \begin{cases} \mathbb{K}^n \times \mathbb{K} & \textit{if } \mathbb{V} \cap \mathbb{K} = \mathbb{K} \\ \mathbb{K}^n \times \{0\} & \textit{otherwise.} \end{cases}$$

*Proof.* We have

$$\mathrm{V}(\mathbf{a}, \mathbf{0_n}, \mathbb{V}) = \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} \,|\, \sigma(g) - g = c_1\, 0 + \dots + c_n\, 0\}.$$

If $\mathbb{V} \cap \mathbb{K} = \mathbb{K}$,

$$\{g \in \mathbb{V} \,|\, \sigma(g) - g = 0\} = \mathbb{K}$$

and therefore

$$\mathrm{V}((1, -1), \mathbf{0_n}, \mathbb{V}) = \mathbb{K}^n \times \mathbb{K}.$$

Otherwise, we must have $\mathbb{V} = \{0\}$ and therefore it follows that

$$\mathrm{V}((1, -1), \mathbf{0_n}, \mathbb{V}) = \mathbb{K}^n \times \{0\}.$$

$\square$

#### 3.2.6.5 Solving Difference Equations in Mathematica

Let $\mathbb{K} := \mathbb{Q}(n_1, \dots, n_r)$ be a field of rational functions and let $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{K}(t_1, \dots, t_l)$ be a $\Pi\Sigma$-field over the constant field $\mathbb{K}$ canonically defined by

$$\sigma(t_i) = \alpha_i\, t_i + \beta_i, \quad \alpha_i \in \mathbb{K}(t_1, \dots, t_{i-1})^*,\ \beta_i \in \mathbb{K}(t_1, \dots, t_{i-1})$$

for $1 \leq i \leq l$. Let

$$\begin{aligned} \mathbf{0} \neq \mathbf{a} &= (a_1, \dots, a_m) \in \mathbb{F}^m, \\ \mathbf{f} &= (f_1, \dots, f_n) \in \mathbb{F}^n. \end{aligned}$$

Then by the function call

$$\texttt{SolveDifferenceVectorSpace}[\{\mathtt{a_1}, \dots, \mathtt{a_m}\}, \{\mathtt{f_1}, \dots, \mathtt{f_n}\}, \{\{\mathtt{t_1}, \alpha_1, \beta_1\}, \dots, \{\mathtt{t_r}, \alpha_\mathtt{r}, \beta_\mathtt{r}\}\}]$$

one computes a basis of the solution space

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}) = \mathrm{V}((a_1, \dots, a_m), (f_1, \dots, f_n)\,, \mathbb{K}(t_1, \dots, t_r))$$

using the reduction process sketched in Section 3.2.6. In the following sections we are concerned with the missing parts in this reduction process, namely with

- polynomial degree boundings (Sections 3.3 and 3.4)
- and denominator boundings (Section 3.5).

Furthermore, in Section 3.6, we will give some ideas how one can solve difference equations in some special difference rings. More precisely, we consider difference ring extensions $(\mathbb{F}[t], \sigma)$ of the $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ canonically defined by

$$\sigma(t) = \alpha\, t$$

where $\alpha \in \mathbb{K}$ is a primitive $k$-th root of unity and

$$t^k = 1.$$

All ideas in the following sections are used for the implementation of my Mathematica package.

## 3.3 Special Cases For Polynomial Boundings

Let $(\mathbb{A}[t], \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ with constant field $\mathbb{K}$ and $t$ transcendental over $\mathbb{A}$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}[t]^m$ and $\mathbf{f} \in \mathbb{A}[t]^n$. In this section we will deal with the problem to find a bound $b \in \mathbb{N}_0 \cup \{-1\}$ such that

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]) = \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_b)$$

for some special cases. As one can see for instance in Section 3.1.3.2 this bound is needed for the incremental reduction method.

### 3.3.1 A Lower Bound

**Proposition 3.3.1.** *Let $(\mathbb{A}[t], \sigma)$ be a difference ring with constant field $\mathbb{K}$ and $t$ transcendental over $\mathbb{A}$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}[t]^m$ and $f \in \mathbb{A}[t]$. If there is a $g \in \mathbb{A}[t]$ such that*

$$\sigma_{\mathbf{a}} g = f$$

*then*

$$\|g\| \geq \max(\|f\| - \|\mathbf{a}\|, -1).$$

*Proof.* By Lemma 3.2.4 we have

$$\|f\| - \|\mathbf{a}\| \leq \|g\|$$

and since $\|g\| \geq -1$, it follows that

$$\|g\| \geq \max(\|f\| - \|\mathbf{a}\|, -1).$$

□

For the general setting under discussion one does not know an algorithm to determine a polynomial degree bounding $b$ such that

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]) = \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t]_b)$$

for given $\mathbf{0} \neq \mathbf{a} \in \mathbb{A}[t]^m$ and $\mathbf{f} \in \mathbb{A}[t]^n$. In this case, the previous proposition motivates to choose heuristically a bound

$$\max(\|\mathbf{f}\| - \|\mathbf{a}\|, -1) + \texttt{plusBound}$$

where `plusBound` $\geq 0$ has to be chosen by the user and must be incremented if the desired solution cannot be found.

### 3.3.2 A Bounding Criterion

**Definition 3.3.1.** Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. $b \in \mathbb{N}_0 \cup \{-1\}$ is called *bound* for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$ if

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W}) = \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W}_b).$$

$\Diamond$

**Lemma 3.3.1.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$, $b \in \mathbb{N}_0 \cup \{-1\}$ and $f \in \mathbb{F}[t]$. If $b$ is a bound for $\mathrm{V}(\mathbf{a}, (f), \mathbb{W})$ then for all $g \in \mathbb{W}$ with $\sigma_{\mathbf{a}} g = f$ we have $\|g\| \leq b$.*

**Theorem 3.3.1.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$, let $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$ and $b \in \mathbb{N}_0 \cup \{-1\}$. If for all $f \in \mathbb{F}[t]$ with $\|f\| \leq \|\mathbf{f}\|$ it follows that $b$ is a bound for $\mathrm{V}(\mathbf{a}, (f), \mathbb{W})$ then*

$$b \textit{ is a bound for } \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W}).$$

*Proof.* Assume $b$ is a bound for $\mathrm{V}(\mathbf{a}, (f), \mathbb{W})$ for all $f \in \mathbb{F}[t]$ with $\|f\| \leq \|\mathbf{f}\|$. Let $\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$, i.e.

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}. \tag{3.13}$$

Take $f := \mathbf{c}\,\mathbf{f}$. By

$$\|f\| = \|\mathbf{c}\,\mathbf{f}\| \leq \|\mathbf{f}\|$$

and (3.13) we may conclude that $b$ is a bound for $\mathrm{V}(\mathbf{a}, (f), \mathbb{W})$ and it follows that $\|g\| \leq b$ by Lemma 3.3.1. Consequently for all $\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$ we have $\|g\| \leq b$ and thus

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W}) = \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W}_b)$$

which proves the theorem. $\square$

In the next sections the following corollary will be **heavily** used in proofs for checking if a particular $b$ is a bound for a given solution space.

**Corollary 3.3.1.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$, let $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$ and let $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$. Let $b \in \mathbb{N}_0 \cup \{-1\}$ be such that for all $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ with $\|f\| \leq \|\mathbf{f}\|$ and*

$$\sigma_{\mathbf{a}} g = f$$

*we have $\|g\| \leq b$. Then $b$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.*

*Proof.* The corollary follows immediately by Lemma 3.3.1 and Theorem 3.3.1. $\square$

### 3.3.3 A Special Case for $m$-th Order Recurrences

**Theorem 3.3.2.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. Let $\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with*

$$\begin{aligned} \|a_r\| &= \|\mathbf{a}\| \textit{ for some } r \in \{1, \dots, m\}, \\ \|a_i\| &< \|\mathbf{a}\| \ \ \forall i \neq r \end{aligned}$$

*and $\mathbf{f} \in \mathbb{F}(t)^n$. Then $\max(\|\mathbf{f}\| - \|\mathbf{a}\|, -1)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.*

*Proof.* Let $f \in \mathbb{F}(t)$ and $g \in \mathbb{W}$ with $\sigma_{\mathbf{a}} g = f$ and $\|f\| \leq \|\mathbf{f}\|$. If $g \neq 0$ then by Lemma 3.2.4 it follows that

$$\|\mathbf{f}\| \geq \|f\| = \|\sigma_{\mathbf{a}} g\| = \|\mathbf{a}\| + \|g\|$$

and therefore

$$\|g\| \leq \|\mathbf{f}\| - \|\mathbf{a}\|.$$

Otherwise, if $g = 0$ then $\|g\| = -1$. Altogether we have

$$\|g\| \leq \max(\|\mathbf{f}\| - \|\mathbf{a}\|, -1)$$

and therefore by Corollary 3.3.1 $\max(\|\mathbf{f}\| - \|\mathbf{a}\|, -1)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. □

### 3.3.4 Polynomial Boundings for Π-Extensions

#### 3.3.4.1 The First Order Case for Π-Extensions

Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. Let

$$\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{F}[t]^2$$

and $\mathbf{f} \in \mathbb{F}[t]^n$. In this section we will deal with the problem to find a bound $b$ for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. In particular for $\mathbb{W} := \mathbb{F}[t]$ we deal with the problem to find a bound $b$ for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$, i.e.

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]) = \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_b)$$

which is needed for the incremental reduction method as one can see in Section 3.1.3.2.

If $\|a_1\| \neq \|a_2\|$, Theorem 3.3.2 provides a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. What remains to consider is the case $\|a_1\| = \|a_2\| \geq 0$.

This means, without loss of generality, we assume that

$$\begin{aligned} a_1 &= t^p + r_1, \\ a_2 &= -c\, t^p + r_2 \end{aligned} \tag{3.14}$$

for $c \in \mathbb{F}^*$, $p \geq 0$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p$.

The result of this section delivers a bound for exactly that case (3.14). Especially, in order to compute this bound, we must be able to decide, if there exists a $d \geq 0$ for some $c, \alpha \in \mathbb{F}^*$ such that

$$\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

Furthermore, if there exists such a $d$, we must even compute it. As mentioned in Section 2.2.5 these problems can be solved if $(\mathbb{F}, \sigma)$ is a ΠΣ-field.

The main idea of the following section is taken from Theorem 15 of [Kar81]. Whereas in Karr's version theoretical and computational aspects are mixed, I tried to separate his theorem in several parts to achieve more transparency.

**Theorem 3.3.3.** *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t$. Let $a_1, a_2 \in \mathbb{F}[t]$ as in (3.14). If there exists a $g \in \mathbb{F}(t)$ with $\|g\| \geq 0$ such that*

$$\|a_1\, \sigma(g) + a_2\, g\| < \|g\| + p \tag{3.15}$$

*then*

$$\frac{c}{\alpha^{\|g\|}} \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

**Example 3.3.1.** Consider the Π-extension $(\mathbb{Q}(t_1, t_2), \sigma)$ of $(\mathbb{Q}(t_1), \sigma)$ canonically defined by

$$\begin{aligned} \sigma(t_1) &= t_1 + 1, \\ \sigma(t_2) &= (t_1 + 1)\, t_2 \end{aligned}$$

and the difference equation

$$t_2\, \sigma(g) - \overbrace{(t_1 + 1)^4}^{c}\, t_2\, g = -t_1\, (2 + t_1)\, t_2\, (2 + t_2^2 + 2\, t_1\, (1 + t_2^2) + t_1^2\, (1 + t_2^2)).$$

There is the solution

$$g = t_2^4 + t_2^2 + 1;$$

therefore inequality (3.15) is satisfied and it follows by Theorem 3.3.3 that

$$\frac{c}{\alpha^4} = \frac{(t_1+1)^4}{(t_1+1)^4} = 1 \in \mathrm{H}_{(\mathbb{Q}(t_1),\sigma)}.$$

$\Diamond$

*Proof.* Let

$$g = \sum_{i=0}^{d} g_i t^i + r$$

where $g_i \in \mathbb{F}$, $g_d \neq 0$ and $r \in \mathbb{F}(t)$ with $\|r\| = -1$. We have

$$\begin{aligned} a_1\,\sigma(g) + a_2\,g &= a_1\,\sigma(\sum_{i=0}^{d} g_i\,t^i + r) + a_2\,(\sum_{i=0}^{d} g_i\,t^i + r) \\ &= a_1(\sum_{i=0}^{d} \sigma(g_i)\,(\alpha\,t)^i + \sigma(r)) + a_2\,(\sum_{i=0}^{d} g_i\,t^i + r) \\ &= (t^p + r_1)\,(\sum_{i=0}^{d} \sigma(g_i)\,(\alpha\,t)^i + \sigma(r)) - c\,(t^p - r_2)\,(\sum_{i=0}^{d} g_i\,t^i + r) \end{aligned}$$

and thus

$$\begin{aligned} [a_1\,\sigma(g) + a_2\,g]_{p+d} = 0 &\Leftrightarrow \left[t^p\,\sigma(g_d)\,(\alpha\,t)^d - c\,t^p\,g_d\,t^d\right]_{p+d} = 0 \\ &\Leftrightarrow \sigma(g_d)\,\alpha^d - c\,g_d = 0 \\ &\Leftrightarrow \frac{c}{\alpha^d} = \frac{\sigma(g_d)}{g_d} \\ &\Leftrightarrow \frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}. \end{aligned}$$

$\square$

**Lemma 3.3.2.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\,t$, $\alpha \in \mathbb{F}^*$. Assume there exists a $d \in \mathbb{Z}$ for $c \in \mathbb{F}^*$ such that*

$$c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

*Then $d$ is uniquely determined.*

*Proof.* Assume there are $d_1, d_2 \in \mathbb{Z}$ with

$$d_1 < d_2$$

and

$$c\,\alpha^{d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}, \quad c\,\alpha^{d_2} \in \mathrm{H}_{(\mathbb{F},\sigma)},$$

i.e. there are $g_1, g_2 \in \mathbb{F}^*$ such that

$$\frac{\sigma(g_1)}{g_1} = c\,\alpha^{d_1},$$

$$\frac{\sigma(g_2)}{g_2} = c\,\alpha^{d_2}.$$

Since $d_2 - d_1 > 0$, it follows that

$$\alpha^{d_2-d_1} = \frac{\sigma(g_2)/g_2}{\sigma(g_1)/g_1} = \frac{\sigma(g_2/g_1)}{g_2/g_1}$$

and thus

$$\alpha^{d_2-d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

By Corollary 2.2.2 $(\mathbb{F}(t),\sigma)$ is not a $\Pi$-extension of $(\mathbb{F},\sigma)$, a contradiction. □

**Theorem 3.3.4.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t$, $\alpha\in\mathbb{F}^*$, and $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. Let $\mathbf{f}\in\mathbb{F}[t]^n$ and assume $a_1,a_2\in\mathbb{F}[t]$ as in (3.14). If there exists a $d\geq 0$ such that*

$$\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$$

*then $d$ is uniquely determined and $\max(d,\|\mathbf{f}\|-p)$ is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$. If there does not exist such a $d$ then $\max(\|\mathbf{f}\|-p,-1)$ is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.*

*Proof.* We will proof the theorem by Corollary 3.3.1. Let $f\in\mathbb{F}[t]$ and $g\in\mathbb{W}$ be arbitrary but fixed such that

$$a_1\,\sigma(g) + a_2\,g = f \quad \text{and} \quad \|f\|\leq\|\mathbf{f}\|.$$

We will show by case distinction that for an appropriate $b\in\mathbb{N}_0\cup\{-1\}$ it follows that

$$\|g\|\leq b$$

which will prove that $b$ for the particular case is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.

1. Assume there exists a $d\geq 0$ such that

   $$\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

   Then $d$ is uniquely determined by Lemma 3.3.2.

   If $\|g\|+p>\|f\|$ and $\|g\|\geq 0$, it follows by Theorem 3.3.3 that $\|g\|=d$ and consequently

   $$\|g\| = d = \max(\|f\|-p,d) \leq \max(\|\mathbf{f}\|-p,d).$$

   Otherwise, if $\|g\|+p\leq\|f\|$ or $\|g\|=-1$, we have

   $$\|g\| \leq \max(\|f\|-p,d) \leq \max(\|\mathbf{f}\|-p,d).$$

   Thus for both cases we may apply Corollary 3.3.1 and it follows that $\max(\|\mathbf{f}\|-p,d)$ is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.

2. Assume there does not exist such a $d$. Then by Theorem 3.3.3 it follows that

   $$\|g\|+p = \|f\| \leq \|\mathbf{f}\| \;\text{ or }\; \|g\| = -1$$

   and thus by Corollary 3.3.1 $\max(\|\mathbf{f}\|-p,-1)$ is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.

□

**Corollary 3.3.2.** *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi$*-extension of* $(\mathbb{F},\sigma)$ *and let* $\mathbb{W}$ *be a subspace of* $\mathbb{F}(t)$ *over the constant field* $\mathbb{K}$ *and* $\mathbf{f}\in\mathbb{F}[t]^n$*. Then* $\max(0,\|\mathbf{f}\|)$ *is a bound for* $\mathrm{V}((1,-1),\mathbf{f},\mathbb{W})$*.*

*Proof.* Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t$, $\alpha\in\mathbb{F}^*$. Since $\frac{1}{\alpha^0}=1\in\mathrm{H}_{(\mathbb{F},\sigma)}$, it follows by Theorem 3.3.4 that $\max(0,\|\mathbf{f}\|)$ is a bound for $\mathrm{V}((1,-1),\mathbf{f},\mathbb{W})$. □

#### 3.3.4.2 A Generalization for $m$-th Order Recurrences

Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$ and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$.

Let $\mathbf{0}\neq\mathbf{a}=(a_1,\dots,a_\lambda,\dots,a_\mu\ \dots,a_m)\in\mathbb{F}[t]^m$ with $\lambda<\mu$,

$$\begin{aligned}\|a_\lambda\|&=\|a_\mu\|=p,\\ \|a_i\|&<p\ \forall i\neq\lambda,\mu\end{aligned}$$

and

$$\begin{aligned}a_\lambda&=t^p+r_1,\\ a_\mu&=-c\,t^p+r_2\end{aligned}\tag{3.16}$$

for $c\in\mathbb{F}^*$, $p\geq 0$ and $r_1,r_2\in\mathbb{F}[t]$ with $\|r_1\|,\|r_2\|<p$.

Let $\mathbf{f}\in\mathbb{F}[t]^n$. In this section we will deal with the problem to find a bound $b$ for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.

**Theorem 3.3.5.** *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi$*-extension of* $(\mathbb{F},\sigma)$ *and set*

$$\sigma^k(t)=\alpha_k\,t,\quad \alpha\in\mathbb{F}^*$$

*for all* $k\in\mathbb{Z}^*$*. Assume* $\mathbf{a}\in\mathbb{F}[t]^m$ *as in (3.16) and suppose that* $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ *is a* $\Pi$*-extension of* $(\mathbb{F},\sigma^{\mu-\lambda})$*. If there exists a* $g\in\mathbb{F}(t)$ *with* $\|g\|\geq 0$ *such that*

$$\|\sigma_{\mathbf{a}}g\|<\|g\|+p$$

*then*

$$\frac{\sigma^{\mu-m}(c)}{\alpha_{\mu-\lambda}^{\|g\|}}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}.$$

*Proof.* Let $d:=\|g\|\geq 0$. We have

$$0=[\sigma_{\mathbf{a}}g]_{p+d}=\left[\sum_{i=1}^m a_i\,\sigma^{m-i}(g)\right]_{p+d}=\Big[a_\lambda\,\sigma^{m-\lambda}(g)+a_\mu\,\sigma^{m-\mu}(g)\Big]_{p+d}$$

because of (3.16) and thus

$$0=\Big[\sigma^{\mu-m}(a_\lambda)\,\sigma^{\mu-\lambda}(g)+\sigma^{\mu-m}(a_\mu)\,g\Big]_{p+d}.$$

By

$$\begin{aligned}\sigma^{\mu-m}(a_\lambda)&=\alpha_{\mu-m}^p\,t^p+\sigma^{\mu-m}(r_1),\\ \sigma^{\mu-m}(a_\mu)&=-\sigma^{\mu-m}(c)\,\alpha_{\mu-m}^p\,t^p+\sigma^{\mu-m}(r_2)\end{aligned}$$

it follows that

$$\Big[b_1\,\sigma^{\mu-\lambda}(g)+b_2\,g\Big]_{p+d}=0$$

for

$$\begin{aligned}b_1 &:= t^p+\sigma^{\mu-m}(r_1)/\alpha^p_{\mu-m},\\ b_2 &:= -\sigma^{\mu-m}(c)\,t^p+\sigma^{\mu-m}(r_2)/\alpha^p_{\mu-m}.\end{aligned}$$

As $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ is a $\Pi$-extension of $(\mathbb{F},\sigma^{\mu-\lambda})$ we may apply Theorem 3.3.3 and thus we obtain

$$\frac{\sigma^{\mu-m}(c)}{\alpha^d_{\mu-\lambda}}\in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}.$$

$\square$

**Theorem 3.3.6.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ and set*

$$\sigma^k(t)=\alpha_k\,t,\quad \alpha_k\in\mathbb{F}^*$$

*for all $k\in\mathbb{Z}^*$. Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Let $\mathbf{f}\in\mathbb{F}[t]^n$, assume $\mathbf{a}\in\mathbb{F}[t]^m$ as in (3.16) and suppose that $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ is a $\Pi$-extension of $(\mathbb{F},\sigma^{\mu-\lambda})$. If there exists a $d\geq 0$ such that*

$$\frac{\sigma^{\mu-m}(c)}{\alpha^d_{\mu-\lambda}}\in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$$

*then $d$ is uniquely determined and $\max(d,\|\mathbf{f}\|-p)$ is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$. If there does not exist such a $d$ then $\max(\|\mathbf{f}\|-p,-1)$ is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.*

*Proof.* We will proof the theorem by Corollary 3.3.1. Let $f\in\mathbb{F}[t]$ and $g\in\mathbb{W}$ be arbitrary but fixed such that

$$\sigma_{\mathbf{a}}g=f\quad\text{ and }\quad\|f\|\leq\|\mathbf{f}\|.$$

We will show by case distinction that for an appropriate $b\in\mathbb{N}_0\cup\{-1\}$ it follows that

$$\|g\|\leq b$$

which will prove that $b$ for the particular case is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.

1. Assume there exists a $d\geq 0$ such that

$$\frac{\sigma^{\mu-m}(c)}{\alpha^d_{\mu-\lambda}}\in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}.$$

Then by Lemma 3.3.2 $d$ is uniquely determined. If $\|g\|+p>\|f\|$ and $\|g\|\geq 0$, by Theorem 3.3.5 it follows that $\|g\|=d$ and therefore

$$\|g\|=d=\max(\|f\|,d)\leq\max(\|\mathbf{f}\|,d).$$

Otherwise, if $\|g\|+p\leq\|f\|$ or $\|g\|=-1$, we have

$$\|g\|\leq\max(\|f\|,d)\leq\max(\|\mathbf{f}\|,d).$$

Consequently in both cases we may apply Corollary 3.3.1 and $\max(\|\mathbf{f}\|,d)$ is a bound for $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{W})$.

2. Assume there does not exist such a $d$. Then by Theorem 3.3.5 it follows that

$$\|g\| + p = \|f\| \leq \|\mathbf{f}\| \text{ or } \|g\| = -1$$

and thus by Corollary 3.3.1 $\max(\|\mathbf{f}\| - p, -1)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

$\square$

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field and $(\mathbb{F}(t), \sigma)$ a $\Pi$-extension of $(\mathbb{F}, \sigma)$. Then the following theorem guarantees that for any $k \neq 0$ the difference field $(\mathbb{F}(t), \sigma^k)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma^k)$. Therefore we can apply Theorem 3.3.6 to get a polynomial degree bound.

**Theorem 3.3.7.** *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field. If $(\mathbb{F}(t), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ then $(\mathbb{F}(t), \sigma^k)$ is for all $k \in \mathbb{Z} \setminus \{0\}$ a $\Pi\Sigma$-extension of $(\mathbb{F}^k, \sigma)$.*

The theorem is taken from [Kar85, Theorem 4] which is one of the main results of this article.

### 3.3.5 Polynomial Boundings for $\Sigma$-Extensions

#### 3.3.5.1 The First Order Case for $\Sigma$-Extensions

Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. Let

$$\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{F}[t]^2$$

and $\mathbf{f} \in \mathbb{F}[t]^n$. In this section we will deal with the problem to find a bound $b$ for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. In particular for $\mathbb{W} := \mathbb{F}[t]$ we will deal with the problem to find a bound $b$ for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$, i.e.

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]) = \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_b)$$

which is needed for the incremental reduction method as one can see in Section 3.1.3.2.

If $\|a_1\| \neq \|a_2\|$, Theorem 3.3.2 provides a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. What remains to consider is the case $\|a_1\| = \|a_2\| \geq 0$.

The main idea of the following section is taken from Theorem 14 of [Kar81]. Whereas in Karr's version theoretical and computational aspects are mixed, I tried to separate his theorem in several parts to achieve more transparency.

**Lemma 3.3.3.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, $(\alpha, \beta \in \mathbb{F}^*)$, and $\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{F}$. If there exists a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and*

$$\|a_1\, \sigma(g) - a_2\, g\| < \|g\| + \|\mathbf{a}\| - 1$$

*then*

$$\|a_1\| = \|a_2\| > 0.$$

*Proof.* Let $g \in \mathbb{F}(t)$ as stated above. Due to Theorem 3.3.2 it follows that

$$\|a_1\| = \|a_2\|.$$

Now assume $\|a_1\| = \|a_2\| = 0$, i.e. $a_1, a_2 \in \mathbb{F}$. Thus there is a $u \in \mathbb{F}$ with

$$\|\sigma(g) - u\, g\| < \|g\| - 1. \tag{3.17}$$

Let

$$g = \sum_{i=0}^{d} g_i\, t^i + r$$

where $d \geq 1$, $g_i \in \mathbb{F}$ for all $0 \leq i \leq d$ with $g_d \neq 0$ and $r \in \mathbb{F}(t)$ with $\|r\| = -1$. By (3.17) it follows that

$$\sigma(g_d\, t^d) - u\, g_d\, t^d = 0$$

and thus

$$\frac{\sigma(g_d\, t^d)}{g_d\, t^d} = u \in \mathbb{F}.$$

By Theorem 2.2.1 $(\mathbb{F}(t), \sigma)$ is a homogeneous extension of $(\mathbb{F}, \sigma)$ and therefore not a $\Sigma$-extension of $(\mathbb{F}, \sigma)$, a contradiction. $\square$

**Corollary 3.3.3.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, $(\alpha, \beta \in \mathbb{F}^*)$, and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. Let $\mathbf{f} \in \mathbb{F}[t]^n$ and $\mathbf{a} \in (\mathbb{F}^*)^2$. Then $\|\mathbf{f}\| + 1$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.*

Therefore for the case $\|a_1\| = \|a_2\| = 0$ Corollary 3.3.3 delivers a bound. What remains to consider is the case $\|a_1\| = \|a_2\| > 0$.

This means, without loss of generality, we assume that

$$\begin{aligned} a_1 &= (t^p + u_1\, t^{p-1} + r_1), \\ a_2 &= c\,(t^p + u_2\, t^{p-1} + r_2) \end{aligned} \tag{3.18}$$

for some $c \in \mathbb{F}^*$, $u_1, u_2 \in \mathbb{F}$, $p \geq 1$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p - 1$.

Then we are able to find a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$ if one can compute the solution space $\mathrm{V}(\mathbf{b}, \mathbf{v}, \mathbb{F})$ for any $\mathbf{b}, \mathbf{v} \in \mathbb{F}^2$. As mentioned in Section 2.2.5 this problem can be solved if $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field.

**Theorem 3.3.8.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, $\alpha, \beta \in \mathbb{F}^*$. Assume $a_1, a_2 \in \mathbb{F}[t]$ as in (3.18). If there is a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and*

$$\|a_1 \sigma(g) - a_2\, g\| < \|g\| + p - 1$$

*then there exists a $w \in \mathbb{F}$ such that*

$$\sigma(w) - \alpha\, w = \alpha\,(u_2 - u_1) - \|g\|\, \beta.$$

*Proof.* Let $g \in \mathbb{F}(t)$ with $d = \|g\| > 0$ as stated above. By

$$\|a_1 \sigma(g) - a_2\, g\| < d + p - 1$$

it follows that

$$[a_1\, \sigma(g) + a_2\, g]_{p+d} = 0 \quad \text{and} \quad [a_1\, \sigma(g) + a_2\, g]_{p+d-1} = 0.$$

Let

$$g = \sum_{i=0}^{d} g_i\, t^i + r$$

where $g_i \in \mathbb{F}$, $g_d \neq 0$ and $r \in \mathbb{F}(t)$ with $\|r\| = -1$. We have

$$\begin{aligned} a_1\, \sigma(g) - a_2\, g =& a_1\, \sigma(\sum_{i=0}^{d} g_i\, t^i + r) + a_2\,(\sum_{i=0}^{d} g_i\, t^i + r) \\ =& a_1\,(\sum_{i=0}^{d} \sigma(g_i)\,(\alpha\, t + \beta)^i + \sigma(r)) + a_2\,(\sum_{i=0}^{d} g_i\, t^i + r) \\ =& (t^p + u_1\, t^{p-1} + r_1)(\sum_{i=0}^{d} \sigma(g_i)\,(\alpha\, t + \beta)^i + \sigma(r)) \\ & + c\,(t^p + u_2\, t^{p-1} + r_2)(\sum_{i=0}^{d} g_i\, t^i + r). \end{aligned}$$

Therefore,

$$\begin{aligned} [a_1\, \sigma(g) + a_2\, g]_{p+d} = 0 &\Leftrightarrow \sigma(g_d)\, \alpha^d + c\, g_d = 0 \\ &\Leftrightarrow c = -\frac{\sigma(g_d)}{g_d}\, \alpha^d. \end{aligned} \tag{3.19}$$

By

$$0 = [a_1\,\sigma(g) + a_2\,g]_{p+d-1} = [a_1\,\sigma(g)]_{p+d-1} + [a_2\,g]_{p+d-1}$$

and looking closer at the two summands

$$\begin{aligned}
[a_2\,g]_{p+d-1} =& [a_2]_p\,[g]_{d-1} + [a_2]_{p-1}\,[g]_d = c\,g_{d-1} + c\,u_2\,g_d = c\,(g_{d-1} + u_2\,g_d),\\
[a_1\,\sigma(g)]_{p+d-1} =& [a_1]_{p-1}\,[\sigma(g)]_d + [a_1]_p\,[\sigma(g)]_{d-1}\\
=& u_1\,\alpha^d\,\sigma(g_d) + \Big[\sigma(g_d)\,(\alpha\,t+\beta)^d + \sigma(g_{d-1})\,(\alpha\,t+\beta)^{d-1}\Big]_{d-1}\\
=& u_1\,\alpha^d\,\sigma(g_d) + d\,\alpha^{d-1}\,\beta\,\sigma(g_d) + \alpha^{d-1}\,\sigma(g_{d-1})
\end{aligned}$$

we obtain

$$u_1\,\alpha^d\,\sigma(g_d) + d\,\alpha^{d-1}\,\beta\,\sigma(g_d) + \alpha^{d-1}\,\sigma(g_{d-1}) + c\,(g_{d-1} + u_2\,g_d) = 0.$$

By (3.19) it follows that

$$u_1\,\alpha^d\,\sigma(g_d) + d\,\alpha^{d-1}\,\beta\,\sigma(g_d) + \alpha^{d-1}\,\sigma(g_{d-1}) - \frac{\sigma(g_d)}{g_d}\,\alpha^d\,(g_{d-1} + u_2\,g_d) = 0.$$

$$\Updownarrow$$

$$\sigma(g_d)\,(u_1\,\alpha + d\,\beta - \alpha\,\frac{g_{d-1}}{g_d} - \alpha\,u_2) = -\sigma(g_{d-1})$$

$$\Updownarrow$$

$$\sigma(\frac{g_{d-1}}{g_d}) - \alpha\,\frac{g_{d-1}}{g_d} = (u_2 - u_1)\,\alpha - d\,\beta$$

and thus for $w := \frac{g_{d-1}}{g_d}$ the theorem is proven. □

**Corollary 3.3.4.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\,t+\beta$, $\alpha,\beta \in \mathbb{F}^*$. Assume $a_1, a_2 \in \mathbb{F}[t]$ as in (3.18). If there exists a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and*

$$\|a_1\sigma(g) - a_2\,g\| < \|g\| + p - 1$$

*then $u_1 \neq u_2$.*

*Proof.* Assume $u_1 = u_2$. Then by Theorem 3.3.8 there is a $w \in \mathbb{F}$ such that

$$\sigma(w) - \alpha\,w = -\|g\|\,\beta$$

and thus

$$\sigma(\frac{w}{-\|g\|}) - \alpha\,\frac{w}{-\|g\|} = \beta.$$

By Corollary 2.2.3 $(\mathbb{F}(t),\sigma)$ is not a $\Sigma$-extension of $(\mathbb{F},\sigma)$, a contradiction. □

**Lemma 3.3.4.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\,t+\beta$, $\alpha,\beta \in \mathbb{F}^*$. Assume there exist a $d \in \mathbb{Z}$, a $w \in \mathbb{F}$ and a $u \in \mathbb{F}$ such that*

$$\sigma(w) - \alpha\,w = u\,\alpha + d\,\beta.$$

*Then $d$ is uniquely determined.*

*Proof.* Assume there are $w_1, w_2 \in \mathbb{F}$ and

$$d_1 < d_2$$

with

$$\sigma(w_1) - \alpha\, w_1 = u\,\alpha + d_1\,\beta \text{ and}$$
$$\sigma(w_2) - \alpha\, w_2 = u\,\alpha + d_2\,\beta.$$

Then it follows that

$$\sigma(w_2 - w_1) - \alpha\,(w_2 - w_1) = (d_2 - d_1)\,\beta$$
$$\Updownarrow$$
$$\sigma(\frac{w_2 - w_1}{d_2 - d_1}) - \alpha\,\frac{w_2 - w_1}{d_2 - d_1} = \beta.$$

By Corollary 2.2.3 $(\mathbb{F}(t), \sigma)$ is not a $\Sigma$-extension of $(\mathbb{F}, \sigma)$, a contradiction. □

**Theorem 3.3.9.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, $(\alpha, \beta \in \mathbb{F}^*)$, and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$. Let $\mathbf{f} \in \mathbb{F}[t]^n$ and assume $a_1, a_2 \in \mathbb{F}[t]$ as in (3.18). If $u_1 = u_2$ then $\max(\|\mathbf{f}\| - p + 1, 0)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. Otherwise, if there exist a $d \geq 0$ and a $w \in \mathbb{F}$ such that*

$$\sigma(w) - \alpha\, w = (u_2 - u_1)\,\alpha - d\,\beta$$

*then $d$ is uniquely determined and $\max(d, \|\mathbf{f}\| - p + 1)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. If there does not exist such a $d$ then $\max(\|\mathbf{f}\| - p + 1, 0)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.*

*Proof.* We will proof the theorem by Corollary 3.3.1. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ be arbitrary but fixed such that

$$a_1\,\sigma(g) + a_2\, g = f \quad \text{and} \quad \|f\| \leq \|\mathbf{f}\|.$$

We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that

$$\|g\| \leq b$$

which will prove that $b$ for the particular case is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

If $u_1 = u_2$ then by Corollary 3.3.4 it follows that $\|g\| \leq 0$ or

$$\|g\| \leq \|f\| - p + 1 \leq \|\mathbf{f}\| - p + 1$$

and thus by Corollary 3.3.1 $\max(\|\mathbf{f}\| - p + 1, 0)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. Otherwise, assume $u_1 \neq u_2$.

1. Assume there exist a $d \geq 0$ and a $w \in \mathbb{F}$ such that

$$\sigma(w) - \alpha\, w = (u_2 - u_1)\,\alpha - d\,\beta.$$

Then by Lemma 3.3.4 $d$ is uniquely determined.

If $\|g\| + p - 1 > \|f\|$ and $\|g\| > 0$, by Theorem 3.3.4 it follows that

$$\|g\| = d = \max(\|f\| - p + 1, d) \leq \max(\|\mathbf{f}\| - p + 1, d).$$

Otherwise, if $\|g\| + p - 1 \leq \|f\|$ or $\|g\| \leq 0$ then clearly we have

$$\|g\| \leq \max(\|f\| - p + 1, d) \leq \max(\|\mathbf{f}\| - p + 1, d).$$

Thus in both cases, by Corollary 3.3.1, $\max(\|\mathbf{f}\| - p + 1, d)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

2. Assume there do not exist such a $d$ and a $w$. Then by Theorem 3.3.8 it follows that

$$\|g\| \leq \|f\| - p + 1 \leq \|\mathbf{f}\| - p + 1 \;\text{ or }\; \|g\| \leq 0$$

and thus by Corollary 3.3.1 $\max(\|\mathbf{f}\| - p + 1, 0)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

□

#### 3.3.5.2 A Generalization for $m$-th Order Recurrences

Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over the constant field $\mathbb{K}$.

Let $\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_\lambda, \dots, a_\mu \; \dots, a_m) \in \mathbb{F}[t]^m$ with $\lambda < \mu$,

$$\begin{aligned} \|a_\lambda\| &= \|a_\mu\| = p, \\ \|a_i\| &< p - 1 \; \forall i \neq \lambda, \mu \end{aligned}$$

and

$$\begin{aligned} a_\lambda &= (t^p + u_1 \, t^{p-1} + r_1), \\ a_\mu &= c \, (t^p + u_2 \, t^{p-1} + r_2) \end{aligned} \tag{3.20}$$

for some $c \in \mathbb{F}^*$, $u_1, u_2 \in \mathbb{F}$, $p > 0$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p - 1$.

Let $\mathbf{f} \in \mathbb{F}[t]^n$. In this section we will deal with the problem to find a bound $b$ for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

**Lemma 3.3.5.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ and set*

$$\sigma^k(t) = \alpha_k \, t + \beta_k, \quad \alpha_k \in \mathbb{F}^*, \beta_k \in \mathbb{F}$$

*for all $k \in \mathbb{Z}^*$. Assume $\mathbf{a} \in \mathbb{F}[t]^m$ as in (3.20) and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a $\Sigma$-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. If there exists a $g \in \mathbb{F}(t)$ with $\|g\| \geq 0$ and*

$$\|\sigma_{\mathbf{a}} g\| < \|g\| + p - 1$$

*then we have*

$$\|b_1 \, \sigma^{\mu-\lambda}(g) + b_2 \, g\| < \|g\| + p - 1$$

*where*

$$\begin{aligned} b_1 &:= t^p + t^{p-1} \, (p \, \beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\alpha_{\mu-m}, \\ b_2 &:= \sigma^{\mu-m}(c) \, t^p + t^{p-1} \, (p \, \beta_{\mu-m} \, \sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\alpha_{\mu-m}. \end{aligned}$$

*Proof.* Let $d := \|g\|$. For $i \in \{0,1\}$ we have

$$0 = [\sigma_{\mathbf{a}} g]_{p+d-i} = \left[\sum_{i=1}^{m} a_i\, \sigma^{m-i}(g)\right]_{p+d-i} = \left[a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)\right]_{p+d-i}$$

because of (3.20) and thus

$$0 = \left[\sigma^{\mu-m}(a_\lambda)\, \sigma^{\mu-\lambda}(g) + \sigma^{\mu-m}(a_\mu)\, g\right]_{p+d-i}.$$

By

$$\begin{aligned}
\sigma^{\mu-m}(a_\lambda) &= (\alpha_{\mu-m}\, t + \beta_{\mu-m})^p + \sigma^{\mu-m}(u_1)\, (\alpha_{\mu-m}\, t + \beta_{\mu-m})^{p-1} + \sigma^{\mu-m}(r_1)\\
&= \alpha_{\mu-m}^p\, t^p + t^{p-1}\, (p\, \alpha_{\mu-m}^{p-1}\, \beta_{\mu-m} + \sigma^{\mu-m}(u_1)\, \alpha_{\mu-m}^{p-1}) + \tilde{r}_1,\\
\sigma^{\mu-m}(a_\mu) &= \sigma^{\mu-m}(c)\, (\alpha_{\mu-m}\, t + \beta_{\mu-m})^p + \sigma^{\mu-m}(u_2)\, (\alpha_{\mu-m}\, t + \beta_{\mu-m})^{p-1} + \sigma^{\mu-m}(r_1)\\
&= \sigma^{\mu-m}(c)\, \alpha_{\mu-m}^p\, t^p + t^{p-1}\, (p\, \alpha_{\mu-m}^{p-1}\, \beta_{\mu-m}\, \sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2)\, \alpha_{\mu-m}^{p-1}) + \tilde{r}_2
\end{aligned}$$

for some $\tilde{r}_1, \tilde{r}_2 \in \mathbb{F}[t]$ with $\|\tilde{r}_i\| < p-2$ it follows that

$$\left[b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\right]_{p+d-i} = 0$$

for

$$\begin{aligned}
b_1 &:= t^p + t^{p-1}\, (p\, \beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\alpha_{\mu-m},\\
b_2 &:= \sigma^{\mu-m}(c)\, t^p + t^{p-1}\, (p\, \beta_{\mu-m}\, \sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\alpha_{\mu-m}
\end{aligned}$$

and thus

$$\|b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\| < \|g\| + p - 1.$$

□

**Theorem 3.3.10.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$ and set*

$$\sigma^k(t) = \alpha_k\, t + \beta_k, \quad \alpha_k \in \mathbb{F}^*, \beta_k \in \mathbb{F}$$

*for all $k \in \mathbb{Z}^*$. Assume $\mathbf{a} \in \mathbb{F}[t]^m$ as in (3.20) and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a $\Sigma$-extension of $(\mathbb{F}^{\mu-\lambda}, \sigma)$. Define*

$$\begin{aligned}
v_1 &: = (p\, \beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\alpha_{m-\mu},\\
v_2 &: = (p\, \beta_{\mu-m}\, \sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\alpha_{\mu-m}.
\end{aligned}$$

*If there exists a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and*

$$\|\sigma_{\mathbf{a}} g\| < \|g\| + p - 1$$

*then there exists a $w \in \mathbb{F}$ such that*

$$\sigma^{\mu-\lambda}(w) - \alpha_{\mu-\lambda}\, w = \alpha_{\mu-\lambda}\, (v_2 - v_1) - \|g\|\, \beta_{\mu-\lambda}.$$

*Proof.* Assume there exists a $g \in \mathbb{F}(t)$ with

$$\|\sigma_{\mathbf{a}} g\| < \|g\| + p - 1.$$

Then by Lemma 3.3.5 there are

$$\begin{aligned} b_1 &:= t^p + t^{p-1}\,(p\,\beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\alpha_{\mu-m},\\ b_2 &:= \sigma^{\mu-m}(c)\,t^p + t^{p-1}\,(p\,\beta_{\mu-m}\,\sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\alpha_{\mu-m} \end{aligned}$$

such that

$$\|b_1\,\sigma^{\mu-\lambda}(g) + b_2\,g\| < \|g\| + p - 1.$$

As $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a $\Sigma$-extension, in particular $\alpha_{\mu-\lambda}, \beta_{\mu-\lambda} \in \mathbb{F}^*$, we may apply Theorem 3.3.8 and obtain

$$\sigma^{\mu-\lambda}(w) - \alpha_{\mu-\lambda}\,w = \alpha_{\mu-\lambda}\,(v_2 - v_1) - \|g\|\,\beta_{\mu-\lambda}$$

for some $w \in \mathbb{F}$. □

**Corollary 3.3.5.** *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Sigma$*-extension of* $(\mathbb{F}, \sigma)$ *and set*

$$\sigma^k(t) = \alpha_k\,t + \beta_k, \quad \alpha_k \in \mathbb{F}^*, \beta_k \in \mathbb{F}$$

*for all* $k \in \mathbb{Z}^*$*. Assume* $\mathbf{a} \in \mathbb{F}[t]^m$ *as in (3.20) and suppose that* $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ *is a* $\Sigma$*-extension of* $(\mathbb{F}, \sigma^{\mu-\lambda})$*. Define*

$$\begin{aligned} v_1 &:= (p\,\beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\alpha_{m-\mu},\\ v_2 &:= (p\,\beta_{\mu-m}\,\sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\alpha_{\mu-m}. \end{aligned}$$

*If there exists a* $g \in \mathbb{F}(t)$ *with* $\|g\| > 0$ *and*

$$\|\sigma_{\mathbf{a}} g\| < \|g\| + p - 1$$

*then* $v_1 \neq v_2$*.*

*Proof.* Assume there is a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and

$$\|\sigma_{\mathbf{a}} g\| < \|g\| + p - 1.$$

By Theorem 3.3.10 there exists a $w \in \mathbb{F}$ such that

$$\sigma^{\mu-\lambda}(w) - \alpha_{\mu-\lambda}\,w = -\|g\|\,\beta_{\mu-\lambda}$$

and therefore

$$\sigma^{\mu-\lambda}\Big(\frac{w}{-\|g\|}\Big) - \alpha_{\mu-\lambda}\,\frac{w}{-\|g\|} = \beta_{\mu-\lambda}.$$

By Corollary 2.2.3 $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is not a $\Sigma$-extension of $(\mathbb{F}^{\mu-\lambda}, \sigma)$, a contradiction. □

**Theorem 3.3.11.** *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Sigma$*-extension of* $(\mathbb{F}, \sigma)$ *and set*

$$\sigma^k(t) = \alpha_k\, t + \beta_k, \quad \alpha_k \in \mathbb{F}^*, \beta_k \in \mathbb{F}$$

*for all* $k \in \mathbb{Z}^*$. *Let* $\mathbb{W}$ *be a subspace of* $\mathbb{F}(t)$ *over* $\mathbb{K}$. *Let* $\mathbf{f} \in \mathbb{F}[t]^n$, *assume* $\mathbf{a} \in \mathbb{F}[t]^m$ *as in (3.20) and suppose that* $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ *is a* $\Sigma$*-extension of* $(\mathbb{F}, \sigma^{\mu-\lambda})$, *in particular* $\beta_{\mu-\lambda} \in \mathbb{F}^*$. *Define*

$$\begin{aligned} v_1 &:= (p\, \beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\alpha_{m-\mu}, \\ v_2 &:= (p\, \beta_{\mu-m}\, \sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\alpha_{\mu-m}. \end{aligned}$$

*If* $v_1 = v_2$ *then* $\max(\|\mathbf{f}\| - p + 1, 0)$ *is a bound for* $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. *Otherwise, if there exist a* $d \geq 0$ *and a* $w \in \mathbb{F}$ *such that*

$$\sigma^{\mu-\lambda}(w) - \alpha_{\mu-\lambda}\, w = (v_2 - v_1)\, \alpha_{\mu-\lambda} - d\, \beta_{\mu-\lambda}$$

*then* $d$ *is uniquely determined and* $\max(d, \|\mathbf{f}\| - p + 1)$ *is a bound for* $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. *If there does not exist such a* $d$ *then* $\max(\|\mathbf{f}\| - p + 1, 0)$ *is a bound for* $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

*Proof.* We will proof the theorem by Corollary 3.3.1. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ be arbitrary but fixed such that

$$a_1\, \sigma(g) + a_2\, g = f \quad \text{and} \quad \|f\| \leq \|\mathbf{f}\|.$$

We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that

$$\|g\| \leq b$$

which will prove that $b$ for the particular case is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. If $v_1 = v_2$ then by Corollary 3.3.5 it follows that

$$\|g\| \leq \|f\| - p + 1 \leq \|\mathbf{f}\| - p + 1 \;\text{ or }\; \|g\| \leq 0$$

and thus by Corollary 3.3.1 $\max(\|\mathbf{f}\| - p + 1, 0)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$. Otherwise, assume $v_1 \neq v_2$.

1. Assume there exist a $d \geq 0$ and a $w \in \mathbb{F}$ such that

   $$\sigma^{\mu-\lambda}(w) - \alpha_{\mu-\lambda}\, w = (v_2 - v_1)\, \alpha_{\mu-\lambda} - d\, \beta_{\mu-\lambda}.$$

   Then by Lemma 3.3.4 $d$ is uniquely determined.

   If $\|g\| + p - 1 > \|f\|$ and $\|g\| > 0$, by Theorem 3.3.10 it follows that

   $$\|g\| = d = \max(\|f\| - p + 1, d) \leq \max(\|\mathbf{f}\| - p + 1, d).$$

   Otherwise, if $\|g\| + p - 1 \leq \|f\|$ or $\|g\| \leq 0$ then clearly we have:

   $$\|g\| \leq \max(\|f\| - p + 1, d) \leq \max(\|\mathbf{f}\| - p + 1, d).$$

   Thus by Corollary 3.3.1 $\max(\|\mathbf{f}\| - p + 1, d)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

2. Assume there do not exist such a $d$ and a $w$. Then by Theorem 3.3.10 it follows that $\|g\| \leq 0$ or
$$\|g\| \leq \|f\| - p + 1 \leq \|\mathbf{f}\| - p + 1$$
and therefore by Corollary 3.3.1 $\max(\|\mathbf{f}\| - p + 1, 0)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W})$.

□

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field and $(\mathbb{F}(t), \sigma)$ a $\Sigma$-extension of $(\mathbb{F}, \sigma)$. Then Theorem 3.3.7 guarantees that for any $k \neq 0$ the difference field $(\mathbb{F}(t), \sigma^k)$ is a $\Sigma$-extension of $(\mathbb{F}, \sigma^k)$. Therefore we can apply Theorem 3.3.11 to get a polynomial degree bound.

## 3.4 Polynomial Degree Boundings for Proper Sum Extensions

Let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$ and $\mathbf{f} \in \mathbb{F}[t]^n$. In the following we look at the problem to find a bound $b$ for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$ which is needed for the incremental reduction method introduced in Section 3.1.3.2. Whereas in Section 3.4.7 we take into consideration the general situation, we regard a polynomial degree bound for the special case $\mathbf{a} \in \mathbb{F}^m$ in Section 3.4.9.

In particular, for the general situation I find a method to determine a polynomial degree bound; although I failed to prove termination of this method, I achieved an important step to solve difference equations where proper sum extensions are involved. If the method terminates for a specific difference equation, it is guaranteed that the polynomial degree bound is correct. Otherwise, by an upper bound of computation steps in this method, it is guaranteed that the method will stop; in this situation a warning[7] will be printed out with the hint to increase the bound in order to find more solutions.

Moreover, I consider the special case $\mathbf{a} \in \mathbb{F}^m$ for which I find a simple polynomial degree bound. In addition, for this special case I dig up remarkable results concerning some properties of solutions for a difference equations.

### 3.4.1 A Simple Check

**Proposition 3.4.1.** *Let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$, $\mathbf{f} \in \mathbb{F}[t]^n$ and $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$. Let $\mathbf{b} := [\mathbf{a}]_{\|a\|}$. If there does not exist a $g \in \mathbb{F}^*$ such that*

$$\sigma_{\mathbf{b}} g = 0$$

*then $\|\mathbf{f}\|$ is a bound for $\mathrm{V}(\mathbf{b}, \mathbf{f}, \mathbb{F}[t])$.*

*Proof.* Assume there exist a $g \in \mathbb{F}[t]$ and a $\mathbf{c} \in \mathbb{K}^n$ with $d := \|g\| > \|\mathbf{f}\|$ and

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}.$$

Take such a $g$, i.e. there are a $w \in \mathbb{F}^*$ and an $r \in \mathbb{F}[t]_{d-1}$ such that

$$g = w\,t^d + r.$$

Let $l := \|\mathbf{a}\|$. We have

$$\begin{gathered}
\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f} \in \mathbb{F}[t]_{l+d-1} \\
\Downarrow \\
\sigma_{\mathbf{a}} g \in \mathbb{F}[t]_{l+d-1} \\
\Updownarrow \\
a_1\,\sigma^{m-1}(w\,t^d + r) + \dots + a_m\,(w\,t^d + r) \in \mathbb{F}[t]_{d+l-1} \\
\Updownarrow \\
a_1\,\sigma^{m-1}(w)\,t^d + \dots + a_m\,w\,t^d \in \mathbb{F}[t]_{d+l-1} \\
\Updownarrow
\end{gathered}$$

[7] This situation has never been occurred up to now. I strongly believe that this method must terminate in each situation. Nevertheless, since I have not found a proof for termination yet, I have to introduce this bound for the maximal computation steps in order to turn the method to an heuristical algorithm.

$$\left[a_1\,\sigma^{m-1}(w)\,t^d + \cdots + a_m\,w\,t^d\right]_{d+l} = 0$$
$$\Updownarrow$$
$$[a_1]_l\;\sigma^{m-1}(w) + \cdots + [a_m]_l\;w = 0$$
$$\Updownarrow$$
$$\sigma_{\mathbf{b}} w = 0.$$

Assume there does not exist a $w \in \mathbb{F}^*$ such that $\sigma_{\mathbf{b}} w = 0$. Then by the above considerations there do not exist a $g \in \mathbb{F}[t]$ and a $\mathbf{c} \in \mathbb{K}^n$ with $\|g\| > \|\mathbf{f}\|$ such that

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$$

and consequently $\|\mathbf{f}\|$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. □

### 3.4.2 The Truncated Solution Space and Polynomial Degree Bounds

From now on we will assume for the whole Section 3.4 that $(\mathbb{F}(t), \sigma)$ is a **proper sum extension** of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbb{W}$ is a subspace of $\mathbb{F}[t]$ over $\mathbb{K}$ and

$$\begin{aligned} \mathbf{0} \neq \mathbf{a} &= (a_1, \ldots, a_n) \in \mathbb{F}[t]^n, \\ \mathbf{f} &= (f_1, \ldots, f_\lambda) \in \mathbb{F}[t]^\lambda. \end{aligned}$$

We already know the definition of the solution space

$$\boxed{\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{W}) = \left\{\mathbf{c} \wedge g \in \mathbb{K}^\lambda \times \mathbb{W} \,|\, \sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}\right\}}$$

and as already indicated in Section 3.2.3 we may write

$$\mathbb{W} = t^r\,\mathbb{W} \oplus \mathbb{W}_{r-1}$$

for $r \geq 0$. In the following we are not interested in all

$$g = g_1 \oplus g_2 \in t^r\,\mathbb{W} \oplus \mathbb{W}_{r-1}$$

and all $\mathbf{c} \in \mathbb{K}^\lambda$ such that

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$$

but only in that solution part $g_1 \in t^r\,\mathbb{W}$, i.e. in the solution space

$$\boxed{\mathrm{V}(\mathbf{a}, \mathbf{f}, r, \mathbb{W}) := \left\{g \in t^r\,\mathbb{W} \,|\, \exists\, \mathbf{c} \in \mathbb{K}^\lambda,\, h \in \mathbb{W}_{r-1} : \sigma_{\mathbf{a}}(g + h) = \mathbf{c}\,\mathbf{f}\right\}.}$$

Now we want also to "forget" the vector $\mathbf{f}$. More precisely, we are interested in those solutions $g \in t^r\,\mathbb{W}$ such that there is an $f \in \mathbb{F}[t]_{\|\mathbf{a}\|+r-1}$ with

$$\sigma_{\mathbf{a}} g = f.$$

This means we consider the vector space

$$\boxed{\mathrm{V}_r(\mathbf{a}, \mathbb{W}) := \left\{g \in t^r\,\mathbb{W} \,|\, \sigma_{\mathbf{a}} g \in \mathbb{F}[t]_{\|\mathbf{a}\|+r-1}\right\}.}$$

This vector space is also called the *truncated solution space*.

**Lemma 3.4.1.** *Let* $r \geq \max(\|\mathbf{f}\| - \|\mathbf{a}\| + 1, 0)$. *Then*

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, r, \mathbb{W}) \subseteq \mathrm{V}_r(\mathbf{a}, \mathbb{W}).$$

*Proof.* Let $g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, r, \mathbb{W})$, i.e. there are a $\mathbf{c} \in \mathbb{K}^\lambda$ and an $h \in \mathbb{W}_{r-1}$ such that

$$\sigma_{\mathbf{a}}(g + h) = \mathbf{c}\,\mathbf{f}.$$

Since $\|\mathbf{f}\| \leq r + \|\mathbf{a}\| - 1$, it follows that

$$\sigma_{\mathbf{a}}(g + h) \in \mathbb{W}_{\|\mathbf{a}\|+r-1}$$

and as

$$\sigma_{\mathbf{a}} h \leq \|\mathbf{a}\| + \|h\| = \|\mathbf{a}\| + r - 1,$$

we have

$$\sigma_{\mathbf{a}} g \in \mathbb{W}_{\|\mathbf{a}\|+r-1}.$$

Consequently $g \in \mathrm{V}_r(\mathbf{a}, \mathbb{W})$ and therefore $\mathrm{V}(\mathbf{a}, \mathbf{f}, r, \mathbb{W}) \subseteq \mathrm{V}_r(\mathbf{a}, \mathbb{W})$. □

**Theorem 3.4.1.** *Let* $r \geq \max(\|\mathbf{f}\| - \|\mathbf{a}\| + 1, 0)$ *and* $m \geq 0$ . *If*

$$\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m)$$

*then* $m + r$ *is a bound for* $\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t])$.

*Proof.* By Lemma 3.4.1 we have $\mathrm{V}(\mathbf{a}, \mathbf{f}, r, \mathbb{F}[t]) \subseteq \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t])$ and therefore by assumption it follows that

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, r, \mathbb{F}[t]) \subseteq \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m). \tag{3.21}$$

Let $\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$ with

$$g = g_1 \oplus g_2 = t^r\, \mathbb{F}[t] \oplus \mathbb{F}[t]_{r-1}.$$

This means we have

$$\sigma_{\mathbf{a}}(g_1 + g_2) = \mathbf{c}\,\mathbf{f}$$

and therefore it follows by definition that

$$g_1 \in \mathrm{V}(\mathbf{a}, \mathbf{f}, r, \mathbb{F}[t]).$$

But by (3.21) we have $\|g_1\| \leq r + m$, therefore $\|g\| \leq r + m$ and thus

$$\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_{r+m}).$$

Since

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_{m+r}) \supseteq \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]),$$

it follows equality and therefore $m + r$ is a bound of $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. □

**Definition 3.4.1.** Let $\mathbb{V}$ be a subspace of $\mathbb{F}[t]$ over $\mathbb{K}$. We say that $\mathbf{g} = (g_1, \ldots, g_l) \in \mathbb{F}[t]^l$ generates $\mathbb{V}$ over $\mathbb{K}$, if

$$\mathbb{V} = \{k_1\, g_1 + \cdots + k_l\, g_l \mid k_i \in \mathbb{K}\}.$$

◊

**Definition 3.4.2.** We define for $m \geq -1$ the function

$$f_m : \begin{cases} \mathbb{N}_0 & \to & \mathcal{P}(\mathbb{F}[t]) \\ r & \mapsto & \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m). \end{cases}$$

◊

**Proposition 3.4.2.** *Let*

$$\mathrm{V}(\mathbf{b}, (0), \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{C} \wedge \mathbf{g}$$

*where* $\mathbf{b} = [\mathbf{a}]_{\|a\|}$. *Then* $t^r\,\mathbf{g}$ *generates* $f_0(r)$ *over* $\mathbb{K}$ *for all* $r \geq 0$, *i.e.*

$$f_0(r) = \{k_1\, t^r\, g_1 + \cdots + k_l\, t^r\, g_l \,|\, k_i \in \mathbb{K}\}$$

*where* $\mathbf{g} = (g_1, \ldots, g_l)$.

*Proof.* We have

$$p \in f_0(r) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}) = \left\{g \in t^r\, \mathbb{F} \,|\, \sigma_{\mathbf{a}}(g) \in \mathbb{F}[t]_{\|\mathbf{a}\|+r-1}\right\}$$
$$\Updownarrow$$
$$\exists h \in \mathbb{F}: \; p = h\, t^r \;\&\; \sigma_{\mathbf{b}} h = 0$$
$$\Updownarrow$$
$$\exists c \in \mathbb{K}\; \exists h \in \mathbb{F}: \; p = h\, t^r \;\&\; (c, h) \in \mathrm{V}(\mathbf{b}, (0), \mathbb{F})$$

where the first equivalence can be easily extracted from the proof of Proposition 3.4.1. It follows immediately[8] that $\mathbf{g}$ generates $f_0(r)$. □

**Lemma 3.4.2.** *Let* $m \geq -1$ *and* $r \geq 0$. *Then*

$$f_m(r) = f_{m+1}(r) \cap \mathbb{F}[t]_{m+r}.$$

*Proof.* We have

$$\begin{aligned} f_{m+1}(r) \cap \mathbb{F}[t]_{m+r} &= \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_{m+1}) \cap \mathbb{F}[t]_{m+r} \\ &= \left\{g \in t^r\, \mathbb{F}_{m+1} \,|\, \sigma_{\mathbf{a}} g \in \mathbb{F}[t]_{\|\mathbf{a}\|+r-1}\right\} \cap \mathbb{F}[t]_{m+r} \\ &= \left\{g \in t^r\, \mathbb{F}_m \,|\, \sigma_{\mathbf{a}} g \in \mathbb{F}[t]_{\|\mathbf{a}\|+r-1}\right\} \\ &= \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m) = f_m(r). \end{aligned}$$

□

**Lemma 3.4.3.** *Let* $m \geq -1$. $f_m(r)$ *is a subspace of* $f_{m+1}(r)$ *over* $\mathbb{K}$.

*Proof.* $f_m(r)$ and $f_{m+1}(r)$ are vector spaces over $\mathbb{K}$ and we have $f_m(r) \subseteq f_{m+1}(r)$ by Lemma 3.4.2. □

**Lemma 3.4.4.** *For* $m \geq -1$ *and* $r \geq 0$ *we have*[9]

$$f_{m+1}(r) \subseteq f_m(r+1) + t^r\, \mathbb{F}.$$

[8] In particular, if $f_0(r) = \{0\}$ then by convention $\mathbf{C} \wedge \mathbf{g} = (0, 0)$ and thus we have $\mathbf{g} = (0) \in \mathbb{F}[t]$ which clearly generates $f_0(r) = \{0\}$.

[9] See Section 3.1.2.1 for the definition of the sum of two vector spaces over the same field $\mathbb{K}$.

*Proof.* We have

$$\begin{aligned} g \in f_{m+1}(r) &\Leftrightarrow g \in \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_{m+1}) \\ &\Leftrightarrow g \in t^r\,\mathbb{F}[t]_{m+1} \;\&\; \sigma_{\mathbf{a}} g \in \mathbb{F}[t]_{\|a\|+r-1} \\ &\Rightarrow g \in t^{r+1}\,\mathbb{F}[t]_m + t^r\,\mathbb{F} \;\&\; \sigma_{\mathbf{a}} g \in \mathbb{F}[t]_{\|a\|+r} \\ &\Leftrightarrow g \in \mathrm{V}_{r+1}(\mathbf{a}, \mathbb{F}[t]_m) \\ &\Leftrightarrow g \in f_{m+1}(r+1) + t^r\,\mathbb{F}. \end{aligned} \tag{3.22}$$

□

**Corollary 3.4.1.** *For $m \geq -1$ and $r \geq 0$ we have*

$$f_{m+1}(r) = \Big\{ g \in f_m(r+1) + t^r\,\mathbb{F} \mid [\sigma_{\mathbf{a}} g]_{\|\mathbf{a}\|+r} = 0 \Big\}.$$

*Proof.* By assuming $[\sigma_{\mathbf{a}} g]_{\|\mathbf{a}\|+r} = 0$ we may replace the left-right implication in line (3.22) by an equivalence. □

**Proposition 3.4.3.** *Let $m \geq -1$ and $r \geq 0$. Then $f_m(r)$ is a finite dimensional subspace of $\mathbb{F}[t]_m$ over $\mathbb{K}$.*

*Proof.* We will proof the proposition by induction on $m$. For $m = -1$ we have $f_{-1}(r) = \mathrm{V}_r(\mathbf{a}, \{0\}) = \{0\}$ and by Proposition 3.4.2 we get a finite set which generates $f_0(r)$ over $\mathbb{K}$ for all $r \geq 0$. Consequently the induction base holds. Now let $m \geq 0$ and assume that for all $r \geq 0$ the vector space $f_m(r)$ is finite dimensional, i.e. let $\mathbf{g} = (g_1, \ldots, g_l)$ be a basis of $f_m(r+1)$ for an arbitrary but fixed $r \geq 0$. Now consider

$$\mathbf{f} := [\sigma_{\mathbf{a}} \mathbf{g}]_{\|a\|}$$

and let

$$\mathrm{V}(\mathbf{b}, \mathbf{f}, \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{C} \wedge \mathbf{h}$$

for $\mathbf{b} := [\mathbf{a}]_{\|a\|}$. Then one can see by Corollary 3.4.1 that $\mathbf{C}\,\mathbf{g} + \mathbf{h}$ generates the vector space $f_{m+1}(r)$. □

**Notation 3.4.1.** Let $r, m \geq 0$ and $\{g_1, \ldots, g_k\} \subseteq t^r\,\mathbb{F}[t]_m$ be a basis of $f_m(r) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m)$. We write

$$f_m(r) \xleftrightarrow{\text{basis}} (g_1, \ldots, g_k).$$

In particular, if $f_m(r) = \{0\}$, we write

$$f_m(r) \xleftrightarrow{\text{basis}} (0).$$

◊

Given the truncated solution space $\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t])$ with

$$\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m)$$

for a specific $m$, the following theorem provides a shortcut in order to compute the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$ for a given $\mathbf{f}$. I will skip the corresponding proof.

**Remark 3.4.1.** Let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^n$ and $\mathbf{f} \in \mathbb{F}[t]^\lambda$. Let $r \geq \max(\|\mathbf{f}\| - \|\mathbf{a}\| + 1, 0)$ and assume

$$\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m).$$

Let

$$\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m) \xleftrightarrow{\text{basis}} \mathbf{g_1}.$$

with $\mathbf{g_1} \in (t^r\, \mathbb{F}[t]_m)^l$ for some $l \geq 1$. Let

$$\mathrm{V}(\mathbf{a}, \mathbf{f} \wedge (\sigma_{\mathbf{a}} \mathbf{g_1}), \mathbb{F}[t]_{r-1}) \xleftrightarrow{\text{basis}} \mathbf{C} \wedge \mathbf{D} \wedge \mathbf{g_0}$$

with $\mathbf{C} \in \mathbb{K}^{s \times \lambda}$, $\mathbf{D} \in \mathbb{K}^{s \times l}$ and $\mathbf{g_0} \in \mathbb{F}[t]_{r-1}^s$ for some $s \geq 1$. Then

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]) \xleftrightarrow{\text{basis}} \mathbf{C} \wedge (\mathbf{D} \cdot \mathbf{g_1} + \mathbf{g_0}).$$

$\diamondsuit$

### 3.4.3 Collecting Further Information for the Truncated Solution Space

As in the previous section we assume that $(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and

$$\begin{aligned} \mathbf{0} \neq \mathbf{a} &= (a_1, \dots, a_n) \in \mathbb{F}[t]^n, \\ \mathbf{f} &= (f_1, \dots, f_\lambda) \in \mathbb{F}[t]^\lambda. \end{aligned}$$

From now on we will assume that $(\mathbb{F}(t)[x], \sigma)$ is a difference ring extension of $(\mathbb{F}(t), \sigma)$ where $x$ is transcendental over $\mathbb{F}(t)$ with $\sigma(x) = x$. In particular, we have

$$\mathrm{const}_\sigma \mathbb{F}(t)[x] = \mathbb{K}[x],$$

i.e. we have to deal with a **constant ring**, not anymore with a constant field. Furthermore for $f = \sum_i f_i\, x^i \in \mathbb{F}(t)[x]$ with $f_i \in \mathbb{F}(t)$ and $p \in \mathbb{F}(t)$ we introduce the notation

$$\boxed{f[p] := \sum_i f_i\, p^i.}$$

In addition, we introduce for $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{F}(t)[x]^r$ the notation

$$\mathbf{w}[p] := (w_1[p], \dots, w_r[p]),$$

for a matrix $\mathbf{M} = \begin{pmatrix} m_{11} & \dots & m_{1\nu} \\ \vdots & & \vdots \\ m_{\mu 1} & \dots & m_{\mu\nu} \end{pmatrix} \in \mathbb{F}(t)[x]^{\mu\times\nu}$ the notation

$$\mathbf{M}[p] := \begin{pmatrix} m_{11}[p] & \dots & m_{1\nu}[p] \\ \vdots & & \vdots \\ m_{\mu 1}[p] & \dots & m_{\mu\nu}[p] \end{pmatrix}$$

and for a set $S \subseteq \mathbb{F}(t)[x]^r$ the notation

$$S[p] := \{\mathbf{w}[p] \mid \mathbf{w} \in S\}.$$

**Definition 3.4.3.** Let $m \geq 0$, $k \geq 1$. We call a map $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ *$m$-polynomial*, if $\mathbf{g}(d) = \mathbf{g}^{(d)}[m + d]$ where

$$\mathbf{g}^{(d)} = \left(g_1^{(d)}, \dots, g_k^{(d)}\right) \in \mathbb{F}[t, x]^k$$

with

$$g_i^{(d)} = t^d \sum_{j=0}^{m} g_{ij}\, t^j$$

and $g_{ij} \in \mathbb{F}[x]$. ◊

**Definition 3.4.4.** Let $m \geq 0$ and $\gamma \geq 0$. We call $f_m$ *$\gamma$-computable* by $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ if

1. $\mathbf{g}$ is $m$-polynomial,
2. for all $r \geq \gamma$ we have that $\mathbf{g}(r)$ generates $f_m(r)$ over $\mathbb{K}$ and
3. for all $r \geq 0$ we have that $\mathbf{g}(r)$ generates a subspace of $f_m(r)$ over $\mathbb{K}$.

$\Diamond$

**Proposition 3.4.4.** *Let*

$$\mathrm{V}(\mathbf{b}, (0), \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{g}$$

*where* $\mathbf{b} := [\mathbf{a}]_{\|a\|}$ *and define*

$$\mathbf{h} : \begin{cases} \mathbb{N}_0 & \to & \mathbb{F}[t]^k \\ d & \mapsto & \mathbf{g}\, t^d. \end{cases}$$

*Then* $f_0(r)$ *is* 0*-computable by* $\mathbf{h}$*.*

*Proof.* This follows by Proposition 3.4.2. $\square$

**Definition 3.4.5.** Let $m \geq 0, r \geq 0$ and $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. We define

$$\mathcal{V}^{\mathbf{g}}_{m,r+1} := \Big\{ \mathbf{c} \wedge h \in \mathbb{K}^k \times \mathbb{F} \mid [\sigma_{\mathbf{a}}(\mathbf{c}\,\mathbf{g}(r+1) + h\, t^r)]_{\|a\|+r} = 0 \Big\}.$$

$\Diamond$

**Remark 3.4.2.** In the following we will write $\mathcal{V}_{m,r+1}$ instead of $\mathcal{V}^{\mathbf{g}}_{m,r+1}$ if it is clear from the context. $\Diamond$

**Lemma 3.4.5.** *Let* $m \geq 0, r \geq 0$ *and let* $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ *be* $m$*-polynomial. Then* $\mathcal{V}_{m,r+1}$ *is a vector space over* $\mathbb{K}$*.*

**Definition 3.4.6.** Let $m \geq 0, r \geq 0$ and $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. We define

$$\mathcal{W}_{m,r+1} := \{ \mathbf{c}\,\mathbf{g}(r+1) + h\, t^r \mid \mathbf{c} \wedge h \in \mathcal{V}_{m,r+1} \}.$$

$\Diamond$

**Lemma 3.4.6.** *Let* $m \geq 0, r \geq 0$ *and* $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ *be* $m$*-polynomial. Then* $\mathcal{W}_{m,r+1}$ *is a vector space over* $\mathbb{K}$*.*

*Proof.* This follows immediately by Lemma 3.4.5. $\square$

**Theorem 3.4.2.** *Let* $m \geq 0$ *and* $f_m$ *be* $\gamma$*-computable by* $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$*. Then for all* $r \geq \gamma$ *we have*

$$f_{m+1}(r) = \mathcal{W}_{m,r+1}.$$

*Proof.* Let $r \geq \gamma$. Then

$$\begin{array}{c} w \in f_{m+1}(r) \\ \Updownarrow \text{ Cor. 3.4.1} \\ w \in f_m(r+1) + t^r\, \mathbb{F} : \; [\sigma_{\mathbf{a}} w]_{\|\mathbf{a}\|+r} = 0 \\ \Updownarrow \\ \exists\, \mathbf{c} \in \mathbb{K}^k\, \exists h \in \mathbb{F} : w = \mathbf{c}\,\mathbf{g}(r+1) + t^r\, h \;\&\; [\sigma_{\mathbf{a}} w]_{\|\mathbf{a}\|+r} = 0 \\ \Updownarrow \\ \exists\, \mathbf{c} \wedge h \in \mathcal{V}_{m,r+1} : \; w = \mathbf{c}\,\mathbf{g}(r+1) + t^r\, h \\ \Updownarrow \\ w \in \mathcal{W}_{m,r+1}. \end{array} \tag{3.23}$$

$\square$

**Corollary 3.4.2.** *Let $m \geq 0$ and $f_m$ be $\gamma$-computable by $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$. Then for all $r \geq 0$ we have*

$$f_{m+1}(r) \supseteq \mathcal{W}_{m,r+1}.$$

*Proof.* The proof is the same as in Theorem 3.4.2 but the equivalence (3.23) must be replaced by a top-down implication. □

**Definition 3.4.7.** Let $m \geq 0$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial and $\mathbf{c}{\wedge}h \in \mathbb{K}[x]^k \times \mathbb{F}[x]$. We define for $d \in \mathbb{N}_0$

$$q^{(d)}_{\mathbf{c}\wedge h} := \left[\sigma_{\mathbf{a}}(\sum_{s=1}^{k} \mathbf{c}\,\mathbf{g}^{(d+1)} + h\,t^d)\right]_{\|\mathbf{a}\|+d}$$

and

$$\mathcal{V}_m := \left\{\mathbf{c}{\wedge}h \in \mathbb{K}[x]^k \times \mathbb{F}[x] \,|\, \forall d \geq 0 :\ q^{(d)}_{\mathbf{c}\wedge h}[m+d+1] = 0\right\}.$$

◊

**Lemma 3.4.7.** *Let $m \geq 0$ and $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. $\mathcal{V}_m$ is a module over $\mathbb{K}[x]$.*

**Remark 3.4.3.** One can immediately see that

$$\mathcal{V}_{m,r+1} = \left\{\mathbf{c}{\wedge}h \in \mathbb{K}^k \times \mathbb{F} \,|\, q^{(r)}_{\mathbf{c}\wedge h}[m+r+1] = 0\right\}.$$

◊

**Lemma 3.4.8.** *Let $m \geq 0$ and let $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. We have*

$$\forall\, r \geq 0 :\ \mathcal{V}_m[m+r+1] \subseteq \mathcal{V}_{m,r+1}.$$

*Proof.* This follows immediately by Remark 3.4.3. □

**Definition 3.4.8.** Let $m, r \geq 0$ and $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. We define

$$\mathcal{W}^{(r)}_m := \left\{\mathbf{c}\,\mathbf{g}^{(r+1)} + t^r\,h \,|\, \mathbf{c}{\wedge}h \in \mathcal{V}_m\right\}.$$

◊

**Lemma 3.4.9.** *Let $m \geq 0$ and $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. We have*

$$\mathcal{W}^{(r)}_m[m+r+1] = \{\mathbf{c}\,\mathbf{g}(r+1) + t^r\,h \,|\, \mathbf{c}{\wedge}h \in \mathcal{V}_m[m+r+1]\}.$$

*Proof.* This follows by

$$\begin{aligned}\mathcal{W}^{(r)}_m[m+r+1] &= \left\{\mathbf{c}[m+r+1]\,\mathbf{g}^{(r+1)}[m+r+1] + t^r\,h[m+r+1] \,|\, \mathbf{c}{\wedge}h \in \mathcal{V}_m\right\}\\ &= \{\mathbf{c}\,\mathbf{g}(r+1) + t^r\,h \,|\, \mathbf{c}{\wedge}h \in \mathcal{V}_m[m+r+1]\}.\end{aligned}$$

□

**Lemma 3.4.10.** *Let $m \geq 0$ and $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. We have*

$$\forall r \geq 0 :\ \mathcal{W}^{(r)}_m[m+r+1] \subseteq \mathcal{W}_{m,r+1}.$$

*Proof.* This follows by Lemmas 3.4.8 and 3.4.9. □

**Corollary 3.4.3.** *Let* $m \geq 0$ *and* $f_m$ *be* $\gamma$*-computable by* $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$*. We have*

$$\forall r \geq 0 : \ \mathcal{W}_m^{(r)}[m+r+1] \subseteq \mathcal{W}_{m,r+1} \subseteq f_{m+1}(r).$$

*Proof.* This a consequence of Lemmas 3.4.10 and 3.4.8. □

**Corollary 3.4.4.** *Let* $m \geq 0$ *and let* $f_m$ *be* $\gamma$*-computable by* $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$*. Then*

$$\forall r \geq \gamma : \ \mathcal{W}_m^{(r)}[m+r+1] \subseteq \mathcal{W}_{m,r+1} = f_{m+1}(r).$$

*Proof.* This follows by Lemma 3.4.10 and Theorem 3.4.2. □

### 3.4.4 The Truncated Solution and Difference Equations

As in the previous sections we assume that $(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$. For $k \in \mathbb{Z}$ we define $\beta_{(k)} \in \mathbb{F}$ such that

$$\beta_{(k)} := \sigma^k(t) - t.$$

As in the previous section we assume that

$$\begin{aligned} \mathbf{0} \neq \mathbf{a} &= (a_1, \dots, a_n) \in \mathbb{F}[t]^n, \\ \mathbf{f} &= (f_1, \dots, f_\lambda) \in \mathbb{F}[t]^\lambda \end{aligned}$$

and that $(\mathbb{F}(t)[x], \sigma)$ is a difference ring extension of $(\mathbb{F}(t), \sigma)$ where $x$ is transcendental over $\mathbb{F}(t)$ with $\sigma(x) = x$. Additionally, recall the notation

$$f[p] = \sum_i f_i\, p^i$$

for $f = \sum_i f_i\, x^i \in \mathbb{F}(t)[x]$ with $f_i \in \mathbb{F}(t)$ and $p \in \mathbb{F}(t)$.

**Lemma 3.4.11.** *Let*

$$\begin{aligned} a &= \sum_{j=0}^{p} a_j\, t^j \in \mathbb{F}[t], \quad a_p \neq 0 \\ g &= \sum_{j=d+1}^{e} g_{j-d-1}\, t^j \in \mathbb{F}[t], \quad g_{e-d-1} \neq 0 \end{aligned}$$

*with $g_j = a_j = 0$ for all $j < 0$. If $0 \leq i \leq e + p$ and $k \geq 0$ then*

$$\Big[a\, \sigma^k(g)\Big]_{e+p-i} = \sum_{s=0}^{i} a_{p+s-i} \sum_{l=0}^{s} \sigma^k(g_{l+e-d-1-s}) \binom{l+e-s}{e-s} \beta_{(k)}^l.$$

*Proof.* We have

$$\Big[a\, \sigma^k(g)\Big]_{e+p-i} = \sum_{s=0}^{i} a_{p+s-i} \Big[\sigma^k(g)\Big]_{e-s}$$

for $0 \leq i \leq e + p$ and

$$\begin{aligned} \Big[\sigma^k(g)\Big]_{e-s} &= \left[\sum_{l=0}^{e} \sigma^k(g_{l-d-1})(t + \beta_{(k)})^l\right]_{e-s} = \sum_{l=e-s}^{e} \sigma^k(g_{l-d-1}) \underbrace{\Big[(t + \beta_{(k)})^l\Big]_{e-s}}_{\binom{l}{e-s} \beta_{(k)}^{l-e+s}} \\ &= \sum_{l=0}^{s} \sigma^k(g_{l-d-1+e-s}) \binom{l+e-s}{e-s} \beta_{(k)}^l \end{aligned}$$

for $0 \leq s \leq e + p$. Combining these two identities, the lemma follows. $\square$

**Proposition 3.4.5.** *Let* $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{F}[t]^n$ *with* $p := \|\mathbf{a}\|$,

$$a_i = \sum_{j=0}^{p} a_{ij}\, t^j \in \mathbb{F}[t],$$
$$g = \sum_{j=d+1}^{e} g_{j-d-1}\, t^j \in \mathbb{F}[t], \quad g_{e-d-1} \neq 0$$

*with* $g_j = 0$ *for all* $j < 0$. *If* $0 \leq i \leq e + p$ *then*

$$\begin{aligned}[\sigma_{\mathbf{a}} g]_{e+p-i} = &\sum_{j=1}^{n} a_{jp}\, \sigma^{n-j}(g_{e-d-1-i}) \\ &+ \sum_{j=1}^{n} \sum_{s=0}^{i-1} a_{j,p+s-i} \sum_{l=0}^{s} \sigma^{n-j}(g_{l+e-d-1-s}) \binom{l+e-s}{e-s} \beta^{l}_{(n-j)} \\ &+ \sum_{j=1}^{n} a_{jp} \sum_{l=1}^{i} \sigma^{n-j}(g_{l+e-d-1-i}) \binom{l+e-i}{e-i} \beta^{l}_{(n-j)}.\end{aligned}$$

*Proof.* By Lemma 3.4.5 it follows that

$$\begin{aligned}[\sigma_{\mathbf{a}} g]_{e+p-i} =&\sum_{j=1}^{n} \sum_{s=0}^{i} a_{j,p+s-i} \sum_{l=0}^{s} \sigma^{n-j}(g_{l+e-d-1-s}) \binom{l+e-s}{e-s} \beta^{l}_{(n-j)} \\ =&\sum_{j=1}^{n} \sum_{s=0}^{i-1} a_{j,p+s-i} \sum_{l=0}^{s} \sigma^{n-j}(g_{l+e-d-1-s}) \binom{l+e-s}{e-s} \beta^{l}_{(n-j)} \\ &+ \sum_{j=1}^{n} a_{jp} \sum_{l=0}^{i} \sigma^{n-j}(g_{l+e-d-1-i}) \binom{l+e-i}{e-i} \beta^{l}_{(n-j)}\end{aligned}$$

and thus by

$$\begin{aligned}&\sum_{j=1}^{n} a_{jp} \sum_{l=0}^{i} \sigma^{n-j}(g_{l+e-d-1-i}) \binom{l+e-i}{e-i} \beta^{l}_{(n-j)} \\ &\qquad = \sum_{j=1}^{n} a_{jp}\, \sigma^{n-j}(g_{e-d-1-i}) + \sum_{j=1}^{n} a_{jp} \sum_{l=1}^{i} \sigma^{n-j}(g_{l+e-d-1-i}) \binom{l+e-i}{e-i} \beta^{l}_{(n-j)}\end{aligned}$$

the proposition follows. □

**Theorem 3.4.3.** *Let* $m \geq 0$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ *be* $m$*-polynomial and define* $\mathbf{b} := [\mathbf{a}]_{\|\mathbf{a}\|}$. *Then there exists a* $\mathbf{w} \in \mathbb{F}[x]^k$ *such that*

$$\mathcal{V}_m = \Big\{ \mathbf{c} \wedge h \in \mathbb{K}[x]^k \times \mathbb{F}[x] \,|\, \sigma_{\mathbf{b}} h = \mathbf{c}\, \mathbf{w} \Big\}.$$

*Proof.* Let $\mathbf{a} = (a_1, \dots, a_n)$ with $p := \|\mathbf{a}\|$ and

$$a_i = \sum_{j=0}^{p} a_{ij}\, t^j \in \mathbb{F}[t].$$

Since

$$\mathbf{g} : \begin{cases} \mathbb{N}_0 & \to & \mathbb{F}[t]^k \\ d & \mapsto & \mathbf{g}^{(d)}[m+d] \end{cases}$$

is $m$-polynomial, we have $\mathbf{g}^{(d)} = \left(g_1^{(d)}, \dots, g_k^{(d)}\right) \in \mathbb{F}[t,x]^k$ with

$$g_i^{(d)} = t^d \sum_{j=0}^{m} g_{ij}\, t^j$$

where $g_{ij} \in \mathbb{F}[x]$. Define

$$\begin{aligned} w_i := &-\sum_{j=1}^{n}\sum_{s=0}^{m} a_{j,p+s-m-1} \sum_{l=0}^{s} \sigma^{n-j}(g_{i,l+m-s}) \binom{l+x-s}{x-s} \beta^l_{(n-j)} \\ &+ \sum_{j=1}^{n} a_{jp} \sum_{l=1}^{m+1} \sigma^{n-j}(g_{i,l-1}) \binom{x+l-i}{x-i} \beta^l_{(n-j)} \in \mathbb{F}[x] \end{aligned} \tag{3.24}$$

for $1 \le i \le k$. Now let $\mathbf{c} \wedge h \in \mathbb{K}[x]^k \times \mathbb{F}[x]$. Then we have

$$\sigma_{\mathbf{b}} h = \mathbf{c}\, \mathbf{w}$$

$$\Updownarrow$$

$$\sigma_{\mathbf{b}} h[m+d+1] = \mathbf{c}[m+d+1]\, \mathbf{w}[m+d+1] \quad \forall\, d \in \mathbb{N}_0.$$

Using (3.24) we obtain

$$\sigma_{\mathbf{b}} h[m+d+1] - \mathbf{c}[m+d+1]\, \mathbf{w}[m+d+1]$$

$$||$$

$$\begin{aligned} &\sum_{\nu=1}^{k} c_\nu[m+d+1] \left( \sum_{j=1}^{n}\sum_{s=0}^{m} a_{j,p+s-m-1} \sum_{l=0}^{s} \sigma^{n-j}(g_{\nu,l+m-s}[m+d+1]) \binom{l+m+d+1-s}{m+d+1-s} \beta^l_{(n-j)} \right. \\ &\left. + \sum_{j=1}^{n} a_{jp} \sum_{l=1}^{m+1} \sigma^{n-j}(g_{i,l-1}[m+d+1]) \binom{m+d+1+l-i}{m+l+1-i} \beta^l_{(n-j)} \right) \quad + \quad \sigma_{\mathbf{b}} h[m+d+1] \end{aligned} \tag{3.25}$$

for $d \geq 0$. By replacing $e$ by $m+d+1$ and $i$ by $m+1$ in Proposition 3.4.5 one sees immediately that (3.25) is equal to

$$\sum_{\nu=1}^{k} c_\nu[m+d+1] \left[ \sigma_{\mathbf{a}} \sum_{j=0}^{m} g_{\nu j}[m+d+1]\, t^{d+1+j} + t^d\, h[m+d+1] \right]_{p+d}$$

$$||$$

$$\left[ \sigma_{\mathbf{a}} \sum_{\nu=1}^{k} c_\nu[m+d+1] \sum_{j=0}^{m} g_{\nu j}[m+d+1]\, t^{d+1+j} + t^d\, h[m+d+1] \right]_{p+d}$$

$$||$$

$$q^{(d)}_{\mathbf{c}\wedge h}[m+d+1].$$

Thus

$$\begin{gathered}\sigma_{\mathbf{b}}h = \mathbf{c}\,\mathbf{w}\\ \Updownarrow\\ \forall d \geq 0 : q^{(d)}_{\mathbf{c}\wedge h}[m+d+1] = 0\end{gathered} \tag{3.26}$$

and consequently

$$\mathcal{V}_m = \Big\{\mathbf{c}\wedge h \in \mathbb{K}[x]^k \times \mathbb{F}[x] \,|\, \sigma_{\mathbf{b}}h = \mathbf{c}\,\mathbf{w}\Big\}.$$

□

**Theorem 3.4.4.** *Let $m \geq 0$ and $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. Define $\mathbf{b} := [\mathbf{a}]_{\|\mathbf{a}\|}$ and let $\mathbf{w} \in \mathbb{F}[x]^k$ be defined by (3.24) in the proof of Theorem 3.4.3. Then for all $r \geq 0$ we have*

$$\mathcal{V}_{m,r+1} = \Big\{\mathbf{c}\wedge h \in \mathbb{K}^k \times \mathbb{F} \,|\, \sigma_{\mathbf{b}}h = \mathbf{c}\,\mathbf{w}[m+r+1]\Big\}.$$

*Proof.* Let $\mathbf{c}\wedge h \in \mathbb{K}^k \times \mathbb{F}$. Then by the proof of Theorem 3.4.3 we obtain immediately that

$$\begin{gathered}\sigma_{\mathbf{b}}h = \mathbf{c}\,\mathbf{w}[m+r+1]\\ \Updownarrow \quad \text{by (3.26)}\\ q^{(r)}_{\mathbf{c}\wedge h}[m+r+1] = 0\\ \Updownarrow \quad \text{Remark 3.4.3}\\ \mathbf{c}\wedge h \in \mathcal{V}_{m,r+1}.\end{gathered}$$

□

**Corollary 3.4.5.** *Let $m \geq 0$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial and $r \geq 0$. Then $\mathcal{V}_{m,r+1}$ is a finite dimensional vector space over $\mathbb{K}$.*

*Proof.* Let $\mathbf{b} := [a]_{\|\mathbf{a}\|}$. By Theorem 3.4.4 it follows that there is a $\mathbf{w} \in \mathbb{F}^k$ such that

$$\mathcal{V}_{m,r+1} = \Big\{\mathbf{c}\wedge h \in \mathbb{K}^k \times \mathbb{F} \,|\, \sigma_{\mathbf{b}}h = \mathbf{c}\,\mathbf{w}\Big\} = \mathrm{V}(\mathbf{b}, \mathbf{w}, \mathbb{F}).$$

Thus by Corollary 3.1.1 $\mathcal{V}_{m,r+1}$ is finite dimensional. □

### 3.4.5 Some Notations and Facts about Modules

Let $\mathbb{A}$ be a module over a ring $\mathbb{B}$ and, more generally, consider $\mathbb{A}^n$ as a module over $\mathbb{B}$. Let $\mathbb{M}$ be a submodule of $\mathbb{A}^n$ over $\mathbb{B}$. We call $\mathbb{M}$ *finitely generated* if there exists a finite set

$$G := \{\mathbf{g_1}, \dots, \mathbf{g_d}\} \subseteq \mathbb{M} \subseteq \mathbb{A}^n$$

such that

$$\mathbb{M} = \{k_1\,\mathbf{g_1} + \dots + k_d\,\mathbf{g_d} \,|\, k_i \in \mathbb{B}\}.$$

In this case we introduce as for the vector space case (Section 3.2.1.2) a *generator matrix*

$$\mathbf{M}_{\mathbb{M}} := \begin{pmatrix} \mathbf{g_1} \\ \vdots \\ \mathbf{g_d} \end{pmatrix},$$

this means we have

$$\mathbb{M} = \left\{\mathbf{k} \cdot \mathbf{M}_{\mathbb{M}} \,|\, \mathbf{k} \in \mathbb{B}^d\right\}.$$

We will write

$$\mathbb{M} \xleftrightarrow{\text{span}} \mathbf{M}_{\mathbb{M}}$$

in order to indicate that $\mathbf{M}_{\mathbb{M}}$ is a generator matrix for $\mathbb{M}$. For the special situation $\mathbb{M} = \{(0, \dots, 0)\} \subseteq \mathbb{A}^n$, the generator matrix is

$$\mathbf{M_B} = (0, \dots, 0) \in \mathbb{A}^{1\times n}.$$

$G = \{\mathbf{g_1}, \dots, \mathbf{g_d}\}$ is a *basis* of the finitely generated module $M$ if the $\mathbf{g_i}$ in $G$ are linearly independent over $\mathbb{B}$. $d$ is called the dimension of $\mathbb{M}$. In this case $\mathbf{M}_{\mathbb{M}}$ is called *basis matrix* of $\mathbb{M}$; we will write

$$\mathbb{M} \xleftrightarrow{\text{basis}} \mathbf{M}_{\mathbb{M}}$$

to indicate this fact. The following lemmas can be found for instance in [Sim84, Coh89, Lan97].

**Lemma 3.4.12.** *Let $\mathbb{M}$ be a module over a ring $\mathbb{A}$. If $\mathbb{M}$ is finitely generated and has a basis of dimension $d$ then all bases of $\mathbb{M}$ have dimension $d$.*

**Lemma 3.4.13.** *Let $\mathbb{M}$ be a finitely generated module over a principle ideal domain $\mathbb{A}$. Then each submodule of $\mathbb{M}$ over $\mathbb{A}$ is finitely generated.*

**Lemma 3.4.14.** *Consider $\mathbb{A}^n$ as a module over a principle ideal domain $\mathbb{A}$ and let $\mathbb{M}$ be a submodule of $\mathbb{A}^n$ over $\mathbb{A}$. Then $\mathbb{M}$ is finitely generated over $\mathbb{A}$ and has a basis over $\mathbb{A}$.*

**Remark 3.4.4.** Consider $\mathbb{A}^n$ as a module over an Euclidean domain $\mathbb{A}$ and let $\mathbb{M}$ be a submodule of $\mathbb{A}^n$ over $\mathbb{A}$. Given

$$\mathbb{M} \xleftrightarrow{\text{span}} \mathbf{M}$$

then the matrix $\mathbf{M}$ can be transformed[10] by row operations

1. Interchange row[11] $r_i$ by row $r_j$.

[10] An algorithm can be found for instance in [Sim84, Theorem 6.1.8].
[11] $r_i$ means the row at position $i$

2. Replace row $r_i$ by row $k\,r_i$ where $k \in \mathbb{A}^*$.

3. Replace row $r_i$ by row $r_i + k\,r_j$ where $i \neq j$ and $k \in \mathbb{A}$.

to a matrix $\mathbf{M}'$ such that[12]

$$\mathbb{M} \xleftrightarrow{\text{basis}} \mathbf{M}'$$

and $\mathbf{M}'$ is in *row-echelon form*; this means that all entries below the left most nonzero entry of a row are zero, in other words all entries below the "stair case" are zero. If $\mathbf{M}, \mathbf{M}' \in \mathbb{A}^{n\times m}$ then we have[13]

$$\{\mathbf{a} \in \mathbb{A}^m \,|\, \mathbf{M}\cdot\mathbf{a} = \mathbf{0}\} = \big\{\mathbf{a} \in \mathbb{A}^m \,|\, \mathbf{M}'\cdot\mathbf{a} = \mathbf{0}\big\}.$$

$\diamondsuit$

**Remark 3.4.5.** Consider $\mathbb{A}^n$ as a module over an Euclidean domain $\mathbb{A}$ and let $\mathbf{A} \in \mathbb{A}^{m\times n}$. Then by row and column[14] operations one can compute[15] a *reduced matrix*

$$\mathbf{D} := \begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & \mathbf{0} & & \\ & & \ddots & & & & \\ & & & d_r & & & \\ & \mathbf{0} & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

with $d_i \in \mathbb{F}^*$ and $d_i \mid d_{i+1}$ for $1 \leq i < r$ and obtains matrices[16] $\mathbf{P} \in \mathrm{GL}_m(\mathbb{A})$ and $\mathbf{Q} \in \mathrm{GL}_n(\mathbb{A})$ such that

$$\mathbf{D} = \mathbf{P}\,\mathbf{A}\,\mathbf{Q}.$$

Now consider the submodule

$$\mathbb{M} := \{\mathbf{f} \in \mathbb{A}^n \,|\, \mathbf{A}\cdot\mathbf{f} = \mathbf{0}\}$$

of $\mathbb{A}^n$ over $\mathbb{A}$. Then by Lemmas 3.4.12 and 3.4.14 $\mathbb{M}$ is finitely generated and has a basis whose dimension $d$ is uniquely determined. By Theorem 6.7.2 in [Sim84] it follows that

$$d = n - r$$

and that the last $d$ columns of $\mathbf{Q}$ are a basis of $\mathbb{M}$. $\diamondsuit$

**Definition 3.4.9.** Let $\mathbb{A}$ be a module over a ring $\mathbb{B}$, consider $\mathbb{A}^n$ as a module over $\mathbb{B}$ and let $\mathbf{f} \in \mathbb{A}^n$. Then we define the *annihilator* of $\mathbf{f}$ over $\mathbb{B}$ by

$$\mathrm{Ann}_{\mathbb{B}}(\mathbf{f}) = \{\mathbf{c} \in \mathbb{B}^n \,|\, \mathbf{c}\,\mathbf{f} = 0\}.$$

$\diamondsuit$

**Lemma 3.4.15.** *Let $\mathbb{A}$ be a module over a ring $\mathbb{B}$ and $\mathbf{f} \in \mathbb{A}^n$. Then $\mathrm{Ann}_{\mathbb{B}}(\mathbf{f})$ is a submodule of $\mathbb{A}^n$ over $\mathbb{B}$.*

---

[12] This property is a direct consequence of Theorem 6.1.2 and Corollary 6.1.7 in [Sim84].

[13] This is a consequence of Theorems 6.4.4 and 6.5.1 in [Sim84].

[14] Column operations are defined in a completely analogous manner by replacing the notion row by the notion column in Remark 3.4.4.

[15] An algorithm is described in the proof of Theorem 6.5.6 in [Sim84].

[16] $\mathrm{GL}_m(\mathbb{A})$ denotes the group of square matrices of length $m$ with entries in $\mathbb{A}$.

### 3.4.6 Some Ideas to Compute the Truncated Solution Space

The following theorem glues the previous results together and gives an idea about how one can find a $m+1$-polynomial $\mathbf{h}:\mathbb{N}_0 \to \mathbb{F}[t]^l$ for some $l \geq 1$ such that $f_{m+1}(r)$ is $\delta$ computable by $\mathbf{h}$ for some $\delta \geq \gamma$.

**Theorem 3.4.5.** *Let $m \geq 0$ and $f_m$ be $\gamma$-computable by $\mathbf{g}:\mathbb{N}_0 \to \mathbb{F}[t]^k$. Assume there is a $\delta \geq \gamma$ such that*

$$\forall r \geq \delta : \ \mathcal{W}_m^{(r)}[m+r+1] = \mathcal{W}_{m,r+1}.$$

*Then*

$$\forall r \geq \delta : \ f_{m+1}(r) = \mathcal{W}_m^{(r)}[m+r+1].$$

*Assume $\mathcal{V}_m$ is a finitely generated, in particular assume*

$$\mathcal{V}_m \xleftrightarrow{\text{span}} \mathbf{C} \wedge \mathbf{q}$$

*with $\mathbf{C} \in \mathbb{K}[x]^{l \times k}$ and $\mathbf{q} \in \mathbb{F}[t][x]^l$ for some $l \geq 1$. Define*

$$\mathbf{h}^{(\mathbf{d})} := \mathbf{C} \cdot \mathbf{g}^{(\mathbf{d+1})} + t^d\, \mathbf{q}.$$

*Then for all $d \geq 0$ it follows that $\mathbf{h}^{(d)}$ generates $\mathcal{W}_m^{(d)}$ and $f_{m+1}$ is $\delta$-computable by*

$$\mathbf{h} : \begin{cases} \mathbb{N}_0 & \to \quad \mathbb{F}[t]^l \\ r & \mapsto \quad \mathbf{h}^{(r)}[m+r+1]. \end{cases}$$

*Proof.* By Corollary 3.4.4 and the assumption $\mathcal{W}_m^{(r)}[m+r+1] = \mathcal{W}_{m,r+1}$ for all $r \geq \delta$ it follows immediately that

$$\forall r \geq \delta : \ f_{m+1}(r) = \mathcal{W}_m^{(r)}[m+r+1].$$

By the Definition 3.4.8 of $\mathcal{W}_m^{(d)}$ we can conclude that $\mathbf{h}^{(\mathbf{d})}$ generates $\mathcal{W}_m^{(d)}$ for all $d \geq 0$. Thus

$$\forall r \geq 0 : \ \mathcal{W}_m^{(r)}[m+r+1] \subseteq f_{m+1}(r)$$

by Corollary 3.4.3 and it follows that $f_{m+1}$ is $\delta$-computable by $\mathbf{h}$. □

This theorem will provide later a stopping condition for Algorithm 3.4.2.

**Theorem 3.4.6.** *Let $m \geq 0$ and $f_{m+1}$ be $\delta$-computable by $\mathbf{g}:\mathbb{N}_0 \to \mathbb{F}[t]^k$, i.e. $f_{m+1}(r)$ is generated by $\mathbf{g}(r) = \mathbf{g}^{(r)}[m+r]$ where $\mathbf{g}^{(r)} = \left(g_1^{(r)}, \ldots, g_k^{(r)}\right) \in \mathbb{F}[t,x]^k$ with*

$$g_i^{(r)} = t^r \sum_{j=0}^{m} g_{ij}\, t^j$$

*and $g_{ij} \in \mathbb{F}[x]$. If*

$$\|\mathbf{g}^{(0)}\| \leq m$$

*then for all $r \geq \delta$ and all $j \geq 0$ we have*

$$f_m(r) = f_{m+j}(r).$$

*Proof.* If $\|\mathbf{g}^{(0)}\| \leq m$ then for all $r \geq \delta$ we have

$$\|\mathbf{g}^{(r)}\| \leq m + r.$$

By Lemma 3.4.2 we have

$$f_m(r) = f_{m+1}(r) \cap \mathbb{F}[t]_{m+r}.$$

Since $f_{m+1}(r)$ is generated by $\mathbf{g}^{(r)}[m+r]$, it follows that

$$f_m(r) = f_{m+1}(r)$$

for all $r \geq \delta$. Thus the induction base holds. Now let $i \geq 1$ and assume $f_{m+i}(r) = f_m(r)$ for all $r \geq \delta$. Then by Corollary 3.4.1 it follows for all $r \geq \delta$ that

$$\begin{aligned} f_{m+i+1}(r) &\overset{\text{Cor. (3.4.1)}}{=} \left\{ g \in f_{m+i}(r+1) + t^r\,\mathbb{F} \mid [\sigma_{\mathbf{a}} g]_{\|\mathbf{a}\|+r} = 0 \right\} \\ &\overset{\text{I.A.}}{=} \left\{ g \in f_m(r+1) + t^r\,\mathbb{F} \mid [\sigma_{\mathbf{a}} g]_{\|\mathbf{a}\|+r} = 0 \right\} \\ &\overset{\text{Cor. (3.4.1)}}{=} f_{m+1}(r) \overset{\text{I.B.}}{=} f_m(r). \end{aligned}$$

$\square$

### 3.4.7 Computing the Truncated Solution Space For Proper Sum Extensions

Also for this Subsection 3.4.7 we will assume that $(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$,

$$\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_n) \in \mathbb{F}[t]^n,$$
$$\mathbf{f} = (f_1, \dots, f_\lambda) \in \mathbb{F}[t]^\lambda$$

and that $(\mathbb{F}(t)[x], \sigma)$ is a difference ring extension of $(\mathbb{F}(t), \sigma)$ where $x$ is transcendental over $\mathbb{F}(t)$ and $\sigma(x) = x$. Additionally, we again recall the notation

$$f[p] = \sum_i f_i\, p^i$$

Furthermore, for $f = \sum_j f_j\, x^j \in \mathbb{F}[t][x]$ with $f_j \in \mathbb{F}[t]$ we write

$$[f]_i^x := f_i;$$

and for $f = \sum_j f_j\, t^j \in \mathbb{F}[x][t]$ with $f_j \in \mathbb{F}[x]$ we write

$$[f]_i^t := f_i.$$

Let $m \geq 0$ and assume we have given a $m$-polynomial $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ such that $f_m$ is $\gamma$-computable by $\mathbf{g}$. If $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$, we will achieve the following two results.

- We explain how one can find a specific $\delta \geq \gamma$ such that for all $r \geq \delta$ we have
$$\forall r \geq \delta : \; \mathcal{W}_m^{(r)}[m + r + 1] = \mathcal{W}_{m,r+1}.$$
- Furthermore we show that $\mathcal{V}_m$ is finitely generated and how one can compute a basis matrix for $\mathcal{V}_m$.

After finding these two ingredients, Theorem 3.4.5 then tells us how one can compute a $m + 1$-polynomial $\mathbf{h} : \mathbb{N}_0 \to \mathbb{F}[t]^l$ such that $f_{m+1}$ is $\delta$-computable by $\mathbf{h}$.

**Lemma 3.4.16.** *Let $\mathbf{0} \neq \mathbf{b} \in \mathbb{F}^n$ and $\sigma_{\mathbf{b}} g = f$ with*

$$g = \sum_{i=0}^{d} g_i\, x^i \in \mathbb{F}[x] \quad \textit{and} \quad f = \sum_{i=0}^{e} f_i\, x^i \in \mathbb{F}[x].$$

*Then $d = e$ and $\sigma_{\mathbf{b}} g_i = f_i$ for $0 \leq i \leq d$.*

*Proof.* We have

$$\sigma_{\mathbf{b}} g = \sigma_{\mathbf{b}}\Big(\sum_{i=0}^{d} g_i\, x^i\Big) = \sum_{i=0}^{d} (\sigma_{\mathbf{b}} g_i)\, x^i = \sum_{i=0}^{e} f_i\, x^i.$$

Due to the fact that $x$ is transcendental over $\mathbb{F}$, it follows that $d = e$ and

$$\sigma_{\mathbf{b}} g_i = f_i$$

for $1 \leq i \leq d$. $\square$

**Lemma 3.4.17.** *Let $\mathbf{0} \neq \mathbf{b} \in \mathbb{F}^n$ and assume that $\{h_1, \dots, h_\rho\}$ is a basis for the vector space $\{h \in \mathbb{F} \mid \sigma_{\mathbf{b}} h = 0\}$. Then*

$$\{g \in \mathbb{F}[x] \mid \sigma_{\mathbf{b}} g = 0\} = h_1 \, \mathbb{K}[x] + \dots + h_\rho \, \mathbb{K}[x].$$

*Proof.* If $g \in h_1 \, \mathbb{K}[x] + \dots + h_\rho \, \mathbb{K}[x]$ then clearly we have $g \in \mathbb{F}[x]$ and $\sigma_{\mathbf{b}} g = 0$. Contrary, let $g \in \mathbb{F}[x]$ with $\sigma_{\mathbf{b}} g = 0$, say $g = \sum_{i=0}^{l} g_i \, x^i$ with $g_i \in \mathbb{F}$. Then by Lemma 3.4.16 it follows that

$$\sigma_{\mathbf{b}} g_i = 0$$

for $0 \leq i \leq l$, i.e. there are $c_{i1}, \dots, c_{i\rho} \in \mathbb{K}$ such that

$$g_i = c_{i1} \, h_1 + \dots + c_{i\rho} \, h_\rho.$$

Hence

$$\begin{aligned} g = \left(1, x, \dots, x^l\right) \begin{pmatrix} g_0 \\ \vdots \\ g_l \end{pmatrix} &= \left(1, x, \dots, x^l\right) \left( \begin{pmatrix} c_{01} & \dots & c_{0\rho} \\ \vdots & & \vdots \\ c_{l1} & \dots & c_{l\rho} \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_\rho \end{pmatrix} \right) \\ &= \underbrace{\left( \left(1, x, \dots, x^l\right) \cdot \begin{pmatrix} c_{01} & \dots & c_{0\rho} \\ \vdots & & \vdots \\ c_{l1} & \dots & c_{l\rho} \end{pmatrix} \right)}_{\in \mathbb{K}[x]^\rho} \begin{pmatrix} h_1 \\ \vdots \\ h_\rho \end{pmatrix} \end{aligned}$$

and hence the lemma is proven. ∎

**Lemma 3.4.18.** *Let $\mathbf{0} \neq \mathbf{b} \in \mathbb{F}^n$ and $\mathbf{w} = (w_1, \dots, w_k) \in \mathbb{F}[x]^k$ with*

$$w_i = \sum_{j=0}^{d} w_{ij} \, x^j$$

*for $1 \leq i \leq k$. Let $g \in \mathbb{F}[x]$ and $\mathbf{c} \in \mathbb{K}[x]^k$ with*

$$\sigma_{\mathbf{b}} g = \mathbf{c} \, \mathbf{w}.$$

*Define*

$$\begin{aligned} \mathbf{v} := (\ & w_{10}, & \dots, & \ w_{k0}, \\ & w_{11}, & \dots, & \ w_{k1}, \\ & \vdots & & \ \vdots \\ & w_{1d}, & \dots, & \ w_{kd}) \in \mathbb{F}^\mu \end{aligned}$$

*with $\mu := k\,(d+1)$ and let*

$$\mathrm{V}(\mathbf{b}, \mathbf{v}, \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{A} \wedge \mathbf{p}$$

*with $\mathbf{A} \in \mathbb{K}^{l \times \mu}$ and $\mathbf{p} \in \mathbb{F}^l$ for some $l \geq 1$. Then there exists an $\mathbf{m} \in \mathbb{K}[x]^l$ such that*

$$\mathbf{m} \, (\mathbf{A} \cdot \mathbf{v}) = \mathbf{c} \, \mathbf{w}.$$

*Proof.* Let $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{K}[x]^k$ with $c_i = \sum_{j=0}^{s} c_{ij}\, x^j$, $1 \le i \le k$. Due to the assumption

$$\sigma_{\mathbf{b}} g = \mathbf{c}\,\mathbf{w}$$

and Lemma 3.4.16 there exist $g_i \in \mathbb{F}$ for $0 \le i \le d+s$ such that

$$\sigma_{\mathbf{b}} g_i = [\mathbf{c}\,\mathbf{w}]_i^x = \sum_{r=1}^{k} \sum_{j=1}^{i} c_{rj}\, w_{r,i-j}.$$

Thus there exist $\mathbf{e_i} \in \mathbb{K}^\mu$ for $1 \le i \le d+s$ with

$$\mathbf{e_i}\,\mathbf{v} = \sum_{r=1}^{k} \sum_{j=1}^{i} c_{rj}\, w_{r,i-j} = \sigma_{\mathbf{b}} g_i =: h_i \tag{3.27}$$

or, in other words,

$$\mathbf{e_i} \wedge g_i \in \mathrm{V}(\mathbf{a}, \mathbf{v}, \mathbb{F}).$$

Since $\mathbf{A} \wedge \mathbf{p}$ is a basis matrix for $\mathrm{V}(\mathbf{a}, \mathbf{v}, \mathbb{F})$, one can find $\mathbf{b_i} \in \mathbb{K}^l$ for $1 \le i \le d+s$ such that

$$\mathbf{b_i} \cdot (\mathbf{A} \wedge \mathbf{p}) = \mathbf{e_i} \wedge g_i.$$

Consequently there is a $\mathbf{B} \in \mathbb{K}^{(d+s)\times l}$ such that

$$\mathbf{B}\,\mathbf{A} = \begin{pmatrix} \mathbf{e_0} \\ \vdots \\ \mathbf{e_{d+s}} \end{pmatrix}.$$

Then by (3.27) we have

$$\begin{pmatrix} \mathbf{e_0} \\ \vdots \\ \mathbf{e_{d+s}} \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} h_0 \\ \vdots \\ h_{d+s} \end{pmatrix}$$

and thus

$$\left(1, x, \dots, x^{d+s}\right) \left( \begin{pmatrix} \mathbf{e_0} \\ \vdots \\ \mathbf{e_{d+s}} \end{pmatrix} \cdot \mathbf{v} \right) = \sum_{j=0}^{d+s} h_j\, x^j = \mathbf{c}\,\mathbf{w}.$$

Therefore

$$\mathbf{c}\,\mathbf{w} = \left(1, x, \dots, x^{d+s}\right) ((\mathbf{B}\,\mathbf{A}) \cdot \mathbf{v}) = (\underbrace{\left(1, x, \dots, x^{d+s}\right) \cdot \mathbf{B}}_{=:\mathbf{m}})\, (\mathbf{A} \cdot \mathbf{v})$$

which proves the lemma. □

Let $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial and define

$$\begin{aligned} \mathbf{v} := (\ & [w_1]_0^x, & \dots, && [w_k]_0^x, \\ & [w_1]_1^x, & \dots, && [w_k]_1^x, \\ & \vdots & && \vdots \\ & [w_1]_d^x, & \dots, && [w_k]_d^x) \in \mathbb{F}^\mu \end{aligned} \tag{3.28}$$

for $\mu := k\,(d+1)$ where

$$\mathbf{w} = (w_1, \dots, w_k) \in \mathbb{F}[x]^k$$

is defined by (3.24) in the proof of Theorem 3.4.3 for that $\mathbf{g}$.

Assuming that $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w})$ is a finitely generated module over the Euclidean domain $\mathbb{K}[x]$, the following theorem tells us that also $\mathcal{V}_m$ is finitely generated over $\mathbb{K}[x]$ and describes how one can compute a generator matrix which spans the vector space $\mathcal{V}_m$.

**Theorem 3.4.7.** *Let $m \geq 0$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial and define $\mathbf{v} \in \mathbb{F}^\mu$ and $\mathbf{w} \in \mathbb{F}[x]^k$ as stated in (3.28) for that $\mathbf{g}$. Assume that $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w})$ is a finitely generated module. Then $\mathcal{V}_m$ is a finitely generated module over $\mathbb{K}[x]$. Let $\mathbf{0} \neq \mathbf{b} := [\mathbf{a}]^t_{\|\mathbf{a}\|} \in \mathbb{F}^n$ and let*

$$\mathrm{V}(\mathbf{b},\mathbf{v},\mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{B}\wedge\mathbf{q}$$

*with $\mathbf{B} \in \mathbb{K}^{s\times\mu}$ and $\mathbf{q} \in \mathbb{F}^s$ for some $s \geq 1$. Furthermore let*

$$\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}) \xleftrightarrow{\text{basis}} \mathbf{C}\wedge\mathbf{D}$$

*with $\mathbf{C} \in \mathbb{K}[x]^{l\times s}$ and $\mathbf{D} \in \mathbb{K}[x]^{l\times k}$ for some $l \geq 1$ such that*

$$\mathbf{C}\cdot(\mathbf{B}\cdot\mathbf{v}) = \mathbf{D}\cdot\mathbf{w}. \tag{3.29}$$

*Then*

$$\mathcal{V}_m \xleftrightarrow{\text{span}} \mathbf{D}\wedge(\mathbf{C}\cdot\mathbf{q}).$$

*Proof.* By Theorem 3.4.3 we have

$$\mathcal{V}_m = \Big\{\mathbf{c}\wedge h \in \mathbb{K}[x]^k \times \mathbb{F}[x] \mid \sigma_{\mathbf{b}} h = \mathbf{c}\,\mathbf{w}\Big\}. \tag{3.30}$$

Consider the finitely generated module

$$\mathbb{V} := \Big\{\mathbf{d} \in \mathbb{K}[x]^l \mid \mathbf{d}\cdot(\mathbf{D}\wedge(\mathbf{C}\cdot\mathbf{q}))\Big\},$$

i.e. $\mathbf{D}\wedge(\mathbf{C}\cdot\mathbf{q})$ is the generator matrix for $\mathbb{V}$. We will show that

$$\boxed{\mathcal{V}_m = \mathbb{V}.}$$

As $\mathbf{B}\wedge\mathbf{q}$ is a basis matrix for $\mathrm{V}(\mathbf{b},\mathbf{v},\mathbb{F})$, we have

$$\mathbf{B}\cdot\mathbf{v} = \sigma_{\mathbf{b}}\mathbf{q}$$

and consequently

$$\mathbf{D}\cdot\mathbf{w} \overset{(3.29)}{=} \mathbf{C}\cdot(\mathbf{B}\cdot\mathbf{v}) = \mathbf{C}\cdot\sigma_{\mathbf{b}}\mathbf{q} = \sigma_{\mathbf{b}}(\mathbf{C}\cdot\mathbf{q}).$$

Thus by (3.30) it follows that

$$\mathbb{V} \subseteq \mathcal{V}_m.$$

Contrary, let $\mathbf{a}\wedge p \in \mathcal{V}_m$ be arbitrary but fixed. Then we have

$$\sigma_{\mathbf{b}} p = \mathbf{a}\,\mathbf{w}$$

and thus there exists an $\mathbf{h} \in \mathbb{K}[x]^s$ by Lemma 3.4.18 such that

$$\mathbf{h}\,(\mathbf{B}\cdot\mathbf{v}) = \mathbf{a}\,\mathbf{w}. \tag{3.31}$$

Since $\mathbf{C}\wedge\mathbf{D}$ is a basis matrix for $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w})$ it follows by (3.31) that there is a $\mathbf{d}\in\mathbb{K}[x]^l$ such that

$$\mathbf{h}\wedge\mathbf{a} = \mathbf{d}\cdot(\mathbf{C}\wedge\mathbf{D}) = (\mathbf{d}\cdot\mathbf{C})\wedge(\mathbf{d}\cdot\mathbf{D}),$$

i.e.

$$\mathbf{h} = \mathbf{d}\cdot\mathbf{C}, \qquad\qquad \mathbf{a} = \mathbf{d}\cdot\mathbf{D}.$$

Therefore

$$\mathbf{a}\wedge(\mathbf{h}\,\mathbf{q}) = (\mathbf{d}\cdot\mathbf{D})\wedge((\mathbf{d}\cdot\mathbf{C})\,\mathbf{q}) = (\mathbf{d}\cdot\mathbf{D})\wedge(\mathbf{d}\,(\mathbf{C}\cdot\mathbf{q})) = \mathbf{d}\cdot(\mathbf{D}\wedge(\mathbf{C}\cdot\mathbf{q}))$$

and thus

$$\mathbf{a}\wedge(\mathbf{h}\,\mathbf{q})\in\mathbb{V}.$$

Now we will show that $\mathbf{a}\wedge p\in\mathbb{V}$. Then it will follow that

$$\mathbb{V}\supseteq\mathcal{V}_m$$

and consequently

$$\mathcal{V}_m = \mathbb{V} \xleftrightarrow{\text{span}} \mathbf{D}\wedge(\mathbf{C}\cdot\mathbf{q}).$$

In particular it will follow that $\mathcal{V}_m$ is a finitely generated module over $\mathbb{K}[x]$.
If $p=\mathbf{h}\,\mathbf{q}$ then we are done. Otherwise, we have

$$\sigma_{\mathbf{b}}(\underbrace{\mathbf{h}\,\mathbf{q}-p}_{\neq 0}) = 0.$$

Assume there are exactly $\{h_1,\dots,h_\eta\}$ linearly independent solutions $h_i\in\mathbb{F}$ over $\mathbb{K}$ such that

$$\sigma_{\mathbf{a}}h_i = 0.$$

Without loss of generality we can assume that $B$ is in row-echelon form such that

$$\mathbf{B}\wedge\mathbf{q} = \left(\begin{array}{c|c} & * \\ E & \vdots \\ & * \\ \hline & h_1 \\ 0 & \vdots \\ & h_\eta \end{array}\right).$$

Then we have

$$\mathbf{p} := \mathbf{B}\cdot\mathbf{v} = (\overbrace{*,\dots,*}^{\nu:=s-\eta\text{ entries}},\overbrace{0,\dots,0}^{\eta\text{ entries}})\in\mathbb{K}^s,$$

in particular for $1\le i\le\eta$

$$(0,\dots,0,\underbrace{1}_{(\nu+i)\text{ pos.}},0,\dots,0)$$

is a solution of $\mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{p}\wedge-\mathbf{w})$. Since $\mathbf{C}\wedge\mathbf{D}$ is a basis matrix for $\mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{p}\wedge-\mathbf{w})$ it follows that there are $\mathbf{x_i}\in\mathbb{K}[x]^l$ for $1\le i\le\eta$ such that

$$\mathbf{x_i}\cdot(\mathbf{D}\wedge\mathbf{C}) = (0,\dots,0,\underbrace{1}_{(\nu+i)\text{ pos.}},0,\dots,0)$$

and consequently

$$\begin{aligned}\mathbf{x_i} \cdot (\mathbf{D} \wedge (\mathbf{C} \cdot \mathbf{q})) &= (\mathbf{x_i} \cdot \mathbf{D}) \wedge (\mathbf{x_i}\, (\mathbf{C} \cdot \mathbf{q})) = \mathbf{0} \wedge ((\mathbf{x_i} \cdot \mathbf{C})\, \mathbf{q}) \\ &= \mathbf{0} \wedge ((0, \dots, 0, \underbrace{1}_{(\nu+i)\text{ pos.}}, 0, \dots, 0)\, (\underbrace{*, \dots, *}_{\nu}, h_1, \dots, h_\eta)) \\ &= \mathbf{0} \wedge h_i.\end{aligned}$$

By Lemma 3.4.17 there is

$$h_1\, c_1 + \dots + h_\eta\, c_\eta = \mathbf{h}\, \mathbf{q} - p$$

for some $c_i \in \mathbb{K}[x]$ and therefore for

$$\mathbf{x} := \mathbf{x_1}\, c_1 + \dots + \mathbf{x}_\eta\, c_\eta \in \mathbb{K}[x]^l$$

we have

$$\mathbf{x} \cdot (\mathbf{D} \wedge (\mathbf{C} \cdot \mathbf{q})) = \mathbf{0} \wedge (h_1\, c_1 + \dots + h_\eta\, c_\eta) = \mathbf{0} \wedge (\mathbf{h}\, \mathbf{q} - p).$$

Thus

$$\mathbf{0} \wedge (\mathbf{h}\, \mathbf{q} - p) \in \mathbb{V}$$

and consequently

$$\mathbf{a} \wedge p = (\underbrace{\mathbf{a} \wedge (\mathbf{h}\, \mathbf{q})}_{\in \mathbb{V}}) - (\underbrace{\mathbf{0} \wedge (\mathbf{h}\, \mathbf{q} - p)}_{\mathbb{V}}) \in \mathbb{V}$$

which finally proves the theorem. □

In the previous Theorem 3.4.7 we assumed that $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B} \cdot \mathbf{v}) \wedge -\mathbf{w})$ is a finitely generated module over $\mathbb{K}[x]$ in order to proof that $\mathcal{V}_m$ is finitely generated over $\mathbb{K}[x]$. Furthermore we needed a basis matrix $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B} \cdot \mathbf{v}) \wedge -\mathbf{w})$ in order to compute $\mathcal{V}_m$. The following lemma gives us a tool to compute such a basis matrix $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B} \cdot \mathbf{v}) \wedge -\mathbf{w})$ if $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$.

**Lemma 3.4.19.** *Let $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{K}(t_1, \dots, t_e)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and let $\mathbf{f} \in \mathbb{F}[x]^n$. Then $\mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{f})$ is a module over $\mathbb{K}[x]$ which is finitely generated and a basis can be computed by linear algebra.*

*Proof.* Let $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}[x]^n$. Since $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$, it follows that $\mathbb{F}$ is the quotient field of the polynomial ring $\mathbb{K}[t_1, \dots, t_e]$. Since $x$ is transcendental over $\mathbb{F}$ it follows in particular that $\mathbb{K}[x][t_1, \dots, t_e]$ is a polynomial ring with coefficients in $\mathbb{K}[x]$, and $\mathbb{K}[x]$ is itself a polynomial ring with coefficients in $\mathbb{K}$. We can find a $d \in \mathbb{K}[t_1, \dots, t_e]^*$ such that

$$\mathbf{g} = (g_1, \dots, g_n) := (f_1\, d, \dots, f_n\, d) \in \mathbb{K}[x][t_1, \dots, t_e].$$

For $\mathbf{c} \in \mathbb{K}[x]^n$ we have

$$\mathbf{c}\, \mathbf{f} = 0 \;\Leftrightarrow\; \mathbf{c}\, \mathbf{g} = 0$$

and therefore

$$\mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{f}) = \mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{g}).$$

Let $c_1, \dots, c_n$ be indeterminates and make the ansatz

$$c_1\, g_1 + \dots + c_n\, g_n = 0.$$

Then the coefficients of each monomial $t_1^{d_1} \dots t_e^{d_e}$ in $c_1 g_1 + \dots + c_n g_n$ must vanish. Therefore we obtain a linear system of equations

$$\begin{array}{cccc} c_1 p_{11}+ & \dots & +c_n p_{1n} & = 0 \\ & \vdots & & \\ c_r p_{r1}+ & \dots & +c_n p_{rn} & = 0 \end{array} \tag{3.32}$$

where each equation corresponds to a coefficient of a monomial which must vanish. Therefore we are interested in finding the set

$$M := \{\mathbf{c} \in \mathbb{K}[x]^n \mid \mathbf{c} \text{ is a solution of } (3.32)\}.$$

Since $p_{ij} \in \mathbb{K}[x]$, $M$ is a submodule of $\mathbb{K}[x]^n$ over the Euclidean domain $\mathbb{K}[x]$. By Lemma 3.4.14 this module $M$ over $\mathbb{K}[x]^n$ is finitely generated and by Remark 3.4.5 a basis can be computed. Therefore we find also a basis for $\mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{g})$ and consequently also for $\mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{f})$. $\square$

**Corollary 3.4.6.** *Let $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{K}(t_1, \dots, t_e)$ be a $\Pi\Sigma$-field over $\mathbb{K}$. Let $m \geq 0$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial. Then $\mathcal{V}_m$ is a finitely generated module over $\mathbb{K}[x]$.*

*Proof.* This is a direct consequence of Theorem 3.4.7 and Lemma 3.4.19. $\square$

In the following we will achieve a similar result for $\mathcal{V}_{m,r+1}$, namely how we can compute a basis for $\mathcal{V}_{m,r+1}$. Having this result in hands we will be able to find a $\delta \geq \gamma$ such that for all $r \geq \delta$ we have

$$\mathcal{V}_m[m + r + 1] = \mathcal{V}_{m,r+1}.$$

Then we have in particular

$$\mathcal{W}_m^{(r)}[m + r + 1] = \mathcal{W}_{m,r+1}$$

- what will be shown later.

**Lemma 3.4.20.** *Let $\mathbf{0} \neq \mathbf{b} \in \mathbb{F}^n$ and $\mathbf{w} = (w_1, \dots, w_k) \in \mathbb{F}[x]^k$ with*

$$w_i = \sum_{j=0}^{d} w_{ij}\, x^j$$

*for $1 \leq i \leq k$. Let $g \in \mathbb{F}$, $\mathbf{c} \in \mathbb{K}^k$ and $d \in \mathbb{N}_0$ with*

$$\sigma_{\mathbf{b}} g = \mathbf{c}\, \mathbf{w}[d]. \tag{3.33}$$

*Define*

$$\begin{array}{llll} \mathbf{v} := ( & w_{10}, & \dots, & w_{k0}, \\ & w_{11}, & \dots, & w_{k1}, \\ & \vdots & & \vdots \\ & w_{1d}, & \dots, & w_{kd}) \in \mathbb{F}^{\mu} \end{array}$$

*with $\mu := k\,(d + 1)$ and assume*

$$\mathrm{V}(\mathbf{b}, \mathbf{v}, \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{A} \wedge \mathbf{p}$$

*with $\mathbf{A} \in \mathbb{K}^{l \times \mu}$ and $\mathbf{p} \in \mathbb{F}^l$ for some $l \geq 1$ . Then there exists an $\mathbf{m} \in \mathbb{K}^l$ such that*

$$\mathbf{m}\, (\mathbf{A} \cdot \mathbf{v}) = \mathbf{c}\, \mathbf{w}[d].$$

*Proof.* Let $\mathrm{Id}_k$ be the identity matrix with length $k$ and define

$$\mathbf{M} := (\mathrm{Id}_k | x\,\mathrm{Id}_k | \dots | x^{d-1}\,\mathrm{Id}_k | x^d\,\mathrm{Id}_k).$$

We have

$$\mathbf{w} \overset{(3.33)}{=} \mathbf{M} \cdot \mathbf{v}$$

and thus

$$\sigma_{\mathbf{b}} g = \mathbf{c}\,\mathbf{w}[d] = \mathbf{c}\,(\mathbf{M}[d] \cdot \mathbf{v}) = (\mathbf{c} \cdot \mathbf{M}[d])\,\mathbf{v}.$$

Since $\mathbf{A} \wedge \mathbf{p}$ is a basis matrix for $\mathrm{V}(\mathbf{b}, \mathbf{v}, \mathbb{F})$, it follows by

$$\sigma_{\mathbf{b}} g = (\mathbf{c} \cdot \mathbf{M}[d])\,\mathbf{v}$$

that there is an $\mathbf{m} \in \mathbb{K}^l$ such that

$$\mathbf{m}\,\mathbf{p} = g.$$

Take such an $\mathbf{m}$. Together with

$$\sigma_{\mathbf{b}} \mathbf{p} = \mathbf{A} \cdot \mathbf{v}$$

it follows that

$$\sigma_{\mathbf{b}} g = \sigma_{\mathbf{b}}(\mathbf{m}\,\mathbf{p}) = \mathbf{m}\,\sigma_{\mathbf{b}}\mathbf{p} = \mathbf{m}\,(\mathbf{A} \cdot \mathbf{v})$$

and therefore

$$\mathbf{m}\,(\mathbf{A} \cdot \mathbf{v}) = \sigma_{\mathbf{b}} g \overset{(3.33)}{=} \mathbf{c}\,\mathbf{w}[d]$$

which proves the lemma. □

The following theorem delivers the corresponding result for $\mathcal{V}_{m,r+1}$ as in Theorem 3.4.7 for $\mathcal{V}_m$.

**Theorem 3.4.8.** *Let $m \geq 0$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial and define $\mathbf{v} \in \mathbb{F}^\mu$ and $\mathbf{w} \in \mathbb{F}[x]^k$ as stated in (3.28) for that $\mathbf{g}$. Let $\mathbf{0} \neq \mathbf{b} := [\mathbf{a}]^t_{\|\mathbf{a}\|} \in \mathbb{F}^n$ and let*

$$\mathrm{V}(\mathbf{b}, \mathbf{v}, \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{B} \wedge \mathbf{q}$$

*with $\mathbf{B} \in \mathbb{K}^{s \times \mu}$ and $\mathbf{q} \in \mathbb{F}^s$ for some $s \geq 1$. Furthermore let $r \geq 0$ and*

$$\mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B} \cdot \mathbf{v}) \wedge -\mathbf{w}[\mathrm{m} + \mathrm{r} + 1]) \xleftrightarrow{\text{basis}} \mathbf{C} \wedge \mathbf{D}$$

*with $\mathbf{C} \in \mathbb{K}[x]^{l \times s}$ and $\mathbf{D} \in \mathbb{K}[x]^{l \times k}$ for some $l \geq 1$ such that*

$$\mathbf{C} \cdot (\mathbf{B} \cdot \mathbf{v}) = \mathbf{D} \cdot \mathbf{w}[m + r + 1]. \tag{3.34}$$

*Then*

$$\mathcal{V}_{m,r+1} \xleftrightarrow{\text{span}} \mathbf{D} \wedge (\mathbf{C} \cdot \mathbf{q}).$$

*Proof.* By Corollary 3.4.5 $\mathcal{V}_{m,r+1}$ is a finite dimensional vector space over $\mathbb{K}$, this means we can take $\mathbf{A} \in \mathbb{K}^{l \times k}$ and $\mathbf{p} \in \mathbb{F}^l$ for some $l \geq 1$ such that

$$\mathcal{V}_{m,r+1} \xleftrightarrow{\text{basis}} \mathbf{A} \wedge \mathbf{p}.$$

Since

$$\mathcal{V}_{m,r+1} = \left\{ \mathbf{c} \wedge h \in \mathbb{K}^k \times \mathbb{F} \,|\, \sigma_{\mathbf{b}} h = \mathbf{c}\,\mathbf{w}[m + r + 1] \right\} \tag{3.35}$$

by Theorem 3.4.4, we have

$$\sigma_{\mathbf{b}}\mathbf{p} = \mathbf{A} \cdot \mathbf{w}$$

and thus there exists an $\mathbf{H} \in \mathbb{K}[x]^{s \times l}$ by Lemma 3.4.20 such that

$$\mathbf{H} \cdot (\mathbf{B} \cdot \mathbf{f}) = \mathbf{A} \cdot \mathbf{w}[m + r + 1]. \tag{3.36}$$

Thus by (3.34), (3.36) there is a vector space $\tilde{V}$ with

$$\begin{array}{ccc} \operatorname{Ann}_{\mathbb{K}}((\mathbf{B} \cdot \mathbf{f} \wedge -\mathbf{w})) & \supseteq & \tilde{V} \\ \updownarrow \text{basis} & & \updownarrow \text{span} \\ \mathbf{C} \wedge \mathbf{D} & & \mathbf{H} \wedge \mathbf{A}. \end{array} \tag{3.37}$$

As $\mathbf{B} \wedge \mathbf{q}$ is a basis matrix for $\mathrm{V}(\mathbf{b}, \mathbf{v}, \mathbb{F})$, we have

$$\mathbf{B} \cdot \mathbf{v} = \sigma_{\mathbf{b}}\mathbf{q}$$

and consequently

$$\mathbf{D} \cdot \mathbf{w}[m + r + 1] \overset{(3.34)}{=} \mathbf{C} \cdot (\mathbf{B} \cdot \mathbf{f}) = \mathbf{C} \cdot \sigma_{\mathbf{b}}\mathbf{q} = \sigma_{\mathbf{b}}(\mathbf{C} \cdot \mathbf{q}),$$

$$\mathbf{A} \cdot \mathbf{w}[m + r + 1] \overset{(3.36)}{=} \mathbf{H} \cdot (\mathbf{B} \cdot \mathbf{f}) = \mathbf{H} \cdot \sigma_{\mathbf{b}}\mathbf{q} = \sigma_{\mathbf{b}}(\mathbf{H} \cdot \mathbf{q}).$$

Together with (3.37) it follows that there is a vector space $\tilde{\tilde{V}}$ with

$$\begin{array}{ccc} \tilde{\tilde{V}} & \supseteq & \mathcal{V}_{m,r+1} \\ \updownarrow \text{span} & & \updownarrow \text{basis} \\ \mathbf{D} \wedge (\mathbf{C} \cdot \mathbf{q}) & & \mathbf{A} \wedge (\mathbf{H} \cdot \mathbf{q}). \end{array}$$

But by (3.35) it follows immediately that

$$\tilde{V} \subseteq \mathcal{V}_{m,r+1}$$

and thus $\mathcal{V}_{m,r+1} = \tilde{\tilde{V}}$. Consequently

$$\mathcal{V}_{m,r+1} \xleftrightarrow{\text{span}} \mathbf{D} \wedge (\mathbf{C} \cdot \mathbf{q}).$$

□

Finally, the following theorem gives us a criterium to determine a $\delta \geq 0$ such that for all $r \geq \delta$ we have

$$\mathcal{V}_m[m + r + 1] = \mathcal{V}_{m,r+1}$$

and

$$\mathcal{W}_m^{(r)}[m + r + 1] = \mathcal{W}_{m,r+1}.$$

**Theorem 3.4.9.** *Let $m \geq 0$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ be $m$-polynomial and define $\mathbf{v} \in \mathbb{F}^\mu$ and $\mathbf{w} \in \mathbb{F}[x]^k$ as stated in (3.28) for that $\mathbf{g}$. Let $\mathbf{0} \neq \mathbf{b} := [\mathbf{a}]^t_{\|\mathbf{a}\|} \in \mathbb{F}^n$ and let*

$$\mathrm{V}(\mathbf{b}, \mathbf{v}, \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{B} \wedge \mathbf{q}$$

*with* $\mathbf{B} \in \mathbb{K}^{s\times\mu}$ *and* $\mathbf{q} \in \mathbb{F}^s$ *for some* $s \geq 1$ *. Assume that* $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{f})\wedge-\mathbf{w})$ *is a finitely generated module over* $\mathbb{K}[x]$ *and let* $\delta \in \mathbb{N}_0$ *such that for all* $d \geq \delta$ *we have*

$$\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w})[d] = \mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}[\mathrm{d}])$$

*with*

$$\dim \mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}) = \dim \mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}[\mathrm{d}]).$$

*Then for all* $r \geq \delta$ *we have*

$$\mathcal{V}_m[m+r+1] = \mathcal{V}_{m,r+1} \quad \textit{and} \quad \mathcal{W}_m^{(r)}[m+r+1] = \mathcal{W}_{m,r+1}.$$

*Proof.* Assume that $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{f})\wedge-\mathbf{w})$ is a finitely generated module over $\mathbb{K}[x]$. This means there exists a matrix $\mathbf{C}\wedge\mathbf{D}$ with

$$\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}) \xleftrightarrow{\text{basis}} \mathbf{C}\wedge\mathbf{D}$$

such that $\mathbf{C}\cdot(\mathbf{B}\cdot\mathbf{v}) = \mathbf{D}\cdot\mathbf{w}$. Let $r \geq \delta$ be arbitrary but fixed. Since

$$\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w})[m+d+1] = \mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{f})\wedge-\mathbf{w}[\mathrm{m}+\mathrm{d}+1])$$

by assumption, we have

$$\mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}[\mathrm{m}+\mathrm{r}+1]) \xleftrightarrow{\text{span}} \mathbf{C}[\mathbf{m}+\mathbf{r}+\mathbf{1}]\wedge\mathbf{D}[\mathbf{m}+\mathbf{r}+\mathbf{1}].$$

By $\dim \mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}) = \dim \mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}[\mathrm{d}])$ it follows in particular that $\mathbf{C}[\mathbf{m}+\mathbf{r}+\mathbf{1}]\wedge\mathbf{D}[\mathbf{m}+\mathbf{r}+\mathbf{1}]$ is a basis matrix for $\mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}[\mathrm{m}+\mathrm{r}+1])$. Applying Theorems 3.4.7 and 3.4.8 it follows that

$$\mathcal{V}_m \xleftrightarrow{\text{span}} \mathbf{D}\wedge(\mathbf{C}\cdot\mathbf{q}),$$

$$\mathcal{V}_{m,r+1} \xleftrightarrow{\text{span}} \mathbf{D}[\mathbf{m}+\mathbf{r}+\mathbf{1}]\wedge(\mathbf{C}[\mathbf{m}+\mathbf{r}+\mathbf{1}]\cdot\mathbf{q}).$$

But this means that

$$\mathcal{V}_m[m+r+1] = \mathcal{V}_{m,r+1}.$$

Looking at Definition 3.4.6 and Lemma 3.4.9 we have

$$\mathcal{W}_{m,r+1} = \{\mathbf{c}\,\mathbf{g}(r+1) + t^r\,h \,|\, \mathbf{c}\wedge h \in \mathcal{V}_{m,r+1}\},$$
$$\mathcal{W}_m^{(r)}[m+r+1] = \{\mathbf{c}\,\mathbf{g}(r+1) + t^r\,h \,|\, \mathbf{c}\wedge h \in \mathcal{V}_m[m+r+1]\}$$

and consequently

$$\mathcal{W}_m^{(r)}[m+r+1] = \mathcal{W}_{m,r+1}.$$

□

The following lemma and proposition provides an algorithm to find such a $\delta$ if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$.

**Lemma 3.4.21.** *Let* $\mathbf{A} \in \mathbb{K}[x]^{m\times n}$ *and consider the module* $\mathbb{M} := \{\mathbf{f} \in \mathbb{K}[x]^n \,|\, \mathbf{A}\cdot\mathbf{f} = \mathbf{0}\}$ *over* $\mathbb{K}[x]$ *and the vector space* $\mathbb{V}_d := \{\mathbf{f} \in \mathbb{K}^n \,|\, \mathbf{A}[d]\cdot\mathbf{f} = \mathbf{0}\}$ *over* $\mathbb{K}$*. Then there exists a* $\delta \geq 0$ *such that for all* $d \in \mathbb{N}_0$ *with* $d \geq \delta$ *we have*

$$\mathbb{M}[d] = \mathbb{V}_d$$

*and* $\dim \mathbb{M} = \dim \mathbb{V}_d$*. If one can compute all roots in* $\mathbb{N}_0$ *of a polynomial in* $\mathbb{K}[x]$ *then* $\delta$ *can be computed.*

*Proof.* Since $\mathbb{K}[x]$ is an Euclidean domain, by Remark 3.4.5 one can transform the matrix $\mathbf{A}$ by row and column operations to a reduced matrix

$$\mathbf{D} := \begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & \mathbf{0} & & \\ & & \ddots & & & & \\ & & & d_r & & & \\ & \mathbf{0} & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

with $d_i \in \mathbb{F}^*$ and $d_i \mid d_{i+1}$ for $1 \leq i < r$. Then it follows that the submodule

$$\mathbb{M} := \{\mathbf{f} \in \mathbb{K}[x]^n \,|\, \mathbf{A} \cdot \mathbf{f} = \mathbf{0}\}$$

of $\mathbb{K}[x]^n$ over $\mathbb{K}[x]$ has a basis with the uniquely determined dimension $n - r$. Furthermore by this row and column operations we obtain $\mathbf{P} \in \mathrm{GL}_m(\mathbb{K}[x])$ and $\mathbf{Q} \in \mathrm{GL}_n(\mathbb{K}[x])$ such that

$$\mathbf{D} = \mathbf{P}\,\mathbf{A}\,\mathbf{Q}$$

and it follows that the last $n - r$ columns $\mathbf{q_{r+1}}, \dots, \mathbf{q_n} \in \mathbb{K}[x]^n$ of $\mathbf{Q}$ form a basis of $\mathbb{M}$.

During this triangularization let $L \subseteq \mathbb{K}[x]^*$ be the finite set of $k$'s used in rule[17] (2) for a row or column operation. Let $\gamma \in \mathbb{N}_0$ be the greatest root in $\mathbb{N}_0$ of all the polynomials in $L$. If there does not exist such a root then take $\gamma := -1$. Furthermore take the greatest root $\delta' \in \mathbb{N}_0$ of $d_r$ and define $\delta := \max(\delta', \gamma)$. If there does not exist such a $\delta' \in \mathbb{N}_0$, take $\delta := \gamma$. Clearly, if we can compute the roots of polynomials in $\mathbb{K}$, we can compute this $\delta$.

Now we will prove that $\delta + 1$ fulfills the above property. Let $d > \delta$. Take the matrix $\mathbf{B} := \mathbf{A}[d]$ with entries in $\mathbb{K}$ and transform it to the matrix $\mathbf{E}$ applying the same steps as for $\mathbf{A}$ but taking the corresponding $k[d]$ instead of $k \in \mathbb{K}[x]$ in a rule (2) or (3) of a column or row operation. Since we did exactly the same operations, it follows that

$$\mathbf{E} = \mathbf{D}[d], \tag{3.38}$$

i.e.

$$\mathbf{E} := \begin{pmatrix} d_1[d] & & & & & & \\ & d_2[d] & & & \mathbf{0} & & \\ & & \ddots & & & & \\ & & & d_r[d] & & & \\ & \mathbf{0} & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0. \end{pmatrix}$$

Since we can guarantee by definition of $\delta$ that $k[d] \in \mathbb{K}^*$ in each step (2), the row and column operations are applied correctly. Furthermore, since $d_r[d] \neq 0$ and $d_i \mid d_{i+1}$, it follows that

$$d_i[d] \neq 0$$

for all $1 \leq i \leq r$. Applying Remark 3.4.5 to $\mathbf{B}$ tells us that

$$\mathbb{V} := \{\mathbf{f} \in \mathbb{K}^n \,|\, \mathbf{B} \cdot \mathbf{f} = \mathbf{0}\}$$

[17] See Remark 3.4.5.

has dimension $n-r$. Furthermore by this correctly applied column and row operations we obtain $\mathbf{P}' \in \mathrm{GL}_m(\mathbb{K})$ and $\mathbf{Q}' \in \mathrm{GL}_n(\mathbb{K})$ such that

$$\mathbf{A}[d] = \mathbf{B} = \mathbf{P}'\,\mathbf{D}[d]\,\mathbf{Q}'$$

where $\mathbf{P}' = \mathbf{P}[d]$ and $\mathbf{Q}' = \mathbf{Q}[d]$. By Remark 3.4.5 it follows that the last $n-r$ columns $\mathbf{q_{r+1}}[d], \dots, \mathbf{q_n}[d] \in \mathbb{K}^n$ of $\mathbf{Q}' = \mathbf{Q}[d]$ are a basis of $\mathbb{V}$. Consequently it follows immediately that

$$\mathbb{M}[d] = \mathbb{V}$$

and hence the lemma is proven. □

**Proposition 3.4.6.** *Let $(\mathbb{F},\sigma)$ with $\mathbb{F} := \mathbb{K}(t_1,\dots,t_e)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and $\mathbf{f} \in \mathbb{F}[x]^n$. Then there exists a $\delta \geq 0$ such that for all $d \geq \delta$ we have*

$$(\mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{f}))[d] = \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}[\mathrm{d}])$$

*and* $\dim \mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{f}) = \dim \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f}[\mathrm{d}])$. *If one can compute all roots in $\mathbb{N}_0$ of a polynomial in $\mathbb{K}[x]$ then $\delta$ can be computed.*

*Proof.* By the same arguments as in the proof of Lemma 3.4.19 we find two linear equation systems

$$\begin{array}{llll} c_1\,p_{11}+ & \dots & +c_n\,p_{1n} & =0 \\ & \vdots & & \\ c_r\,p_{r1}+ & \dots & +c_n\,p_{rn} & =0, \end{array} \qquad \begin{array}{llll} c_1\,p_{11}[d]+ & \dots & +c_n\,p_{1n}[d] & =0 \\ & \vdots & & \\ c_r\,p_{r1}[d]+ & \dots & +c_n\,p_{rn}[d] & =0 \end{array} \tag{3.39}$$

such that

$$\begin{aligned} \mathrm{Ann}_{\mathbb{K}[x]}(\mathbf{f})) &= \{\mathbf{c} \in \mathbb{K}[x]^n \mid \mathbf{c} \text{ is a solution of the left system in (3.39)}\}, \\ \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f})) &= \{\mathbf{c} \in \mathbb{K}^n \mid \mathbf{c} \text{ is a solution of the right system in (3.39)}\}. \end{aligned}$$

Thus the theorem follows by Lemma 3.4.21. □

**Assumption 3.4.1.** In the following we assume that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$. Additionally, we should assume that one can compute for any $\mathbf{a} \in \mathbb{F}^m$ and $\mathbf{f} \in \mathbb{F}^n$ the solution space $\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{F})$. Since there are many unsolved problems for $\Pi\Sigma$-fields to compute this solution space, we restrict ourself to the assumption that for the concrete vectors $\mathbf{b}$ and $\mathbf{v}$ in Algorithm 3.4.1 we are able to solve the solution space $\mathrm{V}(\mathbf{b},\mathbf{v},\mathbb{F})$. Furthermore we assume that one can compute all roots in $\mathbb{N}_0$ of a polynomial in $\mathbb{K}[x]$.

**Algorithm 3.4.1.**

**$(\delta,\mathbf{h})$=FindABasis$(m,\gamma,\mathbf{g})$**

Input: $f_m$ is $\gamma$-computable by $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$, see Assumption 3.4.1.

Output: A pair $(\delta,\mathbf{h})$ such that $f_{m+1}$ is $\delta$-computable by $\mathbf{h}$.

(1) Let $\mathbf{w} = (w_1,\dots,w_k)$ be defined by (3.24) in the proof of Theorem 3.4.3, let $\mathbf{b} := [\mathbf{a}]^t_{\|\mathbf{a}\|}$ and define

$$\begin{array}{rlll} \mathbf{v} := ( & [w_1]^x_0\,, & \dots, & [w_k]^x_0\,, \\ & [w_1]^x_1\,, & \dots, & [w_k]^x_1\,, \\ & \vdots & & \vdots \\ & [w_1]^x_d\,, & \dots, & [w_k]^x_d) \in \mathbb{F}^\mu \end{array}$$

with $\mu := k(d+1)$.

(2) Let

$$\mathrm{V}(\mathbf{b},\mathbf{v},\mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{B}\wedge\mathbf{q}$$

with $\mathbf{B} \in \mathbb{K}^{s\times n}$ and $\mathbf{q} \in \mathbb{F}^s$ for some $s \geq 1$ .

(3) Let

$$\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}) \xleftrightarrow{\text{basis}} \mathbf{C}\wedge\mathbf{D}$$

with $\mathbf{C} \in \mathbb{K}[x]^{l\times s}$ and $\mathbf{D} \in \mathbb{K}[x]^{l\times k}$ for some $l \geq 1$ such that $\mathbf{C}\cdot(\mathbf{B}\cdot\mathbf{f}) = \mathbf{D}\cdot\mathbf{w}$.

(4) Let $\delta \in \mathbb{N}_0$ with $\delta \geq \gamma$ be such that for all $d \geq \delta$ we have

$$\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w})[d] = \mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}[\mathrm{d}])$$

with

$$\dim\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}) = \dim\mathrm{Nullspace}_{\mathbb{K}}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w}[\mathrm{d}]).$$

(5) Define $\mathbf{h} : \mathbb{N}_0 \to \mathbb{F}[t]^l$ by

$$\mathbf{h}(r) = \mathbf{D}\cdot\mathbf{g}^{(m+r+1)}[r+1] + t^r\,\mathbf{C}\cdot\mathbf{q}.$$

(6) RETURN$(\delta,\mathbf{h})$

**Corollary 3.4.7.** *Let $m \geq 0$ and $f_m$ be $\gamma$-computable by $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ and assume that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field. Then there exists a $\delta \geq \gamma$ and a function*

$$\mathbf{h} : \mathbb{N}_0 \to \mathbb{F}[t]^l$$

*such that $f_{m+1}$ is $\delta$-computable by $\mathbf{h}$. This function $\mathbf{h}$ and $\delta$ can be computed by Algorithm 3.4.1 under the Assumption 3.4.1.*

*Proof.* There exists a basis matrix for $\mathrm{V}(\mathbf{b},\mathbf{v},\mathbb{F})$. Take such a basis matrix. By Lemma 3.4.19 we are able to compute a basis matrix for $\mathrm{Ann}_{\mathbb{K}[x]}((\mathbf{B}\cdot\mathbf{v})\wedge-\mathbf{w})$ in Step (3). By Theorem 3.4.7 it follows that

$$\mathcal{V}_m \xleftrightarrow{\text{span}} \mathbf{D}\wedge(\mathbf{C}\cdot\mathbf{q}).$$

By Proposition 3.4.6 there exists a $\delta$ as stated in step (4). Take such a $\delta$. By Theorem 3.4.9 we obtain

$$\forall r \geq \delta:\ \mathcal{W}_m^{(r)}[m+r+1] = \mathcal{W}_{m,r+1}.$$

Thus by Theorem 3.4.5 it follows that $f_{m+1}$ is $\delta$-computable by $\mathbf{h}$.

Under Assumption 3.4.1 we are also able to compute a basis matrix $\mathbf{B}\wedge\mathbf{q}$ for $\mathrm{V}(\mathbf{b},\mathbf{v},\mathbb{F})$ in Step (2) and can compute a $\delta$ by Proposition 3.4.6. Therefore we can compute $\mathbf{h}$ by Algorithm 3.4.1 under Assumption 3.4.1. □

**Corollary 3.4.8.** *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$. There exists a sequence*

$$0 = \delta_0 \leq \delta_1 \leq \cdots \leq \delta_i \leq \delta_{i+1} \leq \ldots$$

*and a sequence of functions $\mathbf{g_i} : \mathbb{N}_0 \to \mathbb{F}[t]^{k_i}$ such that $f_i$ is $\delta_i$-computable by $\mathbf{g_i}$.*

*Proof.* By Proposition 3.4.4 there exists a function $\mathbf{g_0}$ such that $f_0$ is 0-computable by $\mathbf{g}_0$. Using Corollary 3.4.7 it follows by induction that there exists such a sequence of functions $\mathbf{g_i}$ and $\delta_i$ such that $f_i$ is $\delta_i$ computable by $\mathbf{g_i}$. □

**Assumption 3.4.2.** Similar to Assumption 3.4.1 we assume the that for the concrete vector $\mathbf{b}$ in Algorithm 3.4.2 we are able to solve the solution space $\mathrm{V}(\mathbf{b},(0),\mathbb{F})$.

**Algorithm 3.4.2.**

**$(\delta, \mathbf{h})$=FindABasis(a)**

Input: $\mathbf{a} \in \mathbb{F}[t]^m$, see Assumptions 3.4.1 and 3.4.2.

Output: a triple $(\delta, \mathbf{h}, m)$ such that $f_m$ is $\delta$-computable by $\mathbf{h}$.

(1) Let
$$\mathrm{V}(\mathbf{b}, (0), \mathbb{F}) \xleftrightarrow{\text{basis}} \mathbf{C} \wedge \mathbf{g}$$
where $\mathbf{b} := [\mathbf{a}]^t_{\|a\|}$ and define $\mathbf{g}_0^{(r)} := t^r\,\mathbf{g}$.

(2) Let $\delta_0 := 0$.

(3) Let i:=0

(4) While($\|\mathbf{g}_i^{(0)}\| = i$ and $i \leq$ `LoopLimitForSumBound`)

(5) i:=i+1

(6) $(\delta_i, \mathbf{g_i})$=FindABasis$(i, \delta_{i-1}, \mathbf{g_{i-1}})$

(7) RETURN$(\delta_{i-1}, \mathbf{g_{i-1}}, i-1)$

**Corollary 3.4.9.** *Let $(\mathbf{h}, \delta, m)$ be the output of Algorithm 3.4.2 where Assumptions 3.4.1 and 3.4.2 hold. Then $f_m$ is $\delta$-computable by $\mathbf{h}$. Furthermore, if $m <$ `LoopLimitForSumBound` then for all $r \geq \delta$ and all $j \geq 0$ we have*
$$f_m(r) = f_{m+j}(r).$$

*Proof.* By Corollary 3.4.8 and its proof it follows immediately that $f_m$ is $\delta$-computable by $\mathbf{h}$. If additionally $m <$ `LoopLimitForSumBound` is satisfied then in the $m+1$-th iteration step for all $r \geq \delta_{m+1}$ we have
$$\|\mathbf{g_{m+1}}^{(0)}\| = m$$
and thus by Theorem 3.4.6 for all $r \geq \delta$ and all $j \geq 0$ it follows that
$$f_m(r) = f_{m+j}(r).$$
□

**Remark 3.4.6.** In [Sin91] M. Singer deals with the analogous problem to solve linear differential equations in Liouvillian differential field extensions $(\mathbb{F}(t), D)$ of the type $\frac{D(t)}{t} \in \mathbb{F}$. In Lemma 3.8 he sketches an algorithm to find a degree bound for a polynomial solution in $t$. Similarly, he computes incrementally the solution space until he finds a bound of the polynomial solutions. Whereas in my approach I failed to find a termination proof - see the stop condition '$i \leq$ `LoopLimitForSumBound`' in Algorithm 3.4.2 - M. Singer succeeded in doing so in his context. As M. Bronstein pointed out in some fruitful discussions [BS00], there might be hope to analyse Singer's approach and termination proof further to find also a termination condition for my approach. ◊

**Corollary 3.4.10.** *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^n$ and $\mathbf{f} \in \mathbb{F}[t]^\lambda$. Let $(\mathbf{h}, \delta, m)$ be the output of Algorithm 3.4.2 under the Assumptions 3.4.1 and 3.4.2 and $m <$ `LoopLimitForSumBound`. Then*
$$m + \max(\|\mathbf{f}\| - \|\mathbf{a}\| + 1, \delta)$$
*is a bound for* $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$.

*Proof.* By Corollary 3.4.9, it follows for all $r \geq \delta$ and all $j \geq 0$ that

$$f_m(r) = f_{m+j}(r)$$

and thus for all $r \geq \delta$ we have

$$\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]).$$

Thus applying Theorem 3.4.1, $m + \max(\|\mathbf{f}\| - \|\mathbf{a}\| + 1, \delta)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. □

**Implementation Note 3.4.1.** Up to an important variation described in the next section I have implemented Algorithm 3.4.2 in my Mathematica package. By setting the option

$$\texttt{LoopLimitForSumBound} \to \texttt{no}$$

one can define the maximal loops in Algorithm 3.4.2. If the Algorithm terminates before running through the main loop `no`-times then by Corollary 3.4.10 a correct polynomial degree bound is determined.
Otherwise, running through the loop `no`-times, the polynomial degree bound method stops and the found "polynomial degree bound" is used. In this case the solutions for the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$ are still correct, but it might be that some of the solutions are missing; this means we find only a basis for a subspace of $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$. Therefore a warning message will be printed out that suggests the user to increase the value `no` in case the computed solutions are not sufficient.
Furthermore, as suggested in Remark 3.4.1 I used the computed truncated solution space $f_m(r) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_m)$ to compute the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. Therefore I shortened the reduction process sketched in Section 3.2.6.

### 3.4.8 A Speed up

As already mentioned in Implementation Note 3.4.1, I have not implemented Algorithm 3.4.2 but a clever variation of it.

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$. As formulated in Corollary 3.4.8, in the previous sections the main goal was to construct a sequence

$$0 = \delta_0 \leq \delta_1 \leq \dots \leq \delta_m \leq \delta_{m+1} \leq \dots$$

and a sequence of functions $\mathbf{g_m} : \mathbb{N}_0 \to \mathbb{F}[t]^{k_m}$ such that $f_m$ is $\delta_m$-computable by $\mathbf{g_m}$.

Looking closer at $f_m(r)$ we obtain by Lemma 3.4.3 the following inclusion of vector spaces

$$\{0\} = f_{-1}(r) \subseteq f_0(r) \subseteq f_1(r) \subseteq \dots \subseteq f_m(r) \subseteq f_{m+1}(r) \subseteq \dots$$

for all $r \geq 0$. Thus we can define the complement vector space $f_{m+1}^{new}(r)$ in $f_{m+1}(r)$, i.e.

$$f_{m+1}(r) = f_m(r) \oplus f_{m+1}^{new}(r).$$

Then we have

$$f_{m+1}(r) = \overbrace{\overbrace{\overbrace{\overbrace{\{0\}}^{f_{-1}(r)} \oplus f_0^{new}(r)}^{f_0(r)} \oplus f_1^{new}(r)}^{f_1(r)} \oplus \dots \oplus f_m^{new}(r)}^{f_m(r)} \oplus f_{m+1}(r)^{new}.$$

Actually, it is already sufficient to compute a function $\mathbf{g_m}$ and to find a $\delta_m$ such that $f_m^{(new)}$ is $\delta_m$-computable by $\mathbf{g_m}$. In the following we indicate how we can compute a sequence of $m$-polynomial maps $\mathbf{g_m}$ such that $\mathbf{g_m}(r)$ generates a vector space $\mathbb{V}_m$ over $\mathbb{K}$ with

$$f_m^{new}(r) \subseteq \mathbb{V}_m \subseteq f_m(r).$$

In particular we try to choose the dimension of the vector space $\mathbb{V}_m$ as small as possible, i.e. we try to minimalize the length of the vector $\mathbf{g_m}(r)$.

In the following we indicate how we can achieve this. For the base case of this computation process will still use Theorem 3.4.4 and construct a map $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ such that $f_0(r) = f_0^{new}(r)$ is 0-computable by $\mathbf{g}$. In the following let $m \geq 0$ and assume that we have computed already a $m$-polynomial map

$$\mathbf{g}' : \begin{cases} \mathbb{N}_0 & \to & \mathbb{F}[t]^{k+l} \\ d & \mapsto & \mathbf{g}'^{(d)}[m+d] \end{cases}$$

with

$$\mathbf{g}'^{(d)} = \left(g_1^{(d)}, \dots, g_k^{(d)}, p_1^{(d)}, \dots, p_l^{(d)}\right) \in \mathbb{F}[t, x]^{k+l} \tag{3.40}$$

such that $f_m$ is $\gamma$-computable by $\mathbf{g}'$. Assume further that

$$f_m^{new} \subseteq \left\{x_1\, g_1^{(d)}[m+r], \dots, x_k\, g_k^{(d)}[m+r] \,|\, x_i \in \mathbb{K}\right\} (=: \mathbb{V}_m)$$

and for $0 \leq i \leq l$ and all $r \geq \gamma$ that $\|p_i^{(r)}\| < r + m$ and

$$\exists w \in \mathbb{F} : \left[\sigma_{\mathbf{a}} p_i^{(r)}[m+r+1] + t^r\, w\right]_{\|a\|+r} = 0. \tag{3.41}$$

Now consider the $m$-polynomial map

$$\mathbf{g}:\begin{cases}\mathbb{N}_0 & \to & \mathbb{F}[t]^k\\ d & \mapsto & \mathbf{g}^{(d)}[m+d]\end{cases}$$

where $\mathbf{g}^{(d)} = \left(g_1^{(d)}, \dots, g_k^{(d)}\right) \in \mathbb{F}[t,x]^k$ and define $\mathcal{V}_{m,r+1}$, $\mathcal{V}_m$, $\mathcal{W}_{m,r+1}$ and $\mathcal{W}_m$ as in Section 3.4.3 with this $m$-polynomial map $\mathbf{g}$.

**Theorem 3.4.10.** *Let $m \geq 0$ and let $\mathcal{W}_{m,r+1}$ be defined by the $m$-polynomial map $\mathbf{g}$ described above. Then for all $r \geq \gamma$ it follows that*

$$f_{m+1}^{new}(r) \subseteq \mathcal{W}_{m,r+1} + f_m(r).$$

*Proof.* Let $r \geq \gamma$ and assume $\mathbf{g}' : \mathbb{N}_0 \to \mathbb{F}[t]^{k+l}$, $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ as above. Define the $m$-polynomial map

$$\mathbf{p}:\begin{cases}\mathbb{N}_0 & \to & \mathbb{F}[t]^l\\ d & \mapsto & \mathbf{p}^{(d)}[m+d]\end{cases}$$

where $\mathbf{p}^{(d)} = \left(p_1^{(d)}, \dots, p_l^{(d)}\right) \in \mathbb{F}[t,x]^k$. Then

$$w \in f_{m+1}^{new}(r)$$
$$\Downarrow \text{ Cor. 3.4.1}$$
$$w \in f_m(r+1) + t^r\,\mathbb{F} :\ [\sigma_{\mathbf{a}} w]_{\|\mathbf{a}\|+r} = 0 \;\&\; \|w\| = m+r+1$$
$$\Updownarrow$$
$$\exists\, \mathbf{c} \in \mathbb{K}^{k+l}\, \exists\, h \in \mathbb{F} : w = \mathbf{c}\,\mathbf{g}'(r+1) + t^r\, h \;\&\; [\sigma_{\mathbf{a}} w]_{\|\mathbf{a}\|+r} = 0 \;\&\; \|w\| = m+r+1$$
$$\Updownarrow$$
$$\begin{aligned}\exists\, \mathbf{c_1} \in \mathbb{K}^k, \mathbf{c_2} \in \mathbb{K}^l,\ h \in \mathbb{F} : w = \mathbf{c_1}\,\mathbf{g}(r+1) + \mathbf{c_2}\,\mathbf{p}(r+1) + t^r\, h\\ \&\; [\sigma_{\mathbf{a}} w]_{\|\mathbf{a}\|+r} = 0 \;\&\; \|w\| = m+r+1\end{aligned} \tag{3.42}$$

Let $\mathbf{c_1} \in \mathbb{K}^k$,$\mathbf{c_2} \in \mathbb{K}^l$ and $h \in \mathbb{F}$ with

$$w = \mathbf{c_1}\,\mathbf{g}(r+1) + \mathbf{c_2}\,\mathbf{p}(r+1) + t^r\, h.$$

Since (3.41), there exists $h_0 \in \mathbb{F}$ such that

$$\Bigg[\sigma_{\mathbf{a}}(\underbrace{\mathbf{c_2}\,\mathbf{p}(r+1) + t^r\, h_0}_{=:w_0})\Bigg]_{\|\mathbf{a}\|+r} = 0.$$

As $\|\mathbf{p}(r+1)\| \leq m+r$, it follows that $\|w_0\| \leq m+r$ and thus $w_0 \in f_m(r)$. Consequently with $h_1 := h - h_0$ we have

$$w = \mathbf{c_1}\,\mathbf{g}(r+1) + t^r\, h_1 + w_0.$$

Finally, it follows by (3.42) that

$$\exists\, \mathbf{c}{\wedge}h_1 \in \mathcal{V}_{m,r+1}, w_0 \in f_m(r) :\ w = \underbrace{\mathbf{c}\,\mathbf{g}(r+1) + t^r\, h_1}_{\in \mathcal{W}_{m,r+1}} + \underbrace{w_0}_{\in f_m(r)}$$
$$\Updownarrow$$
$$w \in \mathcal{W}_{m,r+1} + f_m(r).$$

□

**Corollary 3.4.11.** *Let $m \geq 0$ and let $\mathcal{W}_{m,r+1}$ be defined by the polynomial map $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ described above. Then*

$$\forall r \geq \gamma : \ \mathcal{W}_m^{(r)}[m+r+1] \subseteq \mathcal{W}_{m,r+1} \supseteq f_{m+1}^{new}(r).$$

*Proof.* This follows by Lemma 3.4.10 and Theorem 3.4.10. □

Please note, that Theorems 3.4.7 and 3.4.8 and Theorem 3.4.9 are true for $m$-polynomial maps and thus for our specific $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$. Therefore, if one applies Algorithm 3.4.1 on the input $(\mathbf{g}, \gamma, m)$ one obtains the output $(\mathbf{h}, \delta)$ such that for all $r \geq \delta$ we have

$$\mathcal{W}_m^{(r)}[m+r+1] = \mathcal{W}_{m,r+1} \supseteq f_{m+1}^{new}(r)$$

and $\mathbf{h}(r)$ generates over $\mathbb{K}$ the vector space $\mathcal{W}_m^{(r)}[m+r+1]$. Thus, given $\mathbf{g}'$ and $\mathbf{h}$, one can construct a map $\mathbf{h}'$ such that $f_{m+1}$ is $\delta$-computable by $\mathbf{h}'$.

Looking closer at Algorithm 3.4.1 in Step (3), one computes

$$\mathcal{V}_m \xleftrightarrow{\text{span}} \mathbf{D} \wedge (\mathbf{C} \cdot \mathbf{q}).$$

Without loss of generality, one can transform (Remark 3.4.5) the generator matrix by row operations in $\mathbb{K}[x]$ such that $\mathbf{D}$ is in row-echelon form. Moreover, all elements above a left most entry have smaller degree in $x$. If a row at position $i$ has just one nonzero entry then we have

$$\exists w \in \mathbb{F} : \ \left[\sigma_{\mathbf{a}} g_i^{(r)}[m+r+1] + t^r \, w\right]_{\|a\|+r} = 0.$$

Consequently we can eliminate all those elements $\mathbf{h}_i^{(r)}$ in $\mathbf{h}^{(r)}$ which satisfy this property and which have $\|\mathbf{h}_i^{(r)}\| \leq m+r$. Finally, we take this refined vector $\mathbf{h}^{(r)}$ and adjoin it with all elements in $\mathbf{g}^{(r)}$ with $\|g_i^{(0)}\| < m+r$.

With this refined $m+1$-polynomial map $\mathbf{h}$ we can repeat this procedure as in Algorithm 3.4.2 to compute step by step maps $\mathbf{g_i}$ and $\delta_i$ such that $f_i$ is $\delta_i$ computable by $\mathbf{g_i}$.

By this strategy we always try to keep the length of the vector $\mathbf{g}^{(r)}$ as small a possible. Consequently also the vector $\mathbf{w}$ and especially $\mathbf{v}$ in the computation step (1) in Algorithm 3.4.1 become shorter. Since the complexity of this algorithm depends highly on the size of this vector $\mathbf{w}$ we can achieve a speed up.

**Implementation Note 3.4.2.** Finally, these ideas are realized in my Mathematica package to determine a polynomial degree bound.

### 3.4.9 The Special Case $\mathbf{a} \in \mathbb{F}^n$ For Proper Sum Extensions

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$. As in the previous sections we suppose that $(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$. In this section we will additionally assume the special case

$$\boxed{\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^n}$$

and $\mathbf{f} \in \mathbb{F}[t]^\lambda$. Furthermore we assume that $(\mathbb{F}(t)[x], \sigma)$ is a difference ring extension of $(\mathbb{F}(t), \sigma)$ where $x$ is transcendental over $\mathbb{F}(t)$ and $\sigma(x) = x$. Additionally, recall the notation

$$f[p] = \sum_i f_i\, p^i$$

for $f = \sum_i f_i\, x^i$ with $f_i \in \mathbb{F}(t)$ and $p \in \mathbb{F}(t)$.

**Lemma 3.4.22.** *For $0 \leq l \leq m < x$ and $r \geq 0$ we have*

$$\binom{x}{m+1} = \binom{x-m+l}{l+1} \frac{l+1}{m+1} \frac{\prod_{i=1}^{m-l}(x+1-i)}{\prod_{i=1}^{m-l}(l+i)}, \tag{3.43}$$

$$\binom{m+r+1}{m+1} = \binom{r+l+1}{r} \frac{l+1}{m+1} \frac{\prod_{i=1}^{m-l}(m+r+2-i)}{\prod_{i=1}^{m-l}(l+i)}. \tag{3.44}$$

*Proof.* We have

$$\begin{aligned}
\binom{x-m+l}{l+1} \frac{l+1}{m+1} \frac{\prod_{i=1}^{m-l}(x+1-i)}{\prod_{i=1}^{m-l}(l+i)} &= \frac{\prod_{i=1}^{l+1}(x-m+l+1-i)}{(l+1)!} \frac{l+1}{m+1} \frac{\prod_{i=1}^{m-l}(x+1-i)}{\prod_{i=1}^{m-l}(l+i)} \\
&= \frac{\prod_{i=m-l+1}^{m+1}(x+1-i)}{l!} \frac{1}{m+1} \frac{\prod_{i=1}^{m-l}(x+1-i)}{\prod_{i=l+1}^{m} i} \\
&= \frac{\prod_{i=1}^{m+1}(x+1-i)}{(m+1)!} = \binom{x}{m+1}
\end{aligned}$$

and

$$\binom{r+l+1}{r}\frac{l+1}{m+1}\frac{\prod_{i=1}^{m-l}(m+r+2-i)}{\prod_{i=1}^{m-l}(l+i)} = \frac{\prod_{i=1}^{l+1}(r+l+2-i)}{(l+1)!}\,\frac{l+1}{m+1}\,\frac{\prod_{i=1}^{m-l}(m+r+2-i)}{\prod_{i=1}^{m-l}(l+i)}$$

$$= \frac{\prod_{i=1}^{l+1}(r+i)}{l!}\,\frac{1}{m+1}\,\frac{\prod_{i=l+2}^{m+1}(r+i)}{\prod_{i=l+1}^{m} i}$$

$$= \frac{\prod_{i=1}^{m+1}(r+i)}{(m+1)!} = \binom{m+r+1}{m+1}.$$

□

**Lemma 3.4.23.** *Let $m \geq 0$ and let $f_m$ be 0-computable by $\mathbf{g}:\mathbb{N}_0 \to \mathbb{F}[t]^k$. In particular assume $\mathbf{g}(d) = \mathbf{g}^{(d)}[m+d]$ where*

$$\mathbf{g}^{(d)} = \left(g_1^{(d)},\dots,g_k^{(d)}\right) \in \mathbb{F}[t,x]^k$$

*with*

$$g_i^{(d)} = t^d \sum_{l=0}^{m} g_{il}\, t^j$$

*and*

$$g_{il} = \tilde{g}_{il}\prod_{j=1}^{m-l}(x+1-j) \in \mathbb{F}[x], \quad \tilde{g}_{il} \in \mathbb{F}.$$

*Then for all $r \geq 0$ we have*

$$\mathcal{W}_m^{(r)}[m+r+1] = f_{m+1}(r)$$

*and there is a module $\tilde{\mathcal{V}}$ over $\mathbb{K}[x]$ with finitely many generators in $\mathbb{K}^n \times \binom{x}{m}\mathbb{F}$ such that for all $r \geq 0$ we have*

$$f_{m+1}(r) = \left\{\mathbf{c}\,\mathbf{g}^{(r+1)} + t^r\, h \,|\, \mathbf{c}\wedge h \in \tilde{\mathcal{V}}[m+r+1]\right\}.$$

*Proof.* Let $\mathbf{g}:\mathbb{N}_0 \to \mathbb{F}[t]^k$ be defined as in the assumption and define $\mathcal{W}_m^{(r)}$, $\mathcal{W}_{m,r+1}$, $\mathcal{V}_m$ and $\mathcal{V}_{m,r+1}$ with this $\mathbf{g}$. By Corollary 3.4.4 we have

$$\forall r \geq 0:\ \mathcal{W}_m^{(r)}[m+r+1] \subseteq \mathcal{W}_{m,r+1} = f_{m+1}(r).$$

In the following we will show equality. Let $r \geq 0$ and

$$\boxed{h \in f_{m+1}(r) = \mathcal{W}_{m,r+1}.}$$

By Definition 3.4.6 there is a $\mathbf{c}\wedge w \in \mathcal{V}_{m,r+1}$ such that

$$\begin{aligned} h &= \mathbf{c}\,\mathbf{g}(r+1) + w\,t^r = \mathbf{c}\,\mathbf{g}^{(\mathbf{r+1})} + w\,t^r \\ &= t^{r+1}\sum_{l=0}^{m} h_l\,t^l \prod_{j=1}^{m-l}(m+r+2-j) + w\,t^r \end{aligned}$$

where

$$h_l := \mathbf{c}\,(\tilde{g}_{1l},\dots,\tilde{g}_{kl}) \in \mathbb{F}[x].$$

Additionally we have

$$[\sigma_{\mathbf{a}}h]_r = 0.$$

Since $\mathbf{a} = (a_1,\dots,a_n) \in \mathbb{F}^n$, it follows that

$$\begin{aligned} 0 = [\sigma_{\mathbf{a}}h]_r &= \left[\sigma_{\mathbf{a}}(t^{r+1}\sum_{l=0}^{m} h_l\,t^l \prod_{j=1}^{m-l}(m+r+2-j))\right]_r + \sigma_{\mathbf{a}}w \\ &= \left[\sum_{i=1}^{n} a_i \sum_{l=0}^{m}\sigma^{n-i}(h_l)\prod_{j=1}^{m-l}(m+r+2-j)\,(t+\beta_{(n-i)})^{r+l+1}\right]_r + \sum_{i=1}^{n} a_i\,\sigma^{n-i}(w) \qquad (3.45) \\ &= \sum_{i=1}^{n} a_i \sum_{l=0}^{m}\sigma^{n-i}(h_l)\binom{r+l+1}{r}\prod_{j=1}^{m-l}(m+r+2-j)\,\beta_{(n-i)}^{l+1} + \sum_{i=1}^{n} a_i\,\sigma^{n-i}(w). \end{aligned}$$

Now we transform the expression[18]

$$\begin{aligned} H := \sum_{i=1}^{n} a_i \sum_{l=0}^{m}\sigma^{n-i}(h_l)\prod_{j=1}^{m-l}(m+r+2-j)\binom{x}{m+1}\binom{r+l+1}{r}\beta_{(n-i)}^{l+1} \\ + \sum_{i=1}^{n} a_i\,\sigma^{n-i}(w)\binom{x}{m+1} \end{aligned}$$

by Lemma 3.4.22.(3.43) to

$$\begin{aligned} H = \sum_{i=1}^{n} a_i \sum_{l=1}^{m}\sigma^{n-i}(h_l)\prod_{j=1}^{m-l}(m+r+2-j)\binom{x-m+l}{l+1}\frac{l+1}{m+1}\times \\ \times \frac{\prod_{j=1}^{m-l}(x+1-j)}{\prod_{j=1}^{m-l}(l+j)}\binom{r+l+1}{r}\beta_{(n-i)}^{l+1} + \sum_{i=1}^{n} a_i\,\sigma^{n-i}(w)\binom{x}{m+1} \end{aligned}$$

and finally by Lemma 3.4.22.(3.44) we can simplify $H$ further to

$$\begin{aligned} H = \sum_{i=1}^{n} a_i \sum_{l=1}^{m}\sigma^{n-i}(h_l)\binom{x-m+l}{l+1}\binom{m+r+1}{m+1}\times \\ \times \prod_{i=1}^{m-l}(x+1-i)\,\beta_{(n-i)}^{l+1} + \sum_{i=1}^{n} a_i\,\sigma^{n-i}(w)\binom{x}{m+1}. \qquad (3.46) \end{aligned}$$

[18]Note that $H = [\sigma_{\mathbf{a}}h]_r\binom{x}{m+1} = 0$.

For

$$\begin{aligned}
p^{(d)} :=& \binom{m+r+1}{m+1} \mathbf{c}\,\mathbf{g}^{(\mathbf{d+1})} + w\,t^d \binom{x}{m+1} \\
=& t^{d+1} \binom{m+r+1}{m+1} \sum_{l=0}^{m} h_l\, t^l \prod_{j=1}^{m-l}(x+1-j) + w \binom{x}{m+1} t^d
\end{aligned} \tag{3.47}$$

we have

$$p^{(r)}[m+r+1] \frac{1}{\binom{m+r+1}{m+1}} = h. \tag{3.48}$$

Furthermore by the same transformation as in (3.45) we obtain

$$\begin{aligned}
P :=& \left[\sigma_{\mathbf{a}} p^{(d)}[m+d+1]\right]_d \\
=& \left[\sum_{i=1}^{n} a_i \sum_{l=0}^{m} \sigma^{n-i}(h_l) \binom{m+r+1}{m+1} \prod_{j=1}^{m-l}(m+d+2-j)\,(t+\beta_{(n-i)})^{d+l+1}\right]_d \\
&+ \sum_{i=1}^{n} a_i\, \sigma^{n-i}(w) \binom{m+r+1}{m+1} \\
=& \sum_{i=1}^{n} a_i \sum_{l=0}^{m} \sigma^{n-i}(h_l) \binom{d+l+1}{d} \binom{m+r+1}{m+1} \prod_{j=1}^{m-l}(m+d+2-j)\,\beta_{(n-i)}^{l+1} \\
&+ \sum_{i=1}^{n} a_i\, \sigma^{n-i}(w) \binom{m+d+1}{m+1}.
\end{aligned}$$

Looking at (3.46), one can see that

$$P = H[m+d+1].$$

Furthermore looking at (3.45) and the definition of $H$ one can see that

$$H = [\sigma_{\mathbf{a}} h]_r \binom{x}{m+1} = 0$$

and it follows that

$$\begin{aligned}
0 =& H[m+d+1] = P = \left[\sigma_{\mathbf{a}} p^{(d)}[m+d+1]\right]_d \\
\overset{(3.47)}{=}& \left[\sigma_{\mathbf{a}}\Big(\binom{m+r+1}{m+1} \mathbf{c}\,\mathbf{g}^{(\mathbf{d+1})}[m+d+1] + w\,t^d \binom{m+d+1}{m+1}\Big)\right]_d
\end{aligned}$$

for all $d \geq 0$. By Definition 3.4.7 we have

$$\mathcal{V}_m = \left\{\mathbf{c}\wedge h \in \mathbb{K}[x]^k \times \mathbb{F}[x] \,|\, \forall d \geq 0 : \left[\sigma_{\mathbf{a}}\Big(\sum_{s=1}^{k} \mathbf{c}\,\mathbf{g}^{(d+1)}[m+d+1] + h\,t^d\Big)\right]_d\right\},$$

and consequently

$$\Big(\binom{m+r+1}{m+1} \mathbf{c}\Big) \wedge \Big(\binom{x}{m+1} w\Big) \in \mathcal{V}_m.$$

But by Definition 3.4.8 it follows that

$$p^{(r)} \in \mathcal{W}_{m+1}^{(r)}$$

and therefore by (3.48) that

$$\boxed{h \in \mathcal{W}_{m+1}^{(r)}[m+r+1].}$$

Hence

$$\forall r \geq 0: \ \mathcal{W}_m^{(r)}[m+r+1] \supseteq f_{m+1}(r)$$

and we have proven

$$\forall r \geq 0: \ \mathcal{W}_m^{(r)}[m+r+1] = f_{m+1}(r).$$

By (3.47) and (3.48) one can immediately see that for all $r \geq 0$ and all $f \in f_{m+1}(r)$ there is a

$$\mathbf{c}\wedge w \in \mathcal{V}_m \cap (\mathbb{K}^k \times \mathbb{F}\binom{x}{m+1}) =: G$$

with

$$f = \mathbf{c}\,\mathbf{g}(r+1) + t^r\, w[m+r+1].$$

Now consider the submodule $\tilde{\mathcal{V}}$ of $\mathcal{V}_m$ over $\mathbb{K}[x]$ which is generated by the infinite set $G$. Then it follows that

$$f_{m+1}(r) \subseteq \left\{\mathbf{c}\,\mathbf{g}^{(r+1)} + t^r\, h \,|\, \mathbf{c}\wedge h \in \tilde{\mathcal{V}}[m+r+1]\right\}$$

for all $r \geq 0$. But since

$$f_{m+1}(r) = \mathcal{W}_m^{(r)}[m+r+1] = \{\mathbf{c}\,\mathbf{g}(r+1) + t^r\, h \,|\, \mathbf{c}\wedge h \in \mathcal{V}_m[m+r+1]\}$$

by Lemma 3.4.9, it follows by $\tilde{\mathcal{V}} \subseteq \mathcal{V}_m$ that

$$f_{m+1}(r) \supseteq \left\{\mathbf{c}\,\mathbf{g}^{(r+1)} + t^r\, h \,|\, \mathbf{c}\wedge h \in \tilde{\mathcal{V}}[m+r+1]\right\}$$

and therefore equality. By Corollary 3.4.6 $\mathcal{V}_m$ is finitely generated over $\mathbb{K}[x]$ and thus by Lemma 3.4.13 the submodule $\tilde{\mathcal{V}}$ of $\mathcal{V}_m$ is also finitely generated over $\mathbb{K}[x]$. Since for any $\mathbf{v} = (v_1, \ldots, v_k, w) \in \tilde{\mathcal{V}}$ we have

$$\binom{x}{m+1} \mid w,$$

there is a finite set of generators in $\mathbb{K} \times \mathbb{F}\binom{x}{m+1}$. $\square$

**Theorem 3.4.11.** *For all $m \geq 0$ there exists a map $\mathbf{g^{(m)}}: \mathbb{N}_0 \to \mathbb{F}[t]^{k_m}$ such that $f_m$ is 0-computable by $\mathbf{g^{(m)}}$. Furthermore, if $\|\mathbf{g^{(m)}}(r)\| = m + r$ for some $m \geq 0$ and $r \geq 0$ then for all $0 \leq i \leq m$ and all $r \geq 0$ we have*

$$\|\mathbf{g^{(i)}}(r)\| = i + r.$$

*Proof.* We will show that for all $0 \leq i \leq m$ there exists a $\mathbf{g}: \mathbb{N}_0 \to \mathbb{F}[t]^{k_m}$ such that $f_m$ is 0-computable by $\mathbf{g}: \mathbb{N}_0 \to \mathbb{F}[t]^k$. In particular we will show that $\mathbf{g}(r) = \mathbf{g}^{(r)}[m+r]$ where

$$\mathbf{g}^{(r)} = \left(g_1^{(r)}, \ldots, g_{k_m}^{(r)}\right) \in \mathbb{F}[t,x]^k$$

with

$$g_i^{(r)} = t^r \sum_{l=0}^{m} g_{il}\, t^j$$

and

$$g_{il} = \tilde{g}_{il} \prod_{j=1}^{m-l} (x+1-j) \in \mathbb{F}[x], \quad \tilde{g}_{il} \in \mathbb{F}.$$

For $m = 0$ the theorem holds by Proposition 3.4.4. Additionally, $\mathbf{g}$ is of the desired form as claimed above. Now assume that there exists a $\mathbf{g} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ with these properties for $m \geq 0$. By Lemma 3.4.23 there is a module $\tilde{\mathcal{V}}$ over $\mathbb{K}[x]$ with

$$\tilde{\mathcal{V}} \xleftrightarrow{\text{span}} \begin{pmatrix} \mathbf{c_1} \wedge (w_1 \binom{x}{m+1}) \\ \vdots \\ \mathbf{c_s} \wedge (w_s \binom{x}{m+1}) \end{pmatrix}$$

where $\mathbf{c_i} = (c_{i1}, \ldots, c_{ik}) \in \mathbb{K}^k$ and $w_i \in \mathbb{F}$ such that for all $r \geq 0$ we have

$$f_{m+1}(r) = \Big\{ \mathbf{c}\, \mathbf{g}(r+1) + w\, t^r \mid \mathbf{c} \wedge w \in \tilde{\mathcal{V}}[m+r+1] \Big\}.$$

Let $r \geq 0$. Then it follows that $f_{m+1}(r)$ is generated by

$$\mathbf{h} : \begin{cases} \mathbb{N}_0 & \to \quad \mathbb{F}[t]^s \\ r & \mapsto \quad (h_1(r), \ldots, h_s(r)) \end{cases}$$

with

$$\begin{aligned} h_i(r) &= (\mathbf{c_i}\, \mathbf{g}(r+1) + w_i\, t^r \binom{x}{m+1})[m+r+1] \\ &= (\sum_{\lambda=1}^{k} c_{i\lambda}\, t^{r+1} \sum_{l=0}^{m} g_{\lambda l}\, t^l + w_i\, t^r \binom{x}{m+1})[m+r+1] \\ &= (t^{r+1} \sum_{l=0}^{m} t^l \sum_{\lambda=1}^{k} c_{i\lambda}\, g_{\lambda l} + w_i\, t^r \binom{x}{m+1})[m+r+1]. \end{aligned}$$

Thus we may write $\mathbf{h}(r) = \mathbf{h^{(r)}}[m+1+r]$ where

$$\mathbf{h^{(r)}} = \left(h_1^{(r)}, \ldots, h_s^{(r)}\right) \in \mathbb{F}[t,x]^s$$

and

$$h_i^{(r)} = t^r \sum_{l=0}^{m+1} h_{il}\, t^l$$

with

$$h_{il} = \sum_{\lambda=1}^{k} c_{i\lambda}\, \tilde{g}_{\lambda,l-1} \prod_{j=1}^{m+1-l} (x+1-j)$$

for $1 \leq l \leq m+1$ and

$$h_{i0} = \frac{w_i}{(m+1)!} \prod_{j=1}^{m+1} (x+1-j).$$

Consequently $f_{m+1}(r)$ is 0-computable by $\mathbf{h}$ and $\mathbf{h}$ is of the desired form. Now assume

$$\|\mathbf{h}(r)\| = m + 1 + r$$

for some $r \geq 0$. Thus there is an $i$ with $1 \leq i \leq s$ and

$$0 \neq h_{i,m+1} = \sum_{\lambda=1}^{k} c_{i\lambda}\, \tilde{g}_{\lambda,m}$$

and therefore it follows that

$$\tilde{g}_{\lambda m} \in \mathbb{F}^*$$

for some $1 \leq \lambda \leq k$. Therefore $g_{\lambda m}[m + d] \neq 0$ for all $d \geq 0$, thus

$$\|\mathbf{g}(d)\| = m + d$$

for all $d \geq 0$ and consequently the second statement is proven by the induction assumption.

□

**Theorem 3.4.12.** *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$; let $\mathbf{a} \in \mathbb{F}^m$. Assume there are $k \geq 0$ linearly independent $g \in \mathbb{F}$ over $\mathbb{K}$ with $\sigma_{\mathbf{a}} g = 0$. Then there exists a $b$ with $-1 \leq b < m - \max(1, k)$ such that for all $r \geq 0$ we have*

$$\{0\} = f_{-1}(r) \subsetneq f_0(r) \subsetneq \cdots \subsetneq f_b(r) = f_{b+1}(r) = \cdots$$

*Proof.* By Theorem 3.4.11 there are maps $\mathbf{g_i} : \mathbb{N}_0 \to \mathbb{F}[t]^k$ such that $f_i$ is 0-computable by $\mathbf{g_i}$. Assume there does not exist a $b \geq 0$ and $r \geq 0$ such that

$$f_{b-1}(r) \subsetneq f_b(r). \tag{3.49}$$

Then the theorem holds. Otherwise take such a $b$ and $r$. Then we have $\|\mathbf{g_b}(r)\| = b + r$ and thus by Theorem 3.4.11 it follows that $\|\mathbf{g_i}(d)\| = i + d$ for all $d \geq 0$ and all $0 \leq i \leq b$. Consequently

$$f_i(d) \subsetneq f_{i+1}(d)$$

for all $d \geq 0$ and $0 \leq i \leq b$ and therefore

$$f_{-1}(d) \subsetneq f_0(d) \subsetneq \cdots \subsetneq f_b(d) \tag{3.50}$$

for all $d \geq 0$. Hence, whenever there are an $r \geq 0$ and a $b \geq 0$ with (3.49), we will have (3.50). Consequently this inclusion chain will either strictly grow to infinity or it will terminate for all $r \geq 0$ at a point $b \geq 0$. Now assume that the chain strictly grows until a $b \geq 0$. In particular for $d = 0$ we find

$$g_i \in f_i(0) \setminus f_{i-1}(0)$$

for $0 \leq i \leq b + 1$ with

$$\|g_i\| = i$$

and

$$\sigma_{\mathbf{a}}(g_i) \in \mathbb{F}[t]_{i+\|\mathbf{a}\|-1} = \{0\}.$$

Therefore we find $b + 1$ linearly independent $g_i \in \mathbb{F}[t]$ with $\sigma_{\mathbf{a}} g_i = 0$ and hence

$$b < m - 1, \tag{3.51}$$

since otherwise Proposition 3.1.2 is violated. In particular we have

$$g_0 \in \mathbb{F} \quad \text{and} \quad g_1, \dots, g_b \in \mathbb{F}[t] \setminus \mathbb{F}.$$

Now assume there are $k \geq 1$ linearly independent $g \in \mathbb{F}$ such that $\sigma_{\mathbf{a}} g = 0$. Then it follows immediately that there are $b + k - 1$ linearly independent $g$ with $\sigma_{\mathbf{a}} g = 0$. Hence

$$b < m - k,$$

since otherwise Proposition 3.1.2 is violated. Together with (3.51) it follows that

$$b < m - \max(k, 1)$$

and thus the theorem is proven. □

**Corollary 3.4.12.** *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$; let $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^m$ and $f \in \mathbb{F}[t]$ with $\|f\| = l$. If there is a $g \in \mathbb{F}[t]$ with*

$$\sigma_{\mathbf{a}} g = f, \quad \deg(g) = n$$

*then there are $g_i \in \mathbb{F}[t]$ with*

$$\sigma_{\mathbf{a}} g_i = 0, \quad \deg(g_i) = i$$

*for $0 \leq i \leq n - l - 1$.*

*Proof.* Assume

$$\sigma_{\mathbf{a}} g = f$$

for some $g, f \in \mathbb{F}[t]$ with $\deg(g) = n \geq \|f\| = l$. The corollary holds, if $n = l$. So suppose $n > l$. Then $g$ can be expressed by

$$g = h + p$$

where $p \in \mathbb{F}[t]_l$ and

$$h \in f_{n-l-1}(l+1) \setminus f_{n-l-2}(l+1)$$

with $\deg(h) = l$. Thus by Theorem 3.4.12 we have

$$f_{-1}(r) \subsetneq f_0(r) \subsetneq \cdots \subsetneq f_{n-l-1}(r)$$

for all $r \geq 0$ and therefore there are

$$h_i \in f_i(0) \setminus f_{i-1}(0)$$

with

$$\sigma_{\mathbf{a}} h_i \in \mathbb{F}[t]_{\|\mathbf{a}\|-1} = \{0\}$$

and $deg(h_i) = i$ for $0 \leq i \leq n - l - 1$. □

**Corollary 3.4.13.** *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^m$, $\mathbf{f} \in \mathbb{F}[t]^n$ and assume there are $k \geq 0$ linearly independent $g \in \mathbb{F}$ over $\mathbb{K}$ with $\sigma_{\mathbf{a}} g = 0$. Then $m + \|\mathbf{f}\| - \max(k, 1)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$.*

*Proof.* By Theorem 3.4.12 there is a $d$ with $-1 \leq d < m - \max(k,1)$ such that

$$f_{-1}(r) \subsetneq f_0(r) \subsetneq \cdots \subsetneq f_d(r) = f_{d+1}(r) = \cdots$$

for $r := \|\mathbf{f}\| + 1$ and thus

$$\mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]_d) = \mathrm{V}_r(\mathbf{a}, \mathbb{F}[t]).$$

Therefore by Theorem 3.4.1 $d + r$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. By

$$d + r = d + \|\mathbf{f}\| + 1 \leq m + \|\mathbf{f}\| - \max(k,1),$$

also $m + \|\mathbf{f}\| - \max(k,1)$ is a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. $\square$

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$, let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^m$ and $\mathbf{f} \in \mathbb{F}[t]^n$. If one wants to compute the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$ then the following reduction process was introduced in Section 3.2.6:

$$\begin{tikzcd}
 & \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]) \arrow[d, equal] & & \text{polynomial bounding} & \\
 & \mathrm{V}(\mathbf{a}, \overbrace{\mathbf{f_b}}^{\mathbf{f}}, \mathbb{F}[t]_b) \arrow[rr, shift right, "1."'] & & \mathrm{I}(\mathbf{a}, \mathbf{f_b}, t^b\,\mathbb{F}) \arrow[ll, shift right, "5."'] \arrow[d, shift right, "2."'] \arrow[dll, "4."] & \\
\text{Theorem 3.2.2} & \mathrm{V}(\mathbf{a}, \mathbf{f_{b-1}}, \mathbb{F}[t]_{b-1}) \arrow[u, "5."'] & & \mathrm{V}(\tilde{\mathbf{a}}_{\mathbf{b}}, \tilde{\mathbf{f}}_{\mathbf{b}}, \mathbb{F}) \arrow[u, shift right, "3."'] & \text{Theorem 3.2.1}
\end{tikzcd}$$

By Corollary 3.4.13 it follows that

$$b := m + \|\mathbf{f}\| - 1 > \|\mathbf{f}\|$$

is a bound. Therefore by Theorem 3.2.1 we have

$$\tilde{\mathbf{f}}_{\mathbf{b}} := \mathbf{0}, \qquad \tilde{\mathbf{a}}_{\mathbf{b}} := \mathbf{a}$$

and hence we obtain the following reduction

$$\begin{tikzcd}
\mathrm{I}(\mathbf{a}, \mathbf{f_b}, t^b\,\mathbb{F}) \arrow[d, shift right, "2."'] \\
\mathrm{V}(\mathbf{a}, \mathbf{0}, \mathbb{F}). \arrow[u, shift right, "3."']
\end{tikzcd} \tag{3.52}$$

Of course, we can compute the solution space $\mathrm{V}(\mathbf{a}, \mathbf{0}, \mathbb{F})$ before we compute a bound for $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$ and can extract the number $k$ of linearly independent solutions $g$ with $\sigma_{\mathbf{a}} g = 0$. Then we can use this number $k$ to improve the bound by Corollary 3.4.13 to

$$b := m + \|\mathbf{f}\| - \max(k,1).$$

Since $k < m$ by Proposition 3.1.2, it follows that $b > \|\mathbf{f}\|$ and therefore as for the old bound $b$ we have to do the reduction step (3.52). Therefore by remembering the result of this reduction step we achieve an improved bounding without any additional computation costs.

## 3.5 Denominator Boundings

In Section 3.1.3.1 the idea of denominator bounding was introduced in order to reduce the problem of finding rational solutions in $\mathbb{F}(t)$ of a difference equation to searching polynomial solutions in $\mathbb{F}[t]$:

$$\text{Find a basis for}\left\{\begin{array}{c} \mathrm{V}(\mathbf{a},\mathbf{f},\overbrace{\boxed{\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}\oplus\mathbb{F}(t)^{(0)}}}^{\mathbb{F}(t)}) \\ \Big\downarrow \begin{array}{c}\text{period 0 and 1}\\ \text{elimination}\end{array} \Big\uparrow \quad \text{by period 1 and 0 denominator bounding} \\ \mathrm{V}(\mathbf{a}',\mathbf{f}',\boxed{\mathbb{F}[t]}) \end{array}\right.$$

As already mentioned in Section 3.1.3.1 we will consider the elimination of the fractional part with period 1 and pure period 0 in Sections 3.5.2 and 3.5.3.

### 3.5.1 The Denominator, Order and $\sigma$-Function

Given the field of rational functions $\mathbb{F}(t)$ over $\mathbb{F}$ and $f \in \mathbb{F}(t)$, we introduce, as in Remark 2.5.1, the *numerator* and *denominator* of $f$ in the following way. Let $f = \frac{a}{b}$ be in reduced representation, i.e. $a, b \in \mathbb{F}[t]$, $\gcd(a, b) = 1$ and $b$ is monic. Then

$$\begin{aligned}\mathrm{num}(f) &= a,\\ \mathrm{den}(f) &= b.\end{aligned}$$

In the following sections the denominator of $f \in \mathbb{F}(t)$ will play an essential role, in particular we will need the following simple lemma.

**Lemma 3.5.1.** *Let $(\mathbb{F}(t), \sigma)$ be a difference field with $t$ transcendental over $\mathbb{F}$ and $f \in \mathbb{F}(t)$. Then*

$$\sigma(\mathrm{den}(f)) = u\,\mathrm{den}(\sigma(f))$$

*for some $u \in \mathbb{F}$*

*Proof.* For $f = 0$ the lemma clearly holds. Let $f = \frac{a}{b} \in \mathbb{F}(t)^*$ and $\sigma(f) = \frac{a'}{b'}$ be in reduced representation. We have

$$\frac{a'}{b'} = \sigma(f) = \frac{\sigma(a)}{\sigma(b)}.$$

Since $\gcd(a, b) = 1$, it follows by Lemma 2.2.1 that

$$\gcd(\sigma(a), \sigma(b)) = 1.$$

As $\gcd(a', b') = 1$, there is a $u \in \mathbb{F}$ with

$$\sigma(\mathrm{den}(f)) = \sigma(b) = u\,b' = \mathrm{den}(\sigma(f)).$$

□

Besides the denominator we will need the order of an element of the polynomial ring $\mathbb{F}[t]$.

**Definition 3.5.1.** Let $\mathbb{F}[t]$ be a polynomial ring and let $f \in \mathbb{F}[t]^*$. We define the *order* of $f$ - in symbols $\mathrm{ord}(f)$ - as the maximal $m \geq 0$ such that

$$t^m \mid q.$$

For the zero-polynomial we define $\mathrm{ord}(0) := -1$. $\diamondsuit$

Later we will need the following simple fact.

**Lemma 3.5.2.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$ and $q \in \mathbb{F}[t]$. Then for all $k \geq 0$ we have*

$$\mathrm{ord}(q) = \mathrm{ord}(\sigma^k(q)).$$

*Proof.* Let $d := \mathrm{ord}(q)$. If $d = -1, 0$, the lemma clearly holds. Now assume $d > 0$ and assume there exists a $k \geq 0$ with

$$\mathrm{ord}(q) \neq \mathrm{ord}(\sigma^k(q)). \tag{3.53}$$

We have

$$q = t^d\, p$$

for some $p \in \mathbb{F}[t]^*$ with $t \nmid p$. Since

$$\sigma^k(q) = \sigma^k(t^d)\, \sigma^k(p) = (\alpha)_k\; t^d \sigma^d(p)$$

where $\sigma^d(p) \in \mathbb{F}[t]$ and $(\alpha)_k \in \mathbb{F}^*$, it follows by (3.53) that

$$t \mid \sigma^k(p)$$

and thus $\sigma^k(p)/t \in \mathbb{F}[t]$. Then

$$\sigma^{-k}(\frac{\sigma^k(p)}{t}) = \frac{p}{\sigma^{-k}(t)} = \frac{p}{c\, t} \in \mathbb{F}[t]$$

for some $c \in \mathbb{F}$ and therefore $t \mid p$, a contradiction. $\square$

**Lemma 3.5.3.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$, let $a \in \mathbb{F}[t]^*$ and let $g \in \mathbb{F}(t)^*$ with $\mathrm{ord}(\mathrm{den}(g)) > 0$. Then*

$$\mathrm{ord}(\mathrm{den}(a\, \sigma^i(g))) = \max(0, \mathrm{ord}(\mathrm{den}(g)) - \mathrm{ord}(a))$$

*for all $i \geq 0$.*

*Proof.* Let $d := \mathrm{ord}(\mathrm{den}(g)) > 0$ and

$$g = \frac{u}{v\, t^d}$$

for some $u, v \in \mathbb{F}[t]^*$ with $t \nmid u, v$ and let

$$a = t^p\, b$$

for some $p \geq 0$ and $b \in \mathbb{F}[t]^*$ with $t \nmid b$. Then

$$a\, \sigma^i(g) = t^p\, b\, \sigma^i(\frac{u}{v\, t^d}) = \frac{b}{t^{d-p}}\, \frac{\sigma^i(u)}{\sigma^i(v)\; (\alpha)_i^d}$$

with $d-p \in \mathbb{Z}$. Clearly we have

$$\sigma^i(u) \in \mathbb{F}[t]^* \quad \text{and} \quad \sigma^i(v)\ (\alpha)_i^d \in \mathbb{F}[t]^*.$$

Since $\operatorname{ord}(u) = \operatorname{ord}(v) = 0$, it follows that

$$t \nmid \sigma^i(u) \quad \text{and} \quad t \nmid \sigma^i(v)\ (\alpha)_i^d$$

by Lemma 3.5.2. If $d-p \geq 0$ then

$$\operatorname{den}(\operatorname{ord}(a\,\sigma^i(g))) = d-p = \max(0, d-p),$$

otherwise

$$\operatorname{den}(\operatorname{ord}(a\,\sigma^i(g))) = 0 = \max(0, d-p).$$

□

**Lemma 3.5.4.** *Let $\mathbb{F}(t)$ be a field of rational functions over $\mathbb{F}$ and let*

$$f = f_1 + f_2 + f_3 \in \mathbb{F}[1/t] \setminus \mathbb{F}^* \oplus \mathbb{F}[t] \oplus \mathbb{F}(t)^{(fracpart)}.$$

*Then* $\operatorname{ord}(\operatorname{den}(f)) = \operatorname{ord}(\operatorname{den}(f_1))$.

*Proof.* Let $f_3 = \frac{p}{q}$ be in reduced representation, in particular $t \nmid q$. If $\operatorname{ord}(\operatorname{den}(f_1)) = 0$ then $f_1 = 0$, hence

$$f = f_2 + f_3 = f_2 + \frac{p}{q} = \frac{f_2\,q + p}{q}$$

and thus $\operatorname{ord}(\operatorname{den}(f)) = 0$. Otherwise, if $\operatorname{ord}(\operatorname{den}(f_1)) > 0$, let $f_1 = \frac{u}{t^d}$ be in reduced representation, in particular $d \geq 1$ and $t \nmid u \neq 0$. Then

$$f = f_1 + f_2 + f_3 = \frac{u}{t^d} + f_2 + \frac{p}{q} = \frac{u\,q + f_2\,t^d\,q + p\,t^d}{t^d\,q}$$

where $t \nmid u\,q$, therefore

$$t \nmid u\,q + f_2\,t^d\,q + p\,t^d$$

and hence $\operatorname{ord}(\operatorname{den}(f)) = d = \operatorname{ord}(\operatorname{den}(f_1))$. □

**Lemma 3.5.5.** *Let $t$ be transcendental over $\mathbb{F}$, $d \geq 1$ and*

$$f = \sum_{i=1}^{d} \frac{f_i}{t^i} \in \mathbb{F}[1/t] \setminus \mathbb{F}^*.$$

*Then* $\operatorname{ord}(\operatorname{den}(f)) = d$ *if and only if* $f_d \neq 0$. *Furthermore,* $\operatorname{ord}(\operatorname{den}(f)) \leq d$.

*Proof.* If $f_d \neq 0$ then

$$f = \sum_{i=1}^{d} \frac{f_i}{t^i} = \frac{f_d + t\,f_{d-1} + \cdots + t^{d-1}\,f_1}{t^d} =: \frac{u}{t^d}$$

where $u \in \mathbb{F}[t]$ with $t \nmid u$. Hence

$$\operatorname{ord}(\operatorname{den}(f)) = d.$$

Contrary, assume that $f_d = 0$. If $f = 0$ then clearly

$$\operatorname{ord}(\operatorname{den}(f)) = \operatorname{ord}(1) = 0 < d.$$

In particular, $d \neq \operatorname{ord}(\operatorname{den}(f))$. Otherwise, if $f \neq 0$, let $l < d$ be maximal such that $f_l \neq 0$. Then

$$f = \sum_{i=1}^{l} \frac{f_i}{t^i} = \frac{f_l + t\, f_{l-1} + \cdots + t^{l-1}\, f_1}{t^l} = \frac{u}{t^l}$$

where $u \in \mathbb{F}[t]$ with $t \nmid u$. Hence

$$\operatorname{ord}(\operatorname{den}(f)) = l < d$$

by the first part of the proof and thus $\operatorname{ord}(\operatorname{den}(f)) \neq d$ . □

**Proposition 3.5.1.** *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi$*-extension of* $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{F}[t]^m$ *and* $p := \min_i(\operatorname{ord}(a_i))$. *Let* $g \in \mathbb{F}(t)$ *with* $d := \operatorname{ord}(\operatorname{den}(g)) > p$ *and define*

$$S := \{a_i \,|\, \operatorname{ord}(a_i) = p\}.$$

*Then* $\operatorname{ord}(\operatorname{den}(\sigma_{\mathbf{a}} g)) < d - p$ *if and only if*

$$\operatorname{ord}(\operatorname{den}(\sum_{i \in S} a_i\, \sigma^{m-i}(g))) < d - p.$$

*Proof.* Take

$$h_i := a_i\, \sigma^{m-i}(g)$$

for all $1 \leq i \leq m$ and write

$$h_i = h_{i1} + h_{i2} + h_{i3} \in \mathbb{F}[1/t] \setminus \mathbb{F}^* \oplus \mathbb{F}[t] \oplus \mathbb{F}(t)^{(fracpart)}.$$

If $a_i = 0$ then $h_i = 0$, therefore

$$o_i := \operatorname{ord}(\operatorname{den}(h_i)) = \operatorname{ord}(1) = 0 < d - p$$

and thus we may write

$$h_{i1} = \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j}.$$

Otherwise, if $a_i \neq 0$ then by Lemma 3.5.3 it follows that

$$o_i := \operatorname{ord}(\operatorname{den}(h_i)) = \max(0, d - p_i) \leq d - p$$

and by Lemma 3.5.5 we may write

$$h_{i1} = \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j}$$

with $h_{ij} \in \mathbb{F}$. Write

$$\sigma_{\mathbf{a}} g = f_1 + f_2 + f_3 \in \mathbb{F}[1/t] \setminus \mathbb{F}^* \oplus \mathbb{F}[t] \oplus \mathbb{F}(t)^{(fracpart)};$$

then

$$f_1 = h_{11} + \cdots + h_{m1} = \sum_{j=1}^{o_1} \frac{\tilde{h}_{1j}}{t^j} + \cdots + \sum_{j=1}^{o_m} \frac{\tilde{h}_{mj}}{t^j}.$$

We have

$$\begin{aligned}
\mathrm{ord}(\mathrm{den}(f)) < d-p \quad & \overset{\text{Lemma (3.5.4)}}{\Leftrightarrow} \quad \mathrm{ord}(\mathrm{den}(f_1)) < d-p \\
& \Leftrightarrow \quad \mathrm{ord}(\mathrm{den}(\sum_{j=1}^{o_1} \frac{\tilde{h}_{1j}}{t^j} + \cdots + \sum_{j=1}^{o_m} \frac{\tilde{h}_{mj}}{t^j})) < d-p \\
& \overset{\text{Lemma (3.5.5)}}{\Leftrightarrow} \quad \sum_{i \in S} \tilde{h}_{io_i} = 0 \\
& \overset{\text{Lemma (3.5.5)}}{\Leftrightarrow} \quad \mathrm{ord}(\mathrm{den}(\sum_{i \in S} \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j})) < d-p \\
& \Leftrightarrow \quad \mathrm{ord}(\mathrm{den}(\sum_{i \in S} a_i\, \sigma^{m-i}(g))) < d-p.
\end{aligned}$$

$\square$

### 3.5.2 Some Special Cases for the Period 1 Denominator Bounding

Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbf{f} \in \mathbb{F}[t]^n$ and $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$. Furthermore let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)^{(0)}$ as a vector space over $\mathbb{K}$. We are interested in finding a $d \in \mathbb{F}[t]^*$ such that for all

$$\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}[t] \oplus \mathbb{W} \oplus \mathbb{F}(t)^{(1)})$$

we have

$$d\, g \in \mathbb{F}[t] \oplus \mathbb{W}. \tag{3.54}$$

Given such a $d$ we can apply the reduction technique described in Section 3.1.3.1. By Corollary 3.1.5 we have

$$\mathbb{F}(t)^{(1)} = \mathbb{F}[1/t] \setminus \mathbb{F}^*.$$

In other words, we are looking for a bound $b \in \mathbb{N}_0$ such that for all $\mathbf{c} \in \mathbb{K}^n$ and $g \in \mathbb{F}(t)$ with

$$\sigma_{\mathbf{a}} g = \mathbf{c}\, \mathbf{f}$$

we have

$$b \geq \mathrm{ord}(\mathrm{den}(g)). \tag{3.55}$$

Then clearly $d := t^b$ satisfies property (3.54).

#### 3.5.2.1 A Lower Bound

**Lemma 3.5.6.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with*

$$p := \min_i(\mathrm{ord}(a_i))$$

*and $f \in \mathbb{F}[t]^*$. If there is a $g \in \mathbb{F}(t)^*$ with $\mathrm{den}(\mathrm{ord}(g)) > 0$ and*

$$\sigma_{\mathbf{a}} g = f$$

*then*

$$\mathrm{ord}(\mathrm{den}(g)) \geq \max(p - \mathrm{ord}(f), 1).$$

*Proof.* We have $d := \mathrm{ord}(\mathrm{den}(g)) > 0$ and

$$g = \frac{u}{v\, t^d}$$

for some $u, v \in \mathbb{F}[t]$ with $t \nmid u, v$. Let

$$a_i = t^{p_i}\, b_i$$

for some $p_i \geq 0$ and $b_i \in \mathbb{F}[t]$ with $t \nmid b_i$ or $b_i = 0$. We have

$$f = \sigma_{\mathbf{a}} g = \sum_{i=1}^{m} t^{p_i}\, b_i\, \sigma^{m-i}\left(\frac{u}{v\, t^d}\right) = \sum_{i=1}^{m} t^{p_i - d}\, b_i\, \frac{\sigma^{m-i}(u)}{\sigma^{m-i}(v)\ (\alpha)^d_{m-i}} \tag{3.56}$$

with $p_i - d \in \mathbb{Z}$. Clearly we have

$$\sigma^{m-i}(u) \in \mathbb{F}[t] \quad \text{and} \quad \sigma^{m-i}(v)\ (\alpha)^d_{m-i} \in \mathbb{F}[t].$$

Since $\mathrm{ord}(v) = 0$, by Lemma 3.5.2 it follows that

$$t \nmid \sigma^{m-i}(v)\ (\alpha)^d_{m-i}\,.$$

If $p_i - d \geq 0$ for all $1 \leq i \leq m$ then by (3.56) it follows that

$$\mathrm{ord}(f) \geq \min(p_1 - d, \dots, p_m - d) = p - d$$

and consequently

$$d \geq p - \mathrm{ord}(f) = \max(p - \mathrm{ord}(f), 1).$$

Otherwise, if there exists an $i$ with $p_i - d < 0$ then $p - d < 0$ and hence

$$d \geq 1 = \max(p - d, 1).$$

□

As in the polynomial case (Section 3.3.1), for the general setting under discussion one does not know an algorithm to determine a bound $b$ for some $\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ and $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}[t]^n$ such that for all $g \in \mathbb{F}(t)$ and $\mathbf{c} \in \mathbb{K}^n$ with

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$$

we have

$$\mathrm{ord}(\mathrm{den}(g)) \leq b.$$

In this case, the previous lemma motivates us to choose heuristically a bound

$$\max(\min_i(\mathrm{ord}(a_i)) - \min_i(\mathrm{ord}(f_i)), 1) + \texttt{plusBound}$$

where $\texttt{plusBound} \geq 0$ has to be chosen by the user and must be incremented if the desired solution cannot be found.

#### 3.5.2.2 A Simple Case

**Theorem 3.5.1.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$, $\mathbf{f} \in \mathbb{F}[t]^n$ and $\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with*

$$\begin{aligned} &\mathrm{ord}(a_r) = p \text{ for some } r \in \{1, \dots, m\},\\ &\mathrm{ord}(a_i) > p \ \ \forall i \neq r \end{aligned}$$

*If $g \in \mathbb{F}(t)$ with $\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$ for some $\mathbf{c} \in \mathbb{K}^n$ then*

$$\mathrm{ord}(\mathrm{den}(g)) \leq p.$$

*Proof.* Suppose $\mathrm{ord}(\mathrm{den}(g)) > p$. Since

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f} \in \mathbb{F}[t],$$

it follows that

$$\mathrm{ord}(\mathrm{den}(\sigma_{\mathbf{a}} g)) = \mathrm{ord}(1) = 0.$$

Therefore, together with Proposition 3.5.1, it follows that

$$\mathrm{ord}(\mathrm{den}(a_r\, \sigma^{m-r}(g))) < \mathrm{ord}(\mathrm{den}(g)) - p.$$

But by Lemma 3.5.3 it follows that

$$\mathrm{ord}(\mathrm{den}(a_r\, \sigma^{m-r}(g))) = \max(0, \mathrm{ord}(\mathrm{den}(g)) - p) = \mathrm{ord}(\mathrm{den}(g)) - p,$$

a contradiction. □

#### 3.5.2.3 The First Order Case

Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$,

$$\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{F}[t]^2$$

and $\mathbf{f} \in \mathbb{F}[t]^n$. In this section we will deal with the problem to find a bound $b$ as in (3.55). Then we have a denominator bounding as stated in (3.54) which is needed for the incremental reduction method as one can see in Section 3.1.3.2.

If $\operatorname{ord}(a_1) \neq \operatorname{ord}(a_2)$, Theorem 3.5.1 provides a bound $b$. What remains to consider is the case $\operatorname{ord}(a_1) = \operatorname{ord}(a_2)$.

This means, without loss of generality, we assume that $\operatorname{ord}(a_1) = \operatorname{ord}(a_2) =: p \geq 0$, i.e.

$$\begin{aligned} a_1 &= t^p(1 + r_1),\\ a_2 &= t^p(-c + r_2) \end{aligned} \tag{3.57}$$

where $r_1, r_2 \in \mathbb{F}[t]$ with $\operatorname{ord}(r_i) > 0$ and $c \in \mathbb{F}^*$.

The result of this section delivers a bound for exactly that case (3.57). Especially, in order to compute this bound, we must be able to decide, if there exists a $d \geq 0$ for some $c, \alpha \in \mathbb{F}^*$ such that

$$c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

Furthermore, if there exists such a $d$, we must even compute it. As mentioned in Section 2.2.5 these problems can be solved if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field.

The main idea of the following section is taken from Theorem 18 of [Kar81]. Whereas in Karr's version theoretical and computational aspects are mixed, I tried to separate his theorem in several parts to achieve more transparency.

**Theorem 3.5.2.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$, $f \in \mathbb{F}[t]$ and assume $a_1, a_2 \in \mathbb{F}[t]$ as in (3.57). If there exists a $g \in \mathbb{F}(t)$ with $d := \operatorname{ord}(\operatorname{den}(g)) > p$ such that*

$$\operatorname{ord}(\operatorname{den}(a_1\,\sigma(g) - a_2\,g)) < d - p$$

*then*

$$c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

*Proof.* Let $g \in \mathbb{F}(t)$ with $d := \operatorname{ord}(\operatorname{den}(g)) > p$, i.e.

$$g = \frac{u}{v\,t^d}$$

for some $u, v \in \mathbb{F}[t]^*$ with $\gcd(u, v) = 1$ and $t \nmid u, v$. We have

$$\begin{aligned} a_1\,\sigma(g) - a_2\,g &= (1 + r_1)\,\frac{\sigma(u)}{\sigma(v)}\,\frac{1}{\alpha^d\,t^{d-p}} - (c - r_2)\frac{u}{v}\frac{1}{t^{d-p}}\\ &= \frac{(1 + r_1)\,\sigma(u)\,v - (c - r_2)\,u\,\sigma(v)\,\alpha^d}{\sigma(v)\,v}\,\frac{1}{\alpha^d\,t^{d-p}}. \end{aligned}$$

As

$$\operatorname{ord}(\operatorname{den}(a_1\,\sigma(g) - a_2\,g)) < d - p,$$

it follows that

$$t \mid ((1+r_1)\,\sigma(u)\,v - (c-r_2)\,u\,\sigma(v)\,\alpha^d)$$
$$\Updownarrow$$
$$\Big[(1+r_1)\,\sigma(u)\,v - (c-r_2)\,u\,\sigma(v)\,\alpha^d\Big]_0 = 0.$$

Let $u_0 := [u]_0 \in \mathbb{F}^*$ and $v_0 := [v]_0 \in \mathbb{F}^*$. As $t \mid r_i$ we get

$$\sigma(u_0)\,v_0 - c\,u_0\,\sigma(v_0)\,\alpha^d = 0 \Leftrightarrow \frac{\sigma(u_0)\,v_0}{u_0\,\sigma(v_0)} = c\,\alpha^d$$
$$\Leftrightarrow \frac{\sigma(h)}{h} = c\,\alpha^d$$

for $h := \frac{u_0}{v_0} \in \mathbb{F}^*$ and thus

$$c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

□

**Theorem 3.5.3.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi$-extension of $(\mathbb{F},\sigma)$, $\mathbf{f} \in \mathbb{F}[t]^n$ and assume $a_1, a_2 \in \mathbb{F}[t]$ as in (3.57). Let $g \in \mathbb{F}(t)$ and $\mathbf{c} \in \mathbb{K}^n$ such that*

$$a_1\,\sigma(g) + a_2\,g = \mathbf{c}\,\mathbf{f}.$$

*If there exists a $d \geq 0$ such that*

$$c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$$

*then $d$ is uniquely determined and we have $\mathrm{ord}(\mathrm{den}(g)) \leq \max(d,p)$. If there does not exist such a $d$ then $\mathrm{ord}(\mathrm{den}(g)) \leq p$.*

*Proof.* Let $g \in \mathbb{F}(t)$ and $\mathbf{c} \in \mathbb{K}^n$ with

$$a_1\,\sigma(g) - a_2\,g = \mathbf{c}\,\mathbf{f} =: f.$$

Since $f \in \mathbb{F}[t]$, we have

$$\mathrm{ord}(\mathrm{den}(f)) = \mathrm{ord}(1) = 0. \tag{3.58}$$

1. Assume there exists a $d \geq 0$ with

   $$c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}.$$

   Then by Lemma 3.3.2 $d$ is uniquely determined. Assume $\mathrm{ord}(\mathrm{den}(g)) > p$. Since (3.58), by Theorem 3.5.2 it follows that $\mathrm{ord}(\mathrm{den}(g)) = d$ and therefore

   $$\mathrm{ord}(\mathrm{den}(g)) = d = \max(p,d).$$

   Otherwise, if $\mathrm{ord}(\mathrm{den}(g)) \leq p$ then we have

   $$\mathrm{ord}(\mathrm{den}(g)) \leq \max(p,d).$$

2. Assume there does not exist such a $d$. Since (3.58), by Theorem 3.5.2 it follows that

   $$\mathrm{ord}(\mathrm{den}(g)) \leq p.$$

□

#### 3.5.2.4 A Generalization for the $m$-th order Case

Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$.

Assume $\mathbf{0} \neq \mathbf{a} = (a_1, \dots, a_\lambda, \dots, a_\mu\ \dots, a_m) \in \mathbb{F}[t]^m$ with

$$\begin{aligned} \operatorname{ord}(a_\lambda) &= \operatorname{ord}(a_\mu) = p, \\ \operatorname{ord}(a_i) &> \operatorname{ord}(a_\lambda) \quad \forall i \neq \lambda, \mu \end{aligned}$$

and

$$\begin{aligned} a_\lambda &= t^p + r_1, \\ a_\mu &= -c\, t^p + r_2 \end{aligned} \tag{3.59}$$

for $c \in \mathbb{F}^*$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\operatorname{ord}(r_1), \operatorname{ord}(r_2) > 0$.

In this section we will deal with the problem to find a bound $b \geq 0$ as in (3.55). Then we have a denominator bounding as stated in (3.54) which is needed for the incremental reduction method as one can see in Section 3.1.3.2.

**Theorem 3.5.4.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extensions of $(\mathbb{F}, \sigma)$ where we set*

$$\sigma^k(t) = \alpha_k\, t$$

*for all $k \in \mathbb{Z}^*$. Let $\mathbf{a} \in \mathbb{F}[t]^m$ as in (3.59) and assume that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a $\Pi$-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. If there exists a $g \in \mathbb{F}(t)$ with $d := \operatorname{ord}(\operatorname{den}(g)) > p$ such that*

$$\operatorname{ord}(\operatorname{den}(\sigma_{\mathbf{a}} g)) < d - p$$

*then*

$$\sigma^{(\mu-m)}(c)\, \alpha^d_{\mu-\lambda} \in \mathrm{H}_{(\mathbb{F}, \sigma^{\mu-\lambda})}.$$

*Proof.* Let $g \in \mathbb{F}(t)$ with $d := \operatorname{ord}(\operatorname{den}(g)) \geq p$ and assume

$$\operatorname{ord}(\operatorname{den}(\sigma_{\mathbf{a}} g)) < d - p.$$

Then by Proposition 3.5.1 and (3.59) it follows that

$$d - p > \operatorname{ord}(\operatorname{den}(a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)))$$

and thus by Lemmas 3.5.1 and 3.5.2 we have

$$\begin{aligned} d - p &> \operatorname{ord}(\sigma^{\mu-m}(\operatorname{den}(a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)))) \\ &= \operatorname{ord}(\operatorname{den}(\sigma^{\mu-m}(a_\lambda)\, \sigma^{\mu-\lambda}(g) + \sigma^{\mu-m}(a_\mu)\, g). \end{aligned}$$

By

$$\begin{aligned} \sigma^{\mu-m}(a_\lambda) &= \alpha^p_{\mu-m}\, t^p + \sigma^{\mu-m}(r_1), \\ \sigma^{\mu-m}(a_\mu) &= -\sigma^{\mu-m}(c)\, \alpha^p_{\mu-m}\, t^p + \sigma^{\mu-m}(r_2) \end{aligned}$$

it follows that

$$\operatorname{ord}(\operatorname{den}(b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g)) < d - p$$

for

$$b_1 := t^p + \sigma^{\mu-m}(r_1)/\alpha^p_{\mu-m},$$
$$b_2 := -\sigma^{\mu-m}(c)\, t^p + \sigma^{\mu-m}(r_2)/\alpha^p_{\mu-m}.$$

As $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a $\Pi$-extension of $(\mathbb{F}^{\mu-\lambda}, \sigma)$, we may apply Theorem 3.5.2 and thus we obtain

$$\sigma^{\mu-m}(c)\, \alpha^d_{\mu-\lambda} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}.$$

$\square$

**Theorem 3.5.5.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$ and set*

$$\sigma^k(t) = \alpha_k\, t$$

*for all $k \in \mathbb{Z}^*$. Let $\mathbf{f} \in \mathbb{F}[t]^n$, assume $\mathbf{a} \in \mathbb{F}[t]^m$ as in (3.59) and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a $\Pi$-extension of $(\mathbb{F}^{\mu-\lambda}, \sigma)$. Let $g \in \mathbb{F}(t)$ and $\mathbf{c} \in \mathbb{K}^n$ such that*

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}.$$

*If there exists a $d \geq 0$ such that*

$$\sigma^{(\mu-m)}(c)\, \alpha^d_{\mu-\lambda} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$$

*then $d$ is uniquely determined and $\mathrm{ord}(\mathrm{den}(g)) \leq \max(d, p)$. Otherwise, if there does not exist such a $d$ then $\mathrm{ord}(\mathrm{den}(g)) \leq p$.*

*Proof.* Let $g \in \mathbb{F}(t)$ and $\mathbf{c} \in \mathbb{K}^n$ with

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f} =: f.$$

Since $f \in \mathbb{F}[t]$, we have

$$\mathrm{ord}(\mathrm{den}(f)) = \mathrm{ord}(1) = 0. \tag{3.60}$$

1. Assume there exists a $d \geq 0$ with

$$\sigma^{(\mu-m)}(c)\, \alpha^d_{\mu-\lambda} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}.$$

   Then by Lemma 3.3.2 $d$ is uniquely determined. Assume $\mathrm{ord}(\mathrm{den}(g)) > p$. Since (3.60), by Theorem 3.5.4 it follows that

$$\mathrm{ord}(\mathrm{den}(g)) = d = \max(p, d).$$

   Otherwise, if $\mathrm{ord}(\mathrm{den}(g)) \leq p$, we have

$$\mathrm{ord}(\mathrm{den}(g)) \leq \max(p, d).$$

2. Assume there does not exist such a $d$. Since (3.60), by Theorem 3.5.4 it follows that

$$\mathrm{ord}(\mathrm{den}(g)) \leq p.$$

$\square$

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field and $(\mathbb{F}(t), \sigma)$ a $\Pi$-extension of $(\mathbb{F}, \sigma)$. Then Theorem 3.3.7 guarantees that for any $k \neq 0$ the difference field $(\mathbb{F}(t), \sigma^k)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma^k)$. Therefore we can apply Theorem 3.5.5 to compute a denominator bounding.

### 3.5.3 M. Bronstein's Period 0 Denominator Bounding

Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbf{f} \in \mathbb{F}[t]^n$ and $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}[t]^m$. Furthermore let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)^{(1)}$ as a vector space over $\mathbb{K}$. We are interested in finding a $d \in \mathbb{F}[t]^*$ such that for all

$$\mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \overbrace{\mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}}^{\mathbb{F}(t)})$$

we have

$$d\,g \in \mathbb{F}[t] \oplus \mathbb{W}.$$

Given such a $d$ we can apply the reduction technique described in Section 3.1.3.1.

#### 3.5.3.1 A Simple Transformation of the Difference Equation

Without loss of generality we may assume that $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ where

$$\boxed{a_1 \neq 0 \neq a_m.}$$

If not, say $0 = a_1 = a_2 = \dots = a_{l-1} \neq a_l$ and $a_k \neq a_{k+1} = \dots = a_m = 0$ with $1 \leq l \leq k \leq m$ then

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$$
$$\Updownarrow$$
$$a_l\, \sigma^{m-l}(g) + \dots + a_k\, \sigma^{m-k}(g) = \mathbf{c}\,\mathbf{f}$$
$$\Updownarrow$$
$$\sigma^{k-m}(a_l)\, \sigma^{k-l}(g) + \dots + \sigma^{k-m}(a_k)\, g = \mathbf{c}\, \sigma^{k-m}(\mathbf{f})$$

where

$$\boxed{\sigma^{k-m}(a_l) \neq 0 \neq \sigma^{k-m}(a_k).}$$

Therefore define

$$\begin{aligned} \mathbf{a}' &:= \Big(\sigma^{k-m}(a_l), \sigma^{k-m}(a_{l+1}), \dots, \sigma^{k-m}(a_k)\Big) \in \mathbb{F}[t]^{k-l+1}, \\ \mathbf{f}' &:= \sigma^{k-m}(\mathbf{f}) \in \mathbb{F}[t]^n, \end{aligned}$$

and solve the problem $\mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}(t))$. Finally we get

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t)) = \Big\{\mathbf{c} \wedge \sigma^{m-k}(g) \,|\, \mathbf{c} \wedge g \in \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}(t))\Big\}. \tag{3.61}$$

#### 3.5.3.2 The Period 0 Denominator Bounding and its Consequences

Let $(\mathbb{A}[t], \sigma)$ be a difference ring with $t$ transcendental over $\mathbb{A}$ and let us recall the *spread* of $a, b \in \mathbb{A}[t]^*$ w.r.t. $\sigma$:

$$\mathrm{spread}_\sigma(a, b) = \{m \geq 0 \,|\, \deg(\gcd(a, \sigma^m(b))) > 0\}.$$

We have the following simple fact.

**Lemma 3.5.7.** *Let $(\mathbb{A}[t], \sigma)$ be a difference ring with $t$ transcendental over $\mathbb{A}$ and $a, b \in \mathbb{A}[t]^*$. If $a \in \mathbb{A}$ or $b \in \mathbb{A}$ then* $\mathrm{spread}_\sigma(a, b) = \emptyset$.

Additionally, if $(\mathbb{F}(t),\sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and $a,b\in\mathbb{F}[t]^*$ then by Theorem 2.2.5 $\operatorname{spread}_\sigma(a,b)$ is a finite set if and only if $(\mathbb{F}(t),\sigma)$ is a $\Sigma$-extension of $(\mathbb{F},\sigma)$ or $t\nmid\gcd(a,b)$. In this case we can define the following sequence.

**Definition 3.5.2.** Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$. Let $a,b\in\mathbb{F}[t]^*$ with $t\nmid\gcd(a,b)$ if $(\mathbb{F}(t),\sigma)$ is a $\Pi$-extension of $(\mathbb{F},\sigma)$. Let

$$\operatorname{spread}_\sigma(a,b)=\{m_1>m_2>\dots>m_s\}.$$

The sequence $\langle(p_i,q_i,u_i)\,|\,1\leq i\leq s+1\rangle$ is called *bounding sequence* of $a$ and $b$ if

1. $p_1:=a$, $q_1:=b$, $u_1:=1$ and
2. for $1\leq i\leq s$ we have iteratively

$$p_{i+1}:=\frac{p_i}{d_i},\qquad q_{i+1}:=\frac{q_i}{\sigma^{-m_i}(d_i)},\qquad u_{i+1}:=u_i\prod_{j=0}^{m_i}\sigma^{-j}d_i$$

where $d_i:=\gcd(p_i,\sigma^{m_i}(q_i))$.

$\Diamond$

In [Kar81] M. Karr developed an algorithm to compute the so called *$\sigma$-factorization* for an element in a $\Pi\Sigma$-field $(\mathbb{F}(t),\sigma)$. Given the $\sigma$-factorizations of $a,b\in\mathbb{F}[t]^*$, one can compute the spread of $a$ and $b$. Then the following theorem provides an algorithm to find a denominator bounding for the solutions of the fractional part with period 0.

**Theorem 3.5.6.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $\mathbf{a}=(a_1,\dots,a_m)\in\mathbb{F}[t]^m$ with $a_1\neq0\neq a_m$ and $\mathbf{f}\in\mathbb{F}[t]^n$. Let*

$$\tilde{a}_1:=\sigma^{m-1}(a_1),\qquad \tilde{a}_m:=\frac{a_m}{t^{\operatorname{ord}(a_m)}},$$

*and consider the bounding sequence*

$$\langle(p_i,q_i,u_i)\,|\,1\leq i\leq s+1\rangle$$

*of $\tilde{a}_1$ and $\tilde{a}_m$. Assume there are a $\mathbf{c}\in\mathbb{K}^n$ and a $g\in\mathbb{F}(t)$ with*

$$\sigma_{\mathbf{a}}g=\mathbf{c}\,\mathbf{f}$$

*where*

$$g=p\oplus g_1\oplus g_0\in\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}\oplus\mathbb{F}(t)^{(0)}.$$

*Then*

$$\operatorname{den}(g_0)\mid u_{s+1}.$$

*Proof.* The Theorem is a direct consequence of Theorems 8 and 10 of [Bro00]. $\square$

**Corollary 3.5.1.** *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $\mathbf{0}\neq\mathbf{a}\in\mathbb{F}^m$ and $\mathbf{f}\in\mathbb{F}[t]^n$. If there are a $\mathbf{c}\in\mathbb{K}^n$ and a $g\in\mathbb{F}(t)$ such that*

$$\sigma_{\mathbf{a}}g=\mathbf{c}\,\mathbf{f}$$

*then $g\in\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}$.*

*Proof.* Without loss of generality we may assume that $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ with $a_1 \neq 0 \neq a_m$. Otherwise transform the problem as described in Section 3.5.3.1 to $\mathbf{a}' = (a'_1, \dots, a'_l) \in \mathbb{F}^l$ with $a'_1 \neq 0 \neq a'_l$ and $\mathbf{f}' \in \mathbb{F}[t]^n$. If we can prove the corollary for this special case then by (3.61) the corollary follows also for the general case $\mathbf{a}$ and $\mathbf{f}$.

Define $\tilde{a}_1, \tilde{a}_m$ as in Theorem 3.5.6. It follows directly that $\tilde{a}_1, \tilde{a}_m \in \mathbb{F}$ and thus by Lemma 3.5.7 it follows that

$$\operatorname{spread}_\sigma(\tilde{a}_1, \tilde{a}_m) = \emptyset.$$

Therefore the bounding sequence of $\tilde{a}_1$ and $\tilde{a}_m$ is $\langle(\tilde{a}_1, \tilde{a}_m, 1)\rangle$, in particular $u_1 = 1$. By Theorem 3.5.6 it follows that for any $g \in \mathbb{F}(t)$ with

$$g = p \oplus g_1 \oplus g_0 \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \oplus \mathbb{F}(t)^{(0)}$$

we have

$$\operatorname{den}(g_0) \mid u_1 = 1$$

and thus $g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}$.

□

**Corollary 3.5.2.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Sigma$-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^m$ and $\mathbf{f} \in \mathbb{F}[t]^n$. If there are a $\mathbf{c} \in \mathbb{K}^n$ and a $g \in \mathbb{F}(t)$ such that*

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$$

*then $g \in \mathbb{F}[t]$.*

*Proof.* As $(\mathbb{F}(t), \sigma)$ is a $\Sigma$-extension of $(\mathbb{F}, \sigma)$, it follows by Corollary 3.1.5 that

$$\mathbb{F}(t)^{(1)} = \emptyset.$$

Thus the corollary is a direct consequence of Corollary 3.5.1. □

**Corollary 3.5.3.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^m$ and $\mathbf{f} \in \mathbb{F}[t]^n$. If there are a $\mathbf{c} \in \mathbb{K}^n$ and a $g \in \mathbb{F}(t)$ such that*

$$\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$$

*then $g = \frac{p}{t^k}$ for some $p \in \mathbb{F}[t]$ and $k \geq 0$.*

*Proof.* As $(\mathbb{F}(t), \sigma)$ is a $\Pi$-extension of $(\mathbb{F}, \sigma)$ it follows by Corollary 3.1.5 that

$$\mathbb{F}(t)^{(1)} = \mathbb{F}[1/t] \setminus \mathbb{F}^*.$$

Thus the corollary is a direct consequence of Corollary 3.5.1. □

## 3.6 Solutions in Some Special Difference Rings

Given a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$ and a primitive $k$-th root of unity $\alpha \in \mathbb{K}$, we want to sketch how one can solve difference equations in a difference ring extension $(\mathbb{F}[y],\sigma)$ with the relation

$$y^k = 1$$

canonically defined by

$$\sigma(y) = \alpha\, y.$$

Intuitively, we try to model the object

$$\alpha^k = \prod_{i=1}^{k} \alpha$$

in the difference field setting.

### 3.6.1 Some Special Difference Rings

Let $\mathbb{F}[x]$ be a polynomial ring with coefficients in a field $\mathbb{F}$ and consider the quotient ring $\mathbb{A} := \mathbb{F}[x]/\langle x^k - 1\rangle$. By the remarks in Section 2.4.8, the ring $\mathbb{A}$ consists of cosets

$$p + \langle x^k - 1\rangle$$

where $p \in \mathbb{F}[x]$. By the division algorithm there are $r, q \in \mathbb{F}[x]$ with $\deg(r) < k$ such that

$$p = q\,(x^k - 1) + r$$

and therefore we have

$$p + \langle x^k - 1\rangle = r + q\,(x^k - 1) + \langle x^k - 1\rangle = r + \langle x^k - 1\rangle.$$

Consequently we can represent the cosets of the ring $\mathbb{A}$ by the elements

$$\mathcal{R} := \{p \in \mathbb{F}[x] \mid \deg(p) < k\}.$$

A classical result is the following lemma.

**Lemma 3.6.1.** *Let $\mathbb{F}$ be a field and $\mathbb{F}[y]$ be a ring with $y^k = 1$. Then*

$$\begin{array}{ccc} \mathbb{F}[y] & \simeq & \mathbb{A} \\ \sum_{i=0}^{k-1} f_i\, y^i & \mapsto & \sum_{i=0}^{k-1} f_i\, x^i. \end{array}$$

**Lemma 3.6.2.** *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$. Let $\alpha \in \mathbb{K}$ be a primitive $k$-th root of unity and consider the ring extension $\mathbb{F}[y]$ of $\mathbb{F}$ with*

$$y^k = 1.$$

*Then $(\mathbb{F}[y],\sigma)$ canonically defined by*

$$\sigma(y) = \alpha\, y$$

*is - up to a difference ring isomorphism - a unique difference ring extension of $(\mathbb{F},\sigma)$ and*

$$\mathrm{const}_\sigma \mathbb{F}[y] = \mathrm{const}_\sigma \mathbb{F}.$$

*Proof.* Let $\mathbb{F}[y]$ be a ring with $y^k = 1$ as stated above. We have to show that $\sigma$ canonically defined by $\sigma(y) = \alpha\, y$ is an homomorphism. So assume $a, b \in \mathbb{F}[y]$ with

$$\begin{aligned} a &= a_0 + a_1\, y + \cdots + a_{k-1}\, y^{k-1}, \\ b &= b_0 + b_1\, y + \cdots + b_{k-1}\, y^{k-1}. \end{aligned}$$

Clearly we have

$$\sigma(a + b) = \sigma(a) + \sigma(b).$$

Now consider

$$a\, b = (a_0 + a_1\, y + \cdots + a_{k-1}\, y^{k-1})\,(b_0 + b_1\, y + \cdots + b_{k-1}\, y^{k-1}).$$

We have

$$[a\, b]_i = a_0\, b_i + a_1\, b_{i-1} + \cdots + a_{n-1}\, b_{i+1}.$$

Looking at

$$\sigma(a)\,\sigma(b) = (\sigma(a_0) + \cdots + \sigma(a_{k-1})\,\alpha^{k-1}\, y^{k-1})\,(\sigma(a_0) + \cdots + \sigma(a_{k-1})\,\alpha^{k-1}\, y^{k-1})$$

we find

$$[\sigma(a)\,\sigma(b)]_i = \sigma(a_0)\,\sigma(b_i)\,\alpha^i + \sigma(a_1)\,\alpha\,\sigma(b_{i-1})\,\alpha^{i-1} + \cdots + \sigma(a_{k-1})\,\alpha^{k-1}\,\sigma(b_{i+1})\,\alpha^{i+1})$$

and thus by using

$$\alpha^k = 1$$

we get

$$\begin{aligned} [\sigma(a)\,\sigma(b)]_i\, y^i &= (\sigma(a_0)\,\sigma(b_i) + \sigma(a_1)\,\sigma(b_{i-1}) + \cdots + \sigma(a_{k-1})\,\sigma(b_{i+1}))\,\alpha^i\, y^i \\ &= \sigma(a_0\, b_i + a_1\, b_{i-1} + \cdots + a_{k-1}\, b_{i+1})\,\alpha^i\, y^i \\ &= \sigma([a\, b]_i\, y^i). \end{aligned}$$

Consequently

$$\sigma(a\, b) = \sigma(a)\,\sigma(b),$$

and therefore $\sigma$ is a difference ring homomorphism. As we can easily construct the inverse by

$$\sigma^{-1}(y) = \alpha^{k-1}\, y,$$

it follows immediately that $\sigma$ is a difference ring automorphism.

Now assume that $\mathrm{const}_\sigma \mathbb{F}[y] \neq \mathrm{const}_\sigma \mathbb{F}$. This means that we can take an $f \in \mathbb{F}[y] \setminus \mathbb{F}$ such that

$$\sigma(f) = f,$$

in particular, $f = \sum_{i=0}^{l} f_i\, y^i \in \mathbb{F}[y] \setminus \mathbb{F}$ with $f_i \in \mathbb{F}$ and $f_l \neq 0$ for some $0 < l < k$. Then

$$\sum_{i=0}^{l} f_i\, y^i = f = \sigma(f) = \sum_{i=0}^{l} \sigma(f_i)\,\alpha^i\, y^i,$$

therefore

$$f_l = \sigma(f_l)\,\alpha^l$$

and consequently

$$\alpha^l \in \mathbb{F},$$

a contradiction to the assumption that $\alpha$ is a primitive $k$-th root of unity. Now assume there is an other difference ring extension $(\mathbb{F}[y'], \sigma)$ with ${y'}^k = 1$ canonically defined by $\sigma(y') = \alpha\, y'$. By Lemma 3.6.1 it follows that

$$\mathbb{F}[y] \simeq \mathbb{F}[x]/\langle x^k - 1\rangle \simeq \mathbb{F}[y']$$

and thus there exists a ring isomorphism $\tau : \mathbb{F}[y] \to \mathbb{F}[y']$ canonically defined by

$$\tau(y) = y'.$$

Since

$$\tau(\sigma(y)) = \tau(\alpha\, y) = \alpha\, y' = \sigma(y') = \sigma(\tau(y))$$

it follows immediately that $\tau$ is a difference ring isomorphism. □

**Lemma 3.6.3.** *The map*

$$\sigma : \begin{cases} \mathbb{A} & \to \quad \mathbb{A} \\ \sum_{i=0}^{k-1} f_i\, x^i & \mapsto \quad \sum_{i=0}^{k-1} \sigma(f_i)\, \alpha^i\, x^i \end{cases}$$

*is a ring automorphism.*

*Proof.* By Lemma 3.6.1 there exists a ring isomorphism $\tau : \mathbb{F}[y] \to \mathbb{A}$ with $\tau(y) = x$ and by Lemma 3.6.2 there exists a difference ring extension $(\mathbb{F}[y], \sigma)$ canonically defined by

$$\sigma(y) = \alpha\, y.$$

Therefore by Lemma 2.4.7 it follows that

$$\sigma' : \begin{cases} \mathbb{F}[t] & \to \quad \mathbb{F}[t] \\ f & \mapsto \quad \tau(\sigma(\tau^{-1}(f))) \end{cases}$$

is a ring automorphism. We have

$$\sigma'(y) = \tau(\sigma(\tau^{-1}(y))) = \tau(\sigma(x)) = \tau(\alpha\, x) = \alpha\, y$$

and thus $\sigma'$ is canonically defined by $\sigma'(y) = \alpha\, y$. □

**Proposition 3.6.1.** *Let $(\mathbb{F}, \sigma)$ be a difference field, $\alpha \in \mathbb{K}$ be a primitive $k$-th root of unity and $\mathbb{F}[y]$ be a ring with $y^k = 1$. Let $(\mathbb{F}[y], \sigma)$ and $(\mathbb{A}, \sigma)$ be the difference ring extensions of $(\mathbb{F}, \sigma)$ canonically defined by*

$$\sigma(x) = \alpha\, x, \qquad\qquad \sigma(y) = \alpha\, y.$$

*Then*

$$(\mathbb{F}[y], \sigma) \simeq (\mathbb{A}, \sigma)$$

*and* $\mathrm{const}_\sigma \mathbb{F}[y] = \mathrm{const}_\sigma \mathbb{A} = \mathrm{const}_\sigma \mathbb{F}$.

*Proof.* By Lemma 3.6.2 the difference ring $(\mathbb{F}[y], \sigma)$ is -up to a difference ring isomorphism- uniquely defined and therefore

$$(\mathbb{F}[y], \sigma) \simeq (\mathbb{A}, \sigma).$$

As a consequence the constant fields must be the same and because of $\mathrm{const}_\sigma \mathbb{F}[y] = \mathrm{const}_\sigma \mathbb{F}$ by Lemma 3.6.2, it follows that

$$\mathrm{const}_\sigma \mathbb{F}[y] = \mathrm{const}_\sigma \mathbb{F}[x] = \mathrm{const}_\sigma \mathbb{F}.$$

□

### 3.6.2 Zero Divisors and Invertible Elements

The ring $\mathbb{A} := \mathbb{F}[x]/\langle x^k - 1\rangle$ has zero divisors: The elements $(x-1)$ and $(1+x+x^2+\cdots x^{k-1})$ in $\mathbb{A}$ are both different from 0, but we have

$$(x-1)\,(1+x+x^2+\cdots x^{k-1}) = x^k - 1$$

where $x^k - 1$ represents the 0 element.

**Lemma 3.6.4.** *Let $u \in \mathcal{R}^*$. Then $u$ is invertible in $\mathbb{A}$ if and only if*

$$\gcd_{\mathbb{F}[x]}(u, x^k - 1) = 1.$$

*Proof.* Let $u, s \in \mathcal{R}^*$. We have

$$\begin{aligned} u\,s =_{\mathbb{A}} 1 &\Leftrightarrow u\,s \equiv 1 \mod x^k - 1 \\ &\Leftrightarrow u\,s + (x^k - 1)\,p = 1 \end{aligned}$$

for some $p \in \mathbb{F}[x]$ and thus equivalently

$$\gcd_{\mathbb{F}[x]}(u, x^k - 1) = 1.$$

□

**Corollary 3.6.1.** *Let $u \in \mathcal{R}^*$. Then $u$ is a zero divisor if and only if*

$$\gcd_{\mathbb{F}[x]}(u, x^k - 1) \neq 1.$$

*Proof.* This is a consequence of Lemma 2.4.6. □

Consequently the invertible elements are exactly those elements which are nonzero divisors.

### 3.6.3 The Reduction Process and Two Base Cases

In the following we assume that $(\mathbb{F}(t), \sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$, $\alpha \in \mathbb{K}$ is a primitive $k$-th root of unity and we consider the difference ring extension $(\mathbb{F}(t)[x]/\langle x^k - 1\rangle, \sigma)$ canonically defined by

$$\sigma(x) = \alpha\, x.$$

In particular we have

$$\mathrm{const}_\sigma(\mathbb{F}(t)[x]/\langle x^k - 1\rangle) = \mathbb{K}.$$

**Lemma 3.6.5.** *Let $\mathbb{F}[t, x]$ be a polynomial ring with coefficients in $\mathbb{F}$. Then*

$$\mathbb{F}[t][x]/\langle x^k - 1\rangle \simeq (\mathbb{F}[x]/\langle x^k - 1\rangle)[t].$$

**Proposition 3.6.2.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\alpha \in \mathbb{K}$ be a primitive $k$-th root of unity. Let $(\mathbb{F}(t)[x]/\langle x^k - 1\rangle, \sigma)$ be the difference ring extension of $(\mathbb{F}(t), \sigma)$ canonically defined by*

$$\sigma(x) = \alpha\, x.$$

*Then $((\mathbb{F}[x]/\langle x^k - 1\rangle)[t], \sigma)$ is a difference ring with*

$$(\mathbb{F}[t][x]/\langle x^k - 1\rangle, \sigma) \simeq ((\mathbb{F}[x]/\langle x^k - 1\rangle)[t], \sigma).$$

*Let $\mathbb{A} := \mathbb{F}[x]/\langle x^k - 1\rangle$. Then $\mathbb{A}[t]$ is a polynomial ring with coefficients in $\mathbb{A}$.*

Let

$$\mathbb{B} := \mathbb{F}(t)[x]/\langle x^k - 1\rangle$$

and let $\mathbf{0} \neq \mathbf{a} \in \mathbb{B}^m$ and $\mathbf{f} \in \mathbb{B}^n$. In the following I will sketched how one can try to apply the reduction technique, described in Section 3.2.6 for $\Pi\Sigma$-fields, to the problem of solving the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{B})$ in the difference ring $(\mathbb{B}, \sigma)$. Let

$$\mathbb{A} := \mathbb{F}[x]/\langle x^k - 1\rangle.$$

Then by Proposition 3.6.2 we have

$$(\mathbb{F}[t][x]/\langle x^k - 1\rangle, \sigma) \simeq (\mathbb{A}[t], \sigma) \tag{3.62}$$

where $\mathbb{A}[t]$ is a polynomial ring with coefficients in $\mathbb{A}$, i.e. $t$ is transcendental over $\mathbb{A}$. By this observation we can try to apply Theorems 3.2.1 and 3.2.2 in order to obtain the following reduction process (see also Section 3.2.6):

$$\begin{tikzcd}
 & \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t)[x]/\langle x^k - 1\rangle) \arrow[d, shift left] & & \boxed{\text{by denominator bounding ??}} & \\
 & \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{F}[t][x]/\langle x^k - 1\rangle) \arrow[u, shift left] \arrow[d, equal] & & \text{by (3.62)} & \\
 & \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{A}[t]) \arrow[d, equal] & & \boxed{\text{by polynomial degree bounding??}} & \\
 & \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{A}[t]_b) \arrow[rr, shift left, "1."] & & & \\
\text{Theorem 3.2.2} & & & \mathrm{I}(\mathbf{a}', \mathbf{f}', t^b\, \mathbb{A}) \arrow[llu, shift left, "5."] \arrow[lld, "4."] \arrow[d, shift right, "2."'] & \text{Theorem 3.2.1} \\
 & \mathrm{V}(\mathbf{a}', \mathbf{f}'', \mathbb{A}[t]_{b-1}) \arrow[uu, "5."] & & \mathrm{V}(\tilde{\mathbf{a}}', \tilde{\mathbf{f}}', \mathbb{A}) \arrow[u, shift right, "3."'] &
\end{tikzcd}$$

### Open Problems

• In order to apply the reduction process sketched above, one has to find a denominator bound $d \in \mathbb{F}[t]^*$ such that for all $g \in \mathbb{F}(t)[x]/\langle x^k - 1\rangle$ and $\mathbf{c} \in \mathbb{K}^n$ with $\sigma_{\mathbf{a}} g = \mathbf{c}\,\mathbf{f}$ we have

$$g\,d \in \mathbb{F}[t][x]/\langle x^k - 1\rangle.$$

• An other open problem is to find a bound $b$ for the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}[t])$, this means to find a $b$ such that

$$\mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{A}[t]_b) = \mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{A}[t]).$$

• Furthermore, we must assume that the solution spaces $\mathrm{V}(\mathbf{a}', \mathbf{f}', \mathbb{A}[t]_b)$ and $\mathrm{V}(\tilde{\mathbf{a}}', \tilde{\mathbf{f}}', \mathbb{A})$ are finite dimensional in order to represent the vector spaces by basis matrices. In other words, one has to guarantee that $\mathbf{a}'$ and $\tilde{\mathbf{a}}'$ are V-finite. Here I have the following conjecture.

**Conjecture 3.6.1.** *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$ and $\alpha \in \mathbb{K}$ be a primitive $k$-th root of unity. Let $(\mathbb{F}[y], \sigma)$ be a difference ring extension of $(\mathbb{F}, \sigma)$ with $\sigma(y) = \alpha\,y$ and $y^k = 1$. Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}[y]^m$ with $a_1 \neq 0 \neq a_m$. Then $\mathbf{a}$ is* V*-finite, if $a_1$ and $a_n$ are units in $\mathbb{F}[y]$.*

If this conjecture holds, then one can easily check by Lemma 3.6.4, if $\mathbf{a}'$ and $\tilde{\mathbf{a}}'$ are V-finite.

Finally, if we succeed in doing all these reduction steps properly, we have to consider two base cases (compare with Section 3.2.6).

### The First Base Case

In the reduction process we finally reach the point to find all solutions $\mathrm{V}(\mathbf{a}', \mathbf{p}, \mathbb{A}[t]_{-1})$ for some $\mathbf{p} \in \mathbb{A}[t]^r_{\|\mathbf{a}\|-1}$ and $r \geq 1$. By Theorem 3.1.4 we have

$$\mathrm{V}(\mathbf{a}, \mathbf{p}, \mathbb{A}[t]_{-1}) = \mathrm{V}(\mathbf{a}, \mathbf{p}, \{0\}) = \mathrm{Nullspace}_{\mathbb{K}}(\mathbf{p}) \times \{0\}$$

and therefore we have to deal with the problem to find all solutions of $\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{p})$. Since

$$\mathbb{A}[t] = (\mathbb{F}[x]/\langle x^k - 1\rangle)[t] \simeq \mathbb{F}[t][x]/\langle x^k - 1\rangle,$$

the following Lemma tells us that $\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{p})$ is a finite dimensional vector space over $\mathbb{K}$ and how one can compute a basis of $\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{p})$.

**Lemma 3.6.6.** *Let $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{K}(t_1, \dots, t_e)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ and consider the difference ring extension $(\mathbb{F}[x]/\langle x^k - 1\rangle, \sigma)$ canonically defined by*

$$\sigma(x) = \alpha\,x.$$

*Let $\mathbf{f} \in (\mathbb{F}[x]/\langle x^k - 1\rangle)^n$. Then* $\mathrm{Nullspace}_{\mathbb{K}}(\mathbf{f})$ *is a finite dimensional subspace of $\mathbb{K}^n$ and a basis can be computed by linear algebra.*

*Proof.* Let $\mathbf{f} = (f_1, \dots, f_n) \in (\mathbb{F}[x]/\langle x^k - 1\rangle)^n$. Since $\mathbb{F}$ is a $\Pi\Sigma$-field, it follows that $\mathbb{F}$ is the quotient field of the polynomial ring $\mathbb{K}[t_1, \dots, t_e]$. We can find a $d \in \mathbb{K}[t_1, \dots, t_e]^*$ such that

$$\mathbf{g} = (g_1, \dots, g_n) := (f_1\,d, \dots, f_n\,d) \in \mathbb{K}[t_1, \dots, t_e][x]/\langle x^k - 1\rangle.$$

Furthermore we can assume that

$$g_i = \sum_{j=0}^{k-1} g_{ij}\, x^j.$$

For $\mathbf{c} \in \mathbb{K}^n$ we have

$$\mathbf{c}\,\mathbf{f} = 0 \;\Leftrightarrow\; \mathbf{c}\,\mathbf{g} = 0$$

and therefore

$$\text{Nullspace}_{\mathbb{K}}(\mathbf{f}) = \text{Nullspace}_{\mathbb{K}}(\mathbf{g}).$$

Let $c_1, \dots, c_n$ be indeterminates and make the ansatz

$$A := c_1\, g_1 + \dots + c_n\, g_n = 0.$$

We have

$$A = \sum_{j=0}^{k-1} x^j \sum_{i=0}^{n} c_i\, g_{ij}$$

and therefore $A = 0$ if and only if

$$A_i := \sum_{i=0}^{n} c_i\, g_{ij}$$

for all $1 \leq j < k$. Since $\mathbb{K}[t_1, \dots, t_e]$ is a polynomial ring over $\mathbb{K}$, the coefficients of each monomial $t_1^{d_1} \dots t_e^{d_e}$ in $A_i$ must vanish. Therefore we obtain a set of linear systems of equations

$$\left\{ \left. \begin{array}{llll} c_1\, p_{i11} + & \dots & + c_n\, p_{i1n} & = 0 \\ & \vdots & & \\ c_r\, p_{ir1} + & \dots & + c_n\, p_{irn} & = 0 \end{array} \right| \; 0 \leq i < k \right\} \tag{3.63}$$

where in the $i$-th subsystem an equation corresponds to a coefficient of a monomial in $A_i$ which must vanish. Since $p_{ijk} \in \mathbb{K}$, finding all $\mathbf{c} \in \mathbb{K}^n$ such that $\mathbf{c}$ is a solution of (3.63) is a simple linear algebra problem. In particular, applying Gaussian elimination we get immediately a basis of the vector space

$$\{\mathbf{c} \in \mathbb{K}^n \,|\, \mathbf{c} \text{ is a solution of (3.63)}\},$$

thus for $\text{Nullspace}_{\mathbb{K}}(\mathbf{g})$ and consequently also for $\text{Nullspace}_{\mathbb{K}}(\mathbf{f})$. □

### The Second Base Case

Let $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{K}(t_1, \dots, t_n)$ be a $\Pi\Sigma$-field over $\mathbb{K}$, $\alpha \in \mathbb{K}$ be a primitive $k$-th root of unity and consider the difference ring extension $(\mathbb{F}(t)[x]/\langle x^k - 1\rangle, \sigma)$ canonically defined by

$$\sigma(x) = \alpha\, x.$$

Then similar as in Section 3.2.6.3 we obtain the following second base case during the reduction process:

$$
\begin{array}{c}
\mathrm{V}(\mathbf{a_e}, \mathbf{f_e}, \mathbb{K}(t_1, \dots, t_{e-1})(t_e)[x]/\langle x^k - 1\rangle) \\
\quad || \qquad \text{by denominator bounding} \\
\mathrm{V}(\mathbf{a}'_\mathbf{e}, \mathbf{f}'_\mathbf{e}, \mathbb{K}(t_1, \dots, t_{e-1})[t_e][x]/\langle x^k - 1\rangle) \\
\quad || \qquad \text{Proposition 3.6.2} \\
\mathrm{V}(\mathbf{a}'_\mathbf{e}, \mathbf{f}'_\mathbf{e}, (\mathbb{K}(t_1, \dots, t_{e-1})[x]/\langle x^k - 1\rangle)[t_e]) \\
\quad || \qquad \text{by polynomial degree bounding} \\
\mathrm{V}(\mathbf{a}'_\mathbf{e}, \mathbf{f}'_\mathbf{e}, (\mathbb{K}(t_1, \dots, t_{e-1})[x]/\langle x^k - 1\rangle)[t_e]_b) \\
\downarrow \quad \uparrow \qquad \text{Incr. Reduction (Theorem 3.2.1)} \\
\mathrm{V}(\mathbf{a_{e-1}}, \mathbf{f_{e-1}}, \mathbb{K}(t_1, \dots, t_{e-1})[x]/\langle x^k - 1\rangle) \\
\downarrow \quad \uparrow \\
\vdots \quad \vdots \\
\downarrow \quad \uparrow \\
\mathrm{V}(\mathbf{a_1}, \mathbf{f_1}, \mathbb{K}[x]/\langle x^k - 1\rangle).
\end{array}
$$

Let

$$\mathbb{A} := \mathbb{K}[x]/\langle x^k - 1\rangle,$$

$\mathbf{0} \neq \mathbf{a} \in \mathbb{A}^m$ and $\mathbf{f} \in \mathbb{A}^n$. In the following we will show that

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}) = \Big\{\mathbf{c}\wedge g \in \mathbb{K}^k \times \mathbb{A} \,|\, \sigma_\mathbf{a} g = \mathbf{c}\,\mathbf{f}\Big\}$$

is a finite dimensional vector space over $\mathbb{K}$ and how we can compute a basis by linear algebra. Consider the vector space

$$\mathbb{V} := \left\{(\mathbf{c}\wedge\mathbf{g}) \in \mathbb{K}^n \times \mathbb{K}^k \,|\, \sigma_\mathbf{a}(\mathbf{g}\begin{pmatrix} x^0 \\ \vdots \\ x^{k-1}\end{pmatrix}) = \mathbf{c}\,\mathbf{f}\right\} \tag{3.64}$$

over $\mathbb{K}$.

**Lemma 3.6.7.** *Let $(\mathbb{K}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\alpha \in \mathbb{K}$ be a primitive $k$-th root of unity and consider the difference ring extension $(\mathbb{K}[x]/\langle x^k - 1\rangle, \sigma)$ of $(\mathbb{K}, \sigma)$ canonically defined by*

$$\sigma(x) = \alpha\, x.$$

*Let $\mathbf{f} \in \mathbb{K}^n$. Then the vector space $\mathbb{V}$ over $\mathbb{K}$ defined by (3.64) is finite dimensional and a basis of $\mathbb{V}$ can be computed by linear algebra.*

*Proof.* Consider the polynomial

$$p = \sigma_\mathbf{a}(\sum_{i=0}^{k-1} g_i\, x^i) - \sum_{i=1}^{n} c_i\, f_i \in \mathbb{K}[c_1 \dots, c_n, g_0 \dots, g_{k-1}][x]$$

in the unknowns $c_1 \dots, c_n$ and $g_0, \dots, g_{k-1}$. By using the relation $x^n = 1$, the polynomial $p$ can be represented by

$$p = \sum_{i=0}^{k-1} p_i\, x^i$$

where $p_i \in \mathbb{K}[c_1 \dots, c_n, g_0 \dots, g_{k-1}]$, in particular

$$p_i = c_1\, p_{i1} + \dots + c_n\, p_{in} + q_{i0}\, g_0 + \dots + q_{i,k-1}\, g_{k-1}$$

for some $p_{ij}, q_{ij} \in \mathbb{K}$. Then we get

$$p = 0 \quad \Leftrightarrow \quad (p_0 = 0 \;\&\; p_1 = 0 \;\&\dots\&\; p_{k-1} = 0).$$

Therefore solving this linear system of equations in the unknowns $c_i$ and $g_i$ delivers a basis of the vector space $\mathbb{V}$. $\square$

Given a basis of $\mathbb{V}$ the following theorem states how we can compute a basis for the solution space $\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A})$.

**Theorem 3.6.1.** *Let $(\mathbb{K}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\alpha \in \mathbb{K}$ be a primitive $k$-th root of unity and consider the difference ring extension $(\mathbb{K}[x]/\langle x^k - 1\rangle, \sigma)$ of $(\mathbb{K}, \sigma)$ canonically defined by*

$$\sigma(x) = \alpha\, x.$$

*Let $\mathbf{f} \in \mathbb{K}^n$ and consider the vector space $\mathbb{V}$ over $\mathbb{K}$ defined by (3.64). Let*

$$\mathbb{V} \xleftrightarrow{\text{basis}} \mathbf{C}{\wedge}\mathbf{G}$$

*where $\mathbf{C} \in \mathbb{K}^{r \times n}$ and $\mathbf{G} \in \mathbb{K}^{r \times k}$ for some $r \geq 1$. Then*

$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}) \xleftrightarrow{\text{basis}} \mathbf{C}{\wedge}\mathbf{g}$$

*where*

$$\mathbf{g} := \mathbf{G} \cdot \begin{pmatrix} x^0 \\ \vdots \\ x^{k-1} \end{pmatrix}.$$

*Proof.* We have

$$\begin{gathered} \mathbf{c}{\wedge}h \in \mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{A}) \\ \Updownarrow \\ \exists \mathbf{p} \in \mathbb{K}^k : \; \sigma_{\mathbf{a}} h = \mathbf{c}\,\mathbf{f} \;\&\; h = \mathbf{p} \begin{pmatrix} x^0 \\ \vdots \\ x^{k-1} \end{pmatrix} \\ \Updownarrow \\ \exists \mathbf{p} \in \mathbb{K}^k : \mathbf{c}{\wedge}\mathbf{p} \in \mathbb{V} : \; h = \mathbf{p} \begin{pmatrix} x^0 \\ \vdots \\ x^{k-1} \end{pmatrix} \end{gathered}$$

and thus the theorem follows. $\square$

**Implementation Note 3.6.1.** Assume one has given a recurrence in terms of sums and products, in particular with a product expression of the form

$$\prod_{i=1}^{r} \alpha$$

where $\alpha$ is a primitive $k$-th root of unity. Moreover, assume this recurrence can be translated during the function call `SolveRecurrence` or `FindSumSolutions` into a difference equation expressed in a difference ring $(\mathbb{F}[y], \sigma)$ of the following form:

- $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$ with $\alpha \in \mathbb{K}$.
- $(\mathbb{F}[y], \sigma)$ is the difference ring extension of $(\mathbb{F}, \sigma)$ canonically defined by

$$\sigma(y) = \alpha\, y.$$

Then by setting the option

$$\texttt{WithMinusPower}- > \texttt{True}$$

in the function `SolveRecurrence` or `FindSumSolutions`, one invokes the reduction technique described above in order to solve the difference equation and therefore to find solutions for the corresponding recurrence in terms of sums and products.
Although there are a lot of open problems (see page 240), in examples this reduction strategy turns out to be very powerful by using heuristical methods. Partially I have already introduced these heuristics in Sections 3.3.1 and 3.5.2.1.

# Chapter 4

# Summation and Difference Field Extensions

In this chapter we are interested in finding appropriate difference field extensions in order to deal with symbolic summation problems. In Section 4.1 we will summarize all results of the previous chapter which give us information about solutions of a given difference equation. These results will be the basis for further investigations.

In Section 4.2 I will consider a special case of Karr's Fundamental Theorem [Kar81, Section 4] for proper product-sum extensions. Loosely speaking, the main result of this section is that looking for an appropriate difference field extension such that there exists a solution of a given difference equation

$$\sigma(g) - g = f$$

means to look for **one** appropriate sum over $\mathbb{F}$.

This result seems in the beginning quite disappointing. Contrary, in Section 4.3 we are able to give new insight for the creative telescoping method in a given difference field by using the Fundamental Theorem.

Furthermore in Section 4.4 the results of the Fundamental Theorem motivates us to search for **one** appropriate sum extension where the "summand" consists of terms as simple as possible in the underlying difference field. In that section we will describe how we can find such appropriate sum extensions by using Karr's reduction method sketched in Section 3.2.6. Additionally, we can use this idea to find recurrences of lower order for a given definite summation problem by choosing an appropriate sum extension.

Finally in Section 4.5 we consider the problem to find appropriate sum extensions in which additional solutions exist for a given difference equation of any order. Additionally, we sketch how one can find d'Alembertian extensions which deliver further solutions of a given difference equation.

### Some Definitions and Remarks

Let $(\mathbb{E}, \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ and let $\mathbf{a} = (a_m, \dots, a_0) \in \mathbb{F}^{m+1}$ with $a_m \neq 0$ . We call

$$\mathcal{L} := a_m \, \sigma^m + a_{m-1} \, \sigma^{m-1} + \cdots + a_0$$

a *difference operator* with *order* $m \geq 0$, in symbols,

$$\text{order}(\mathcal{L}) = m.$$

If we apply $g \in \mathbb{E}$ on $\mathcal{L}$, we write

$$\mathcal{L}(g) = a_m\,\sigma^m(g) + a_{m-1}\,\sigma^{m-1}(g) + \cdots + a_0\,g.$$

In the notation of the previous chapter we have for all $g \in \mathbb{F}$ that

$$\mathcal{L}(g) = \sigma_{\mathbf{a}} g. \tag{4.1}$$

Given the difference operator $\mathcal{L}$, we write

$$\operatorname{vect}(\mathcal{L}) = (a_m, \dots, a_0) = \mathbf{a} \in \mathbb{F}^{\operatorname{order}(\mathcal{L})+1}$$

to extract the vector $\mathbf{a}$ from the difference operator $\mathcal{L}$ such that we have (4.1) for all $g \in \mathbb{F}$. Furthermore we will write

$$\ker \mathcal{L} = \{g \in \mathbb{E} \,|\, \mathcal{L}(g) = 0\}$$

for the *kernel* of the difference operator $\mathcal{L}$ and will consider it as a subspace of $\mathbb{E}$ over $\mathbb{K}$.

As described in [BP96] we can consider

$$\mathbb{F}[\sigma] := \left\{ \sum_{i=0}^{n} a_i\,\sigma^i \,|\, n \in \mathbb{N}_0 \,\&\, a_i \in \mathbb{F} \right\}$$

as a noncommutative polynomial ring with usual addition and multiplication given by

$$\sigma\,a = \sigma(a)\,\sigma.$$

By convention we define $\operatorname{order}(0) = -\infty$. In this polynomial ring we can compute the *Euclidean right division* of $A \in \mathbb{F}[\sigma]$ by $B \in \mathbb{F}[\sigma]^*$ and obtain $Q, R \in \mathbb{F}[\sigma]$ such that

$$A = Q\,B + R$$

where $\operatorname{order}(R) < \operatorname{order}(B)$.

## 4.1 Solutions in Reduced Product-Sum Extensions

In this section we collect all results of Chapter 3 which give us information of the solutions of a given recurrence $\mathcal{L} \in \mathbb{F}[\sigma]$ in a reduced product sum extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$.

**Proposition 4.1.1.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$, $f = \sum_i f_i\, t^i \in \mathbb{F}[t]$ and $g \in \mathbb{F}(t)$ with*

$$\mathcal{L}(g) = f.$$

*Then $g = \sum_i g_i\, t^i \in \mathbb{F}[t, \frac{1}{t}]$ and*

$$\mathcal{L}(g_i\, t^i) = f_i\, t^i$$

*for all $i \in \mathbb{Z}$.*

*Proof.* By Corollary 3.5.3 it follows that $g = \sum_i g_i\, t^i \in \mathbb{F}[t, \frac{1}{t}]$. Therefore

$$\mathcal{L}(g_i\, t^i) = h_i\, t^i$$

for some $h_i \in \mathbb{F}$ and consequently

$$\mathcal{L}(g) = \sum_i h_i\, t^i.$$

As $t$ is transcendental over $\mathbb{F}$, we may conclude by coefficient matching that $h_i = f_i$. □

**Corollary 4.1.1.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$, $f \in \mathbb{F}$ and $g \in \mathbb{F}(t)$ such that*

$$\sigma(g) - g = f.$$

*Then $g \in \mathbb{F}$.*

*Proof.* Assume $g \notin \mathbb{F}$. Then by Proposition 4.1.1 there exists an $i \in \mathbb{Z}^*$ and a $d \in \mathbb{F}^*$ such that

$$\sigma(d\, t^i) - d\, t^i = 0$$

$$\Updownarrow$$

$$\frac{\sigma(d)}{d} = \alpha^{-i}.$$

If $i < 0$ then $(\mathbb{F}(t), \sigma)$ is not a $\Pi$-extension of $(\mathbb{F}, \sigma)$ by Corollary 2.2.2, a contradiction. Otherwise we have

$$\frac{\sigma(h)}{h} = \alpha^i$$

for $h := 1/d$, again a contradiction by Corollary 2.2.2. □

**Proposition 4.1.2.** *Let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and let $\mathcal{L} \in \mathbb{F}[\sigma]^*$ with $m := \operatorname{order}(\mathcal{L}) \geq 0$. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{F}(t)$ with*

$$\mathcal{L}(g) = f.$$

*Then $g \in \mathbb{F}[t]$.*

*Proof.* This is just a reformulation of Corollary 3.5.2. □

**Theorem 4.1.1.** *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field and let $(\mathbb{F}(t),\sigma)$ be a proper sum extension of $(\mathbb{F},\sigma)$; let $\mathcal{L}\in\mathbb{F}[\sigma]^*$ with $m:=\operatorname{order}(\mathcal{L})\geq 0$ and let $f\in\mathbb{F}[t]$, $g\in\mathbb{F}(t)$ with*

$$\mathcal{L}(g)=f.$$

*Then $g\in\mathbb{F}[t]$ with $\deg(g)\leq m+\deg(f)$. Furthermore there is a linearly independent set $\{d_0,\dots,d_{l-1}\}\subseteq\mathbb{F}[t]$ over $\mathbb{K}$ for $l:=\deg(g)-\deg(f)$ with*

$$\begin{aligned}\mathcal{L}(d_i)&=0,\\ \deg(d_i)&=i\end{aligned}$$

*for all $1\leq i<l$.*

*Proof.* By Proposition 4.1.2 it follows that $g\in\mathbb{F}[t]$. If $l=\deg(g)-\deg(f)$ then by Corollary 3.4.12 there is a linearly independent set $\{d_0,\dots,d_{l-1}\}\subseteq\mathbb{F}[t]$ over $\mathbb{K}$ with the above properties. Furthermore by Corollary 3.4.13 we have $\deg(g)\leq\deg(f)+m$. $\square$

If $\mathcal{L}\in\mathbb{F}[\sigma]$ has order 1, we get the following special case in case $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field. Here we will prove a more general statement for any difference field $(\mathbb{F},\sigma)$.

**Lemma 4.1.1.** *Let $(\mathbb{F}(t),\sigma)$ be a proper sum extension $(\mathbb{F},\sigma)$, $\mathcal{L}=a_1\,\sigma+a_2\in\mathbb{F}[\sigma]^*$, $f\in\mathbb{F}[t]$ and $g\in\mathbb{F}(t)$ with*

$$\mathcal{L}(g)=a_1\,\sigma(g)+a_2\,g=f.$$

*Then $g\in\mathbb{F}[t]$ with $\deg(g)\leq\deg(f)+1$. Furthermore, if $\deg(g)=\deg(f)+1$ then there is a $d\in\mathbb{F}^*$ with*

$$a_1\,\sigma(d)-a_2\,d=0.$$

*Proof.* By Proposition 4.1.2 it follows that $g\in\mathbb{F}[t]$. Furthermore by Corollary 3.3.3 we have $\deg(g)\leq\deg(f)+1$. Now assume that $\deg(g)=\deg(f)+1$. Then by Proposition 3.4.1 together with $\mathbf{0}\neq(a_1,a_2)\in\mathbb{F}^2$ it follows that there is a $d\in\mathbb{F}^*$ with $a_1\,\sigma(d)-a_2\,d=0$. $\square$

**Proposition 4.1.3.** *Let $(\mathbb{F}(t),\sigma)$ be a proper sum extension of $(\mathbb{F},\sigma)$, $\mathbf{0}\neq(a_1,a_2)\in\mathbb{F}^2$, $f\in\mathbb{F}$ and $g\in\mathbb{F}(t)\setminus\mathbb{F}$ with*

$$a_1\,\sigma(g)+a_2\,g=f.$$

*Then $a_1\neq 0\neq a_2$ and*

$$g=d\,t+w$$

*for some $w\in\mathbb{F}$ and $d\in\mathbb{F}^*$. Furthermore we have*

$$a_1\,\sigma(d)+a_2\,d=0.$$

*Proof.* By Lemma 4.1.1 it follows that $\deg(g)=1$ and thus

$$g=d\,t+w$$

for some $d\in\mathbb{F}^*$ and $w\in\mathbb{F}$. If $a_1=0$ or $a_2=0$ then $a_1\,\sigma(g)+a_2\,g\notin\mathbb{F}$, a contradiction. Hence $a_1,a_2\in\mathbb{F}^*$ and it follows that

$$a_1\,\sigma(d\,t+w)+a_2\,d\,t=\underbrace{(a_1\,\sigma(d)+a_2\,d)}_{\in\mathbb{F}}\,t+v$$

for some $v \in \mathbb{F}$. Since $f \in \mathbb{F}$ and $t$ is transcendental over $\mathbb{F}$, we have

$$a_1\,\sigma(d) + a_2\,d = 0.$$

$\square$

**Remark 4.1.1.** A similar result like Proposition 4.1.3 is achieved in [Kar85, Lemma 4.2]. $\lozenge$

**Corollary 4.1.2.** *Let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$. Let $f \in \mathbb{F}$ and $g \in \mathbb{F}[t]$ with*

$$\sigma(g) - g = f.$$

*Then*

$$g = c\,t + w$$

*for some $c \in \mathbb{K}$ and $w \in \mathbb{F}$.*

*Proof.* If $g \in \mathbb{F}(t) \setminus \mathbb{F}$ then by Proposition 4.1.3 there is a $g = d\,t + w$ for some $w \in \mathbb{F}$ and $d \in \mathbb{F}^*$ such that $\sigma(g) - g = f$ and

$$\sigma(d) - d = 0.$$

Therefore $d \in \mathbb{K}^*$. Otherwise, if $g \in \mathbb{F}$, the corollary follows immediately. $\square$

## 4.2 The Fundamental Theorem for the First Order Case

### 4.2.1 Finding Solutions for the First Order Case

Let $(\mathbb{F}, \sigma)$ be a difference field, $a_1, a_2 \in \mathbb{F}^*$, $f \in \mathbb{F}$ and assume there does not exist a $g \in \mathbb{F}$ such that

$$a_1\,\sigma(g) + a_2\,g = f. \tag{4.2}$$

How can we find an appropriate reduced product-sum extension to solve this first order difference equation and how does this reduced product-sum extension look like? First we try to solve the homogeneous variation

$$a_1\,\sigma(w) + a_2\,w = 0.$$

If there exists such a $w \in \mathbb{F}$ then we are done. Otherwise, if there does not exist an $n > 1$ with

$$(-\frac{a_2}{a_1})^n \in \mathrm{H}_{(\mathbb{F},\sigma)} \tag{4.3}$$

then by Corollary 2.4.2 we can construct immediately a $\Pi$-extension $(\mathbb{F}[t], \sigma)$ of $(\mathbb{F}, \sigma)$ canonically defined by

$$\sigma(t) = -\frac{a_2}{a_1}\,t$$

such that

$$a_1\,\sigma(t) - a_2 = 0.$$

If there exists an $n > 0$ with (4.3) then we still may change the underlying difference field $(\mathbb{E}, \sigma)$ to avoid this case.

In the following we assume that there exists a difference field $(\mathbb{F}, \sigma)$ and $w \in \mathbb{F}$ such that

$$a_1\,\sigma(w) + a_2\,w = 0.$$

If we look at the difference equation

$$\begin{gathered} a_1\,\sigma(w)\,\sigma(g) + a_2\,w\,g = f \\ \Updownarrow \\ -a_2\,w\,\sigma(g) + a_2\,w\,g = f \\ \Updownarrow \\ \sigma(g) - g = -\frac{f}{a_2\,w} \end{gathered} \tag{4.4}$$

then we see immediately that $w\,g$ is a solution of (4.2) if and only if $g$ is a solution of (4.4). Therefore we are interested in finding an appropriate difference field extension $(\mathbb{G}, \sigma)$ such that there exists a $g \in \mathbb{G}$ with

$$\sigma(g) - g = f. \tag{4.5}$$

If we are able to solve this problem, we also can deal with the problem to find an appropriate difference field extension for the difference equation (4.2) - except we reach the case (4.3) for some $n > 1$.

If there exists a $g \in \mathbb{F}$ with $\sigma(g) - g = f$ then we are done. Otherwise it follows by Proposition 2.4.1 that $(\mathbb{F}(t), \sigma)$ canonically defined by

$$\sigma(t) = t + f \tag{4.6}$$

is –up to a difference field isomorphism– a unique sum extension of $(\mathbb{F}, \sigma)$ which is proper. Of course, $t$ is a solution of the recurrence (4.5).

Now there arises the question if more interesting reduced product-sum extensions $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ exist in which we find a $g \in \mathbb{E}$ with

$$\sigma(g) - g = f$$

and how the solution $g$ looks like.

### 4.2.2 Product-Sum Extensions for the First Order Case

**Definition 4.2.1.** Let $(\mathbb{E}(t), \sigma)$ be a reduced product-sum extension of $(\mathbb{F}, \sigma)$ canonically defined by

$$\sigma(t) = t + \beta.$$

The sum $t$ is called *sum-reduced* w.r.t. $\mathbb{F}$, if whenever there exists an $h \in \mathbb{E}$ with

$$\beta + \sigma(h) - h \in \mathbb{F}$$

then $\beta \in \mathbb{F}$.

Let $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ be a reduced product-sum extension of $(\mathbb{F}, \sigma)$. $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is called *sum-reduced* w.r.t. $\mathbb{F}$, if in each sum extension $(\mathbb{F}(t_1, \dots, t_i), \sigma)$ of $(\mathbb{F}(t_1, \dots, t_{i-1}), \sigma)$ the sum $t_i$ is sum-reduced w.r.t. $\mathbb{F}$. $\diamondsuit$

**Lemma 4.2.1.** *Let $(\mathbb{E}, \sigma)$ be a reduced product-sum extension of $(\mathbb{F}, \sigma)$. Then there exists a reduced product-sum extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ which is sum reduced w.r.t. $\mathbb{F}$ and*

$$(\mathbb{E}, \sigma) \simeq (\mathbb{H}, \sigma).$$

*Proof.* We will prove the corollary by induction on the number $n$ of extensions. For the induction base $n = 0$ nothing has to be shown. Now assume that for the reduced product-sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{E} := \mathbb{F}(t_1, \dots, t_n)$ one has a reduced product-sum extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{H} := \mathbb{F}(t_1, \dots, t_n)$ which is sum-reduced w.r.t. $\mathbb{F}$ and

$$(\mathbb{E}, \sigma) \overset{\tau}{\simeq} (\mathbb{H}, \sigma).$$

Now consider the reduced product-sum extension $(\mathbb{E}(x), \sigma)$ of $(\mathbb{E}, \sigma)$ with

$$\sigma(x) = \alpha\, x + \beta.$$

By Lemma 2.4.1 we can construct a difference field extension $(\mathbb{H}(y), \sigma)$ of $(\mathbb{H}, \sigma)$ with

$$\sigma(y) = \tau(\alpha)\, y + \tau(\beta).$$

By Proposition 2.3.3 we get that $(\mathbb{H}(y), \sigma)$ is a reduced product-sum extension of $(\mathbb{H}, \sigma)$ and that

$$(\mathbb{E}(x), \sigma) \simeq (\mathbb{H}(y), \sigma).$$

If $\beta = 0$ then we are done. Otherwise we have $\alpha = 1$ and $\beta \neq 0$. If there do not exist an $h \in \mathbb{H} \setminus \mathbb{F}$ and a $w \in \mathbb{F}$ such that

$$\tau(\beta) = w + \sigma(h) - h$$

then we are also done. Otherwise, by Lemma 2.4.1 we can construct a difference field extension $(\mathbb{H}(z), \sigma)$ of $(\mathbb{H}, \sigma)$ with

$$\sigma(z) = z + \tau(\beta) - (\sigma(h) - h) = z + w.$$

By Proposition 2.3.1 we get that $(\mathbb{H}(z), \sigma)$ is a proper sum extension and that

$$(\mathbb{H}(y), \sigma) \simeq (\mathbb{H}(z), \sigma).$$

Thus

$$(\mathbb{E}(x), \sigma) \simeq (\mathbb{H}(z), \sigma)$$

where $(\mathbb{H}(z), \sigma)$ is sum-reduced w.r.t. $\mathbb{F}$. $\square$

**Theorem 4.2.1 (Fundamental Theorem).** *Let $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ be a reduced product-sum extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ which is sum-reduced w.r.t. $\mathbb{F}$. Let $f \in \mathbb{F}$ and $g \in \mathbb{E}$ such that*

$$\sigma(g) - g = f.$$

*Then*

$$g = \sum_{i \in S} c_i\, t_i + w$$

*where $w \in \mathbb{F}$, $c_i \in \mathbb{K}$ and*

$$S = \{i \,|\, t_i \text{ is a sum over } \mathbb{F}\}.$$

*Proof.* We will prove the theorem by induction on the number $n$ of extensions. For the induction base $n = 0$ nothing has to be shown. Now assume that the theorem holds for $n$ extensions and consider a reduced product-sum extension $(\mathbb{F}(t_1, \dots, t_{n+1}), \sigma)$ of $(\mathbb{F}, \sigma)$ which is sum-reduced w.r.t. $\mathbb{F}$ and in which we have $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{F}(t_1, \dots, t_{i-1})$. Let $f \in \mathbb{F}$ and $g \in \mathbb{F}(t_1, \dots, t_{n+1})$ such that

$$\sigma(g) - g = f.$$

Then by the induction assumption we have

$$g = \sum_{i=2}^{n} c_i\, t_i + w$$

where $w \in \mathbb{F}(t_1)$, $c_i \in \mathbb{K}$ and furthermore $c_i = 0$ if $t_i$ is not a sum over $\mathbb{F}(t_1)$. Assume there is a $c_j \neq 0$ with $\beta_j \notin \mathbb{F}$ and let $j$ be maximal. Then

$$\begin{aligned}\sigma(g) - g &= \sigma(\sum_{i=2}^{n} c_i\, t_i + w) - (\sum_{i=2}^{n} c_i\, t_i + w)\\ &= \sigma(\sum_{i=2}^{j-1} c_i\, t_i + w) - (\sum_{i=2}^{j-1} c_i\, t_i + w) + c_j\, \beta_j + \sum_{i=j+1}^{n} c_i\, \beta_i = f\end{aligned}$$

$$\Updownarrow$$

$$\beta_j + \sigma(h) - h = f - \sum_{i=j+1}^{n} c_i\, \beta_i \in \mathbb{F}$$

where

$$h := \frac{1}{c_j}(\sum_{i=2}^{j-1} c_i\, t_i + w) \in \mathbb{F}(t_1, \dots, t_{j-1}).$$

As $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ is a sum-reduced extension of $(\mathbb{F},\sigma)$ w.r.t. $\mathbb{F}$, it follows that $\beta_j\in\mathbb{F}$ which contradicts to the assumption. Thus we may assume that for all $j$ with $c_j\neq 0$ we have $\beta_j\in\mathbb{F}$. If $w\in\mathbb{F}$ then the theorem is already proven. Otherwise, assume $w\in\mathbb{F}(t_1)\setminus\mathbb{F}$. We have

$$\sigma(w)-w=f-\sigma(\sum_{i=2}^{n}c_i\,t_i)-\sum_{i=2}^{n}c_i\,t_i=f-\sum_{i=2}^{n}c_i\,\beta_i\in\mathbb{F}.$$

If $(\mathbb{F}(t_1),\sigma)$ is a $\Pi$-extension of $(\mathbb{F},\sigma)$ then by Corollary 4.1.1 $w\in\mathbb{F}$, a contradiction. Therefore $(\mathbb{F}(t_1),\sigma)$ must be a proper sum extension of $(\mathbb{F},\sigma)$. By Corollary 4.1.2 it follows that

$$w=c_1\,t_1+q$$

for some $c_1\in\mathbb{K}$ and $q\in\mathbb{F}$ and consequently we can write

$$g=\sum_{i=1}^{n}c_i\,t_i+q$$

where $c_i=0$ if $t_i$ is not a sum over $\mathbb{F}$. □

M. Karr states[1] in [Kar81, Section 4] and proves in [Kar85, Section 4] the Fundamental Theorem for the somehow more general case

$$a_1\,\sigma(g)+a_2\,g=f \tag{4.7}$$

where $a_1,a_2\in\mathbb{F}^*$ and $g$ is an element of a $\Pi\Sigma$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$. In order to achieve this, he not only has to assume that the difference field $(\mathbb{E},\sigma)$ is sum-reduced w.r.t. $\mathbb{F}$ but he also has to restrict to some additional properties of $\Sigma$-extensions. He also states that one can always transform a difference field extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ to a difference field with the desired properties. But in this case he does not deal anymore with reduced product-sum extensions, which we are basically interested in, but with more general $\Pi\Sigma$-extensions. Furthermore, as stated in Lemma 4.2.1, we can transform the difference field $(\mathbb{E},\sigma)$ to an isomorphic difference field which is sum-reduced w.r.t. $\mathbb{F}$ whereas in Karr's more involved transformation the difference field is in general not anymore isomorphic. On the other side, we can reduce by the remarks of Section 4.2.1 the more general case (4.7) to

$$\sigma(g)-g=f.$$

If we want to focus on solutions which are represented in terms of sum and product extensions then our simpler approach seems much more natural.

[1] Our special case is included Karr's result.

**Corollary 4.2.1.** *Let $(\mathbb{E},\sigma)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$. Let $f \in \mathbb{F}$ and $g \in \mathbb{E}$ such that*

$$\sigma(g) - g = f.$$

*Then there is a reduced product-sum extension $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ of $(\mathbb{F},\sigma)$ which is sum-reduced w.r.t. $\mathbb{F}$ with*

$$(\mathbb{E},\sigma) \overset{\tau}{\simeq} (\mathbb{F}(t_1,\dots,t_n),\sigma).$$

*The corresponding solution $h := \tau(g) \in \mathbb{F}(t_1,\dots,t_n)$ is of the form*

$$h = \sum_{i\in S} c_i\, t_i + w$$

*where $w \in \mathbb{F}$, $c_i \in \mathbb{K}$ and*

$$S = \{i \,|\, t_i \text{ is a sum over } \mathbb{F}\}.$$

*Proof.* By Lemma 4.2.1 there exists a proper sum extension $(\mathbb{F}(t_1,\dots,t_n),\sigma)$ which is sum-reduced w.r.t. $\mathbb{F}$ and for which we have

$$(\mathbb{E},\sigma) \overset{\tau}{\simeq} (\mathbb{F}(t_1,\dots,t_n),\sigma).$$

Therefore by Theorem 4.2.1 the corollary follows. □

**Lemma 4.2.2.** *Let $(\mathbb{E},\sigma)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ which is sum-reduced over $\mathbb{F}$. Let $f \in \mathbb{F}$ and $g \in \mathbb{E}$ such that*

$$\sigma(g) - g = f.$$

*Then one can construct a proper sum extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ and compute a $c \in \mathbb{K}$ and a $w \in \mathbb{F}$ such that*

$$\sigma(c\,s + w) - (c\,s + w) = f.$$

*Moreover there is a difference field extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ such that*

$$(\mathbb{F}(s),\sigma) \simeq (\mathbb{H},\sigma) \le (\mathbb{E},\sigma).$$

*Proof.* By Theorem 4.2.1 we have

$$g = \sum_{i\in S} c_i\, t_i + w$$

where $w \in \mathbb{F}$, $c_i \in \mathbb{K}$ and

$$S = \{i \,|\, t_i \text{ is a sum over } \mathbb{F}\} = \{i_1 < i_2 < \dots < i_d\}.$$

Consequently we have

$$f = \sigma(g) - g = \sum_{i\in S} c_i\, \beta_i + \sigma(w) - w. \tag{4.8}$$

Applying Proposition 2.4.4 we get

$$\begin{aligned}(\mathbb{F}(t_1,\dots,t_n),\sigma) &\simeq (\mathbb{F}(t_{i_1},t_{i_2},\dots,t_{i_d})(t_1,\dots,t_{i_1-1},t_{i_1+1},\dots,t_{i_d+1},\dots,t_n),\sigma)\\ &\ge (\mathbb{F}(t_{i_1},t_{i_2},\dots,t_{i_d}),\sigma)\end{aligned} \tag{4.9}$$

where $(\mathbb{H},\sigma)$ with $\mathbb{H}:=\mathbb{F}(t_{i_1},t_{i_2},\dots,t_{i_d})$ is a proper sum extension of $(\mathbb{F},\sigma)$. Surely we have

$$g\in\mathbb{H}.$$

Let

$$p:=1/c_{i_d}\sum_{j=1}^{d-1}c_{i_j}\,t_{i_j}.$$

Then there is a difference field extension $(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}})(s),\sigma)$ of $(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}}),\sigma)$ with

$$\sigma(s)=s+\beta_{i_d}+\sigma(p)-p=s+\underbrace{\beta_{i_d}+1/c_{i_d}\sum_{j=1}^{d-1}c_{i_j}\,\beta_{i_j}}_{\in\mathbb{F}} \tag{4.10}$$

by Lemma 2.4.1. Furthermore, by Proposition 2.3.1 $(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}})(s),\sigma)$ is a proper sum extension of $(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}}),\sigma)$ and

$$(\mathbb{H},\sigma)\simeq(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}})(s),\sigma). \tag{4.11}$$

Finally, by applying Proposition 2.4.4, $(\mathbb{F}(s)(t_{i_1},\dots,t_{i_{d-1}}),\sigma)$ is a proper sum extension of $(\mathbb{F},\sigma)$ with

$$(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}})(s),\sigma)\simeq(\mathbb{F}(s)(t_{i_1},\dots,t_{i_{d-1}}),\sigma)\geq(\mathbb{F}(s),\sigma) \tag{4.12}$$

and therefore $(\mathbb{F}(s),\sigma)$ is a proper sum extension of $(\mathbb{F},\sigma)$ canonically defined by (4.10). For

$$h:=c_{i_d}\,s+w\in\mathbb{F}(s)$$

we have

$$\sigma(h)-(h)=\sigma(w)-w+c_{i_d}\,\beta_{i_d}+\sum_{j=1}^{d-1}c_{i_j}\,\beta_{i_j}\overset{(4.8)}{=}f.$$

Altogether we get

$$\begin{aligned}
(\mathbb{F}(s),\sigma)&\overset{(4.12)}{\leq}(\mathbb{F}(s)(t_{i_1},\dots,t_{i_{d-1}}),\sigma)\\
&\simeq(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}})(s),\sigma)\\
&\overset{(4.11)}{\simeq}(\mathbb{F}(t_{i_1},\dots,t_{i_{d-1}})(t_d),\sigma)\\
&\overset{(4.9)}{\leq}(\mathbb{F}(t_{i_1},t_{i_2},\dots,t_{i_d})(t_1,\dots,t_{i_1-1},t_{i_1+1},\dots,t_{i_d+1},\dots,t_n),\sigma)\\
&\simeq(\mathbb{F}(t_1,\dots,t_n),\sigma)
\end{aligned}$$

and thus the theorem is proven. □

**Proposition 4.2.1.** *Let $(\mathbb{E},\sigma)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$. Let $f\in\mathbb{F}$ and $g\in\mathbb{E}$ such that*

$$\sigma(g)-g=f.$$

*Then there exists a proper sum extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$, $c\in\mathbb{K}$ and $h\in\mathbb{F}$ such that*

$$\sigma(c\,s+h)-(c\,s+h)=f.$$

*Moreover there is a difference field extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ such that*

$$(\mathbb{F}(s),\sigma)\simeq(\mathbb{H},\sigma)\leq(\mathbb{E},\sigma).$$

*Proof.* By Lemma 4.2.1 there exists a proper sum extension $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ which is sum-reduced w.r.t. $\mathbb{F}$ and for which we have

$$(\mathbb{E}, \sigma) \overset{\tau}{\simeq} (\mathbb{F}(t_1, \dots, t_n), \sigma).$$

Therefore by Lemma 4.2.2 the corollary follows. □

### 4.2.3 Some Misleading Interpretations

Assume there does not exists a $g \in \mathbb{F}$ with

$$\sigma(g) - g = f. \tag{4.13}$$

As already illustrated in Section 4.2.1, we are able to construct immediately a proper sum extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$\sigma(t) = t + f$$

and $g = t$ is a solution of the recurrence (4.13).

If we find any other reduced product-sum extension of $(\mathbb{E}, \sigma)$ which has a solution of (4.13) then Corollary 4.2.1 tells us that there exists an isomorphic difference field in which the solution is of the form

$$g = \sum_{i \in S} c_i\, t_i + w$$

where $w \in \mathbb{F}$, $c_i \in \mathbb{K}$ and

$$S = \{i \,|\, t_i \text{ is a sum over } \mathbb{F}\}.$$

This observation misleads M. Karr to the following remark in [Kar81]:

> Loosely speaking, if $f$ is summable in $E$, then part of it is summable in $F$, and the rest consists of pieces whose formal sums have been adjoined to $F$ in the construction of $E$. This makes the construction of extension fields in which $f$ is summable somewhat uninteresting and justifies the tendency to look for sums of $f \in F$ only in $F$.

Even worse, Proposition 4.2.1 motivates us to search for exactly one proper sum extension. Assume we find such a proper sum extension $(\mathbb{F}(s), \sigma)$ of $(\mathbb{F}, \sigma)$ such that there is an $h \in \mathbb{F}(s)$ with

$$\sigma(h) - h = f$$

then it follows by Proposition 2.3.2 immediately that

$$\tau : \begin{cases} \mathbb{F}(t) & \to & \mathbb{F}(s) \\ t & \mapsto & h \end{cases}$$

is a difference field isomorphism. Therefore, one can take the view that finding any other proper sum extension $(\mathbb{F}(s), \sigma)$ of $(\mathbb{F}, \sigma)$ does not deliver anything new - it is isomorphic to the easy constructible difference field $(\mathbb{F}(t), \sigma)$.

These results seem very disappointing and may unmotivate us to simplify sum and product expressions by appropriate difference field extensions. But in the opposite way, these results are very constructive and tell us for which kind of proper sum extensions we have to look in order to simplify sum expressions. Namely, as will be considered in more details in Section 4.4, one should look for a proper sum extension[2] $(\mathbb{F}(s), \sigma)$ canonically defined by

$$\sigma(s) = s + \beta$$

such that

$$(\mathbb{F}(t), \sigma) \simeq (\mathbb{F}(s), \sigma)$$

where $\beta$ in terms of $\mathbb{F}$ is as "simple" as possible. In particular, "simple" means that $\beta$ should not depend on other sums or hyperexponentials over the difference field $\mathbb{F}$, if possible. In other words, if one wants to simplify expressions in terms of sums and products, it plays a major role to choose the right representation of a difference field extension.

[2]Please note that the sum extension that we are looking for is sum-reduced w.r.t. $\mathbb{F}$. But it might well be that one could simplify a sum expression by looking for an appropriate proper sum-extension which is **not** sum-reduced. This case is not considered further in the following.

## 4.3 Definite Summation in Difference Fields

In this section we will consider in more details how definite summation problems can be treated in difference fields. Additionally, we are able to give new insight for the creative telescoping method in a given difference field by using the Fundamental Theorem.

In the following we will consider concisely what we have already mentioned informally in Section 1.3.3.

### 4.3.1 Finding a Recurrence for a Definite Summation Problem

Let $\mathbb{K}$ be a field with characteristic 0 and consider the sum

$$\mathrm{Sum}(n) = \sum_{k=0}^{n} F(n,k)$$

with

$$\langle \mathrm{Sum}(0), \mathrm{Sum}(1), \dots \rangle \in \mathcal{S}(\mathbb{K}).$$

Assume further we have a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ over the constant field $\mathbb{K}(n)$ and a difference ring homomorphism[3]

$$h : \begin{cases} \mathbb{A} & \to & \mathcal{S}(\mathbb{K}) \\ f & \mapsto & \langle \mathrm{ev}(f,0), \mathrm{ev}(f,1), \dots \rangle \end{cases}$$

for a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : \mathbb{A} \to \mathbb{N}_0$ where $\mathbb{A}$ is a sub-difference ring of $\mathbb{F}$. Assume further there exist an $\mathbf{f} = (f_1, \dots, f_m) \in \mathbb{A}^m$ and a $\gamma \in \mathbb{N}_0$ with

$$\mathrm{ev}(f_i, k) = F(n+i-1, k)$$

for all $k \geq \gamma$ with . Now we solve $\mathrm{V}((1,-1), \mathbf{f}, \mathbb{F})$.

1. Assume we find a $\mathbf{c} \in \mathbb{K}(n)^m$ and $g \in \mathbb{F}$ such that

$$\sigma(g) - g = \mathbf{c}\,\mathbf{f} = c_1\, f_1 + \dots + c_n\, f_m. \tag{4.14}$$

   If $g \in \mathbb{A}$ then there exists a $\delta \geq \gamma$ such that for all $k \geq \delta$ we have

$$\begin{aligned} \mathrm{ev}(f_i, k) &= F(n+i-1, k) \\ \mathrm{ev}(g, k) &= G(k) \\ \mathrm{ev}(c_i, k) &= c_i = c_i(n). \end{aligned}$$

   Therefore we can transform the telescoping equation back to the ring of sequences and get for all $g \geq \delta$ that

$$G(k+1) - G(k) = c_1(n)\, F(n,k) + \dots + c_m(n)\, F(n+m-1,k). \tag{4.15}$$

   Consequently, summing both sides from $\delta$ to $a$ delivers an equation

$$G(a+1) - G(\delta) = c_1(n) \sum_{k=\delta}^{a} F(n,k) + \dots + +c_m(n) \sum_{k=\delta}^{a} F(n+m-1,k) \tag{4.16}$$

[3] See Section 2.5 for more details

which is valid for all $a \geq \delta$. In particular, substituting $a$ by $n$ yields to

$$G(n+1) - G(\delta) = c_1(n) \sum_{k=\delta}^{n} F(n,k) + \dots + c_m(n) \sum_{k=\delta}^{n} F(n+m-1,k)$$

for all $n \geq \delta$. Finally, by

$$\sum_{k=\delta}^{n+i} F(n+i,k) = \sum_{k=\delta}^{n} F(n+i,k) + \sum_{r=1}^{i} F(n+i,n+r)$$

we can compute an expression $B(n)$ with

$$c_1(n) \sum_{k=\lambda}^{n} F(n,k) + \dots + c_m(n) \sum_{k=\lambda}^{n+m-1} F(n+m-1,k) = B(n).$$

Altogether, for the sum

$$\mathrm{Sum}'(n) = \sum_{k=\lambda}^{n} F(n,k)$$

we are able to find the recurrence

$$c_1(n)\,\mathrm{Sum}'(n) + \dots + c_m(n)\,\mathrm{Sum}'(n+m-1) = B(n)$$

where $c_i(n) \in \mathbb{K}(n)$.

2. Otherwise, assume there do not exist a $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}(n)^m$ and a $g \in \mathbb{F}$ with (4.14). Then we can try to find a $\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ and a difference ring homomorphism $h : \mathbb{A}' \to \mathcal{S}(\mathbb{K})$ for a difference ring $(\mathbb{A}', \sigma)$ with

$$(\mathbb{A}, \sigma) \leq (\mathbb{A}', \sigma) \leq (\mathbb{E}, \sigma)$$

such that there exit an $f_{m+1} \in \mathbb{A}$ and a $\gamma' \geq \gamma$ with

$$\mathrm{ev}(f_{m+1}, k) = F(n+m, k)$$

for all $k \geq \gamma'$. Finally we can proceed with $\mathbf{f}' = (f_1, \dots, f_{m+1}) \in \mathbb{A}^{m+1}$ to compute $\mathrm{V}((1,-1), \mathbf{f}', \mathbb{E})$ and goon either with step one or two.

**Implementation Note 4.3.1.** In the function `GenerateRecurrence` introduced in Chapter 1 all these ideas sketched above are realized.

### 4.3.2 A Connection Between the Telescoping Equation and Sum Extensions

By the following theorem we will derive a link between proper sum extensions and solving the creative telescoping equation (4.14).

**Theorem 4.3.1.** *Let* $(\mathbb{F}, \sigma)$ *be a difference field with constant field* $\mathbb{K}$ *and* $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. *Take, up to a difference field isomorphism, the uniquely determined sum-extension* $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ *of* $(\mathbb{F}, \sigma)$ *with*

$$\sigma(t_i) = t_i + f_i$$

*and* $\mathrm{const}_\sigma \mathbb{F}(t_1, \dots, t_n) = \mathbb{K}$.
*Then there do not exist a* $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^n$ *and a* $g \in \mathbb{F}$ *such that*

$$\sigma(g) - g = \mathbf{c}\,\mathbf{f}$$

*if and only if* $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ *is a proper sum extension of* $(\mathbb{F}, \sigma)$.

*Proof.* Assume there is

$$\sigma(g) - g = \mathbf{c}\,\mathbf{f}$$

for some $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}$ and $g \in \mathbb{F}$. Then

$$\sigma(g) - g = \mathbf{c}\,\mathbf{f} = \mathbf{c}\ (\sigma(t_1) - t_1, \dots, \sigma(t_n) - t_n) = \sigma(\sum_{i=1}^{n} c_i\, t_i) - \sum_{i=1}^{n} c_i\, t_i.$$

and thus

$$\sigma(\sum_{i=1}^{n} c_i\, t_i - g) = \sum_{i=1}^{n} c_i\, t_i - g.$$

Since $\mathrm{const}_\sigma \mathbb{F}(t_1, \dots, t_n) = \mathbb{K}$, there is a $k \in \mathbb{K}$ such that

$$\sum_{i=1}^{n} c_i\, t_i - g + k = 0,$$

hence there are algebraic relations in the $t_i$ and consequently $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is not a proper sum extension of $(\mathbb{F}, \sigma)$ by Remark 2.2.4.

Contrary, assume that $(\mathbb{F}(t_1, \dots, t_n), \sigma)$ is not a proper sum extension of $(\mathbb{F}, \sigma)$. Let $i$ be minimal such that $(\mathbb{F}(t_1, \dots, t_{i+1}), \sigma)$ is not a proper sum extension of $(\mathbb{F}(t_1, \dots, t_i), \sigma)$. By Corollary 2.2.4 there exists a $g \in \mathbb{F}(t_1, \dots, t_i)$ such that

$$\sigma(g) - g = f_{i+1}.$$

Since by definition $(\mathbb{F}(t_1, \dots, t_i), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$ which is sum-reduced w.r.t. $\mathbb{F}$, by Theorem 4.2.1 there are $c_j \in \mathbb{K}$ and an $h \in \mathbb{F}$ such that

$$g = h + \sum_{j=1}^{i} c_j\, t_j.$$

We have

$$\sigma(h) - h = f_{i+1} - \sum_{j=1}^{i} c_j(\sigma(t_j) - t_j) = -c_1 f_1 - \cdots - c_i f_i + f_{i+1}$$

and hence there are a $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^n$ and an $h \in \mathbb{F}$ such that

$$\sigma(h) - h = \mathbf{c}\,\mathbf{f}.$$

□

Starting from the context given in the previous Section 4.3.1, we will try to explain a relationship/link between proper sum extensions and the creative telescoping method. Assume that for $\mathbf{f}' = (f_1, \ldots, f_{m-1}) \in \mathbb{F}^{m-1}$ there do not exist a $\mathbf{0} \neq \mathbf{c}' \in \mathbb{K}(n)^{m-1}$ and a $g' \in \mathbb{F}$ with

$$\sigma(g') - g' = \mathbf{c}'\,\mathbf{f}',$$

but we find a $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}(n)^m$ and $g \in \mathbb{F}$ with

$$\sigma(g) - g = \mathbf{c}\,\mathbf{f} = c_1 f_1 + \cdots + c_m f_m. \tag{4.17}$$

This means that $m$ is the minimal order to find the creative telescoping equation (4.17) in the difference field $(\mathbb{F}, \sigma)$. In particular we must have

$$c_m \neq 0$$

since otherwise $m$ is not minimal. By Theorem 4.3.1, we can construct the proper sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}(s_1, \ldots, s_{m-1})$ and

$$\sigma(s_i) = s_i + f_i. \tag{4.18}$$

Furthermore we can define a sum extension $(\mathbb{E}(s_m), \sigma)$ of $(\mathbb{E}, \sigma)$ with

$$\sigma(s_m) = s_m + f_m$$

and $\mathrm{const}_\sigma \mathbb{E}(s_m) = \mathbb{K}(n)$. By Theorem 4.3.1 we get that $(\mathbb{E}(s), \sigma)$ is not a proper sum extension of $(\mathbb{E}, \sigma)$ and therefore there exists an $h \in \mathbb{F}(s_1, \ldots, s_{m-1})$ such that

$$\sigma(h) - h = f_m. \tag{4.19}$$

One can immediately check by (4.17) that

$$h := \frac{g}{c_m} - \sum_{i=1}^{m-1} \frac{c_i}{c_m}\, s_i$$

is a solution of (4.19). Since $\sigma(s_m - h) = s_m - h$, it follows that

$$s_m = \frac{g}{c_m} - \sum_{i=0}^{m-1} \frac{c_i}{c_m}\, s_i + x \tag{4.20}$$

for some $x \in \mathbb{K}$. Consequently, solving the creative telescoping equation (4.17) is equivalent to expressing the sum $s_m$ by the sums $s_1, \ldots, s_{m-1}$ and an element from $\mathbb{F}$. Furthermore by the Fundamental Theorem 4.2.1 we get immediately that the $s_i$ must be linear over the constant field $\mathbb{K}(n)$. In other words, if there exists a creative telescoping equation (4.15) then the $c_i$ must be in $\mathbb{K}(n)$.

The corresponding result of (4.20) in terms of the ring of sequences is reflected by (4.16) which is equivalent to

$$\sum_{k=\delta}^{a} F(n+m-1,k) = \frac{G(a+1)}{c_m(n)} - \sum_{i=1}^{m-1} \frac{c_i(n)}{c_m(n)} \sum_{k=\delta}^{a} F(n+i-1,k) - \frac{G(\delta)}{c_m(n)}$$

where $G(\delta) \in \mathbb{K}$.

## 4.4 Low Nested Sum Extensions for First Order Equations

Here we try to explain how one can find appropriate sum extensions in order to simplify indefinite sums (Sections 1.2.3 and 1.2.4) and how one can generate a recurrence of lower order by creative telescoping using an appropriate sum extension (Section 1.3.5).

Let $(\mathbb{E},\sigma)$ be a difference field, $f \in \mathbb{E}$ and assume there does not exist a $g \in \mathbb{E}$ with

$$\sigma(g) - g = f. \tag{4.21}$$

As already mentioned in Section 4.2.3, we can construct a proper sum extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ with

$$\sigma(t) = t + f$$

and $t$ is a solution of the recurrence (4.21). In order to get this solution, we had to increase the **recursion depth** in this difference field extension.

Now let $(\mathbb{E},\sigma)$ with $\mathbb{E} = \mathbb{F}(t_1,\dots,t_n)$ be a reduced product-sum extension of a difference field $(\mathbb{F},\sigma)$ and let $\mu \in \mathbb{N}$ be minimal such that

$$f \in \mathbb{F}(t_1,\dots,t_\mu).$$

In this section we try to find a proper sum extension $(\mathbb{E}(s),\sigma)$ with

$$\sigma(s) = s + \beta$$

such that there exists a $g \in \mathbb{G}$ with (4.21). **Additionally**, the recursion depth must be as "simple" as possible, i.e. we try to find a $\beta \in \mathbb{F}(t_1,\dots .t_i)$ where $i$ is minimal. Then of course, this difference field extension $(\mathbb{E}(s),\sigma)$ is isomorphic to $(\mathbb{E}(t),\sigma)$ by Proposition 2.3.2, or in other words we did not construct a really "new" difference field. But if $i < \mu$ then from the **summation point of view** we simplify the summation expression by using terms which depend on a smaller difference field. Especially, if $\beta$ consists of less nested sum expressions than $f$, we find a difference field extension $(\mathbb{E}(s),\sigma)$ with a smaller recursion depth w.r.t. sums than the difference field extension $(\mathbb{E}(t),\sigma)$.

Please note that the following method finds such a difference field extension – in case it exists.

### 4.4.1 Finding Low Nested Sum Extensions Automatically to Solve the Solution Space

**Theorem 4.4.1.** *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and let $(\mathbb{E},\sigma)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$, $\mathbb{E} := \mathbb{F}(t_1,\dots,t_e)$ and $\mathbf{f} \in \mathbb{E}^n$. Assume there is a proper sum extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ with $g \in \mathbb{E}(s) \setminus \mathbb{E}$ and $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^n$ such that*

$$\sigma(g) - g = \mathbf{c}\,\mathbf{f}.$$

*Then in the reduction process, given by Theorems 3.2.1 and 3.2.2, for the problem*

$$\mathrm{V}((1,-1),\mathbf{f},\mathbb{E})$$

*one has to solve a subproblem $\mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F})$ with $\mathbf{0} \neq (a_1,a_2) \in \mathbb{F}^2$ and $\mathbf{h} \in \mathbb{F}^m$ such that*

$$\mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F}) \subsetneq \mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F}(s)).$$

*Proof.* By Proposition 2.4.4 there exists a reduced product-sum extension $(\mathbb{F}(t_1,\dots,t_e)(s),\sigma)$ of $(\mathbb{F},\sigma)$ with

$$(\mathbb{F}(t_1,\dots,t_e)(s),\sigma)\simeq(\mathbb{F}(s)(t_1,\dots,t_e),\sigma). \tag{4.22}$$

Assume that for each subproblem $\mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F})$ in the reduction process with $a_1,a_2\in\mathbb{F}$ and $\mathbf{h}\in\mathbb{F}^m$ we have

$$\mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F})=\mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F}(s)).$$

But this means that for each subproblem $\mathrm{V}((b_1,b_2),\mathbf{q},\mathbb{F}(t_1,\dots,t_i))$ in the reduction process for $0\leq i\leq e$ and for some $b_1,b_2\in\mathbb{F}(t_1,\dots,t_i)$ and $\mathbf{q}\in\mathbb{F}^r$ with $r\geq 1$ we have

$$\mathrm{V}((b_1,b_2),\mathbf{q},\mathbb{F}(t_1,\dots,t_i))=\mathrm{V}((b_1,b_2),\mathbf{q},\mathbb{F}(s)(t_1,\dots,t_i)).$$

But then we have

$$\mathrm{V}((1,-1),\mathbf{f},\mathbb{F}(t_1,\dots,t_e))=\mathrm{V}((1,-1),\mathbf{f},\mathbb{F}(s)(t_1,\dots,t_e)).$$

And since (4.22) we get

$$\mathrm{V}((1,-1),\mathbf{f},\mathbb{F}(t_1,\dots,t_e))=\mathrm{V}((1,-1),\mathbf{f},\mathbb{F}(t_1,\dots,t_e)(s)).$$

which contradicts to the assumption that there are a $g\in\mathbb{F}(t_1,\dots,t_e)(s)\setminus\mathbb{F}(t_1,\dots,t_e)$ and a $\mathbf{0}\neq\mathbf{c}\in\mathbb{K}^n$ with

$$\sigma(g)-g=\mathbf{c}\,\mathbf{f}.$$

Thus there must exist a subproblem in the reduction process with

$$\mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F})\subsetneq\mathrm{V}((a_1,a_2),\mathbf{h},\mathbb{F}(s)).$$

By construction of the reduction process we have $(a_1,a_2)\neq\mathbf{0}$. □

**Lemma 4.4.1.** *Let $(\mathbb{F}(s),\sigma)$ be a proper sum extension of $(\mathbb{F},\sigma)$ and assume*

$$\mathrm{V}((a_1,a_2),\mathbf{f},\mathbb{F})\subsetneq\mathrm{V}((a_1,a_2),\mathbf{f},\mathbb{F}(s))$$

*for some $\mathbf{0}\neq(a_1,a_2)\in\mathbb{F}^2$ and $\mathbf{f}\in\mathbb{F}^n$. Then $a_1\neq 0\neq a_2$ and there is a $d\in\mathbb{F}^*$ such that*

$$a_1\,\sigma(d)+a_2\,d=0.$$

*Proof.* By assumption there are a $g\in\mathbb{F}(s)\setminus\mathbb{F}$ and a $\mathbf{c}\in\mathbb{K}^n$ with

$$a_1\,\sigma(g)+a_2\,g=\mathbf{c}\,\mathbf{f}\in\mathbb{F}.$$

Thus by Proposition 4.1.3 it follows that $a_1\neq 0\neq a_2$ and there exists a $d\in\mathbb{F}^*$ with

$$a_1\,\sigma(d)+a_2\,d=0.$$

□

**Theorem 4.4.2.** *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$. Let $(\mathbb{F}(s),\sigma)$ be a proper sum extension of $(\mathbb{F},\sigma)$ and assume*

$$\mathrm{V}((a_1,a_2),\mathbf{f},\mathbb{F}) \subsetneq \mathrm{V}((a_1,a_2),\mathbf{f},\mathbb{F}(s))$$

*for some $\mathbf{0}\neq(a_1,a_2)\in\mathbb{F}^2$ and $\mathbf{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. Then $a_1\neq 0\neq a_2$. Take $d\in\mathbb{F}^*$ such that*

$$a_1\,\sigma(d)+a_2\,d=0$$

*and consider the -up to a difference field isomorphism- uniquely determined difference field extension $(\mathbb{F}(s_1,\dots,s_n),\sigma)$ of $(\mathbb{F},\sigma)$ with*

$$\sigma(s_i)=s_i+\frac{f_i}{a_2\,d}$$

*for $1\leq i\leq n$ and $\mathrm{const}_\sigma\mathbb{F}(s_1,\dots,s_n)=\mathbb{K}$. Then there is a proper sum extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ such that*

$$(\mathbb{F}(s),\sigma)\simeq(\mathbb{F}(t),\sigma)\subseteq(\mathbb{F}(s_1,\dots,s_n),\sigma).$$

*Proof.* By Lemma 4.4.1 we can find a $d\in\mathbb{F}^*$ such that

$$a_1\,\sigma(d)+a_2\,d=0. \tag{4.23}$$

Additionally, by Proposition 4.1.3 there are a $g\in\mathbb{F}(s)\setminus\mathbb{F}$ and a $\mathbf{c}\in\mathbb{K}^n$ such that

$$a_1\,\sigma(g)+a_2\,g=\mathbf{c}\,\mathbf{f}$$

where

$$g=\tilde{d}(s+h)$$

for some $\tilde{d}\in\mathbb{F}^*$ and $h\in\mathbb{F}$. Furthermore we have

$$a_1\,\sigma(\tilde{d})+a_2\,\tilde{d}=0. \tag{4.24}$$

Using equation (4.23) and (4.24) we get

$$\sigma(\frac{\tilde{d}}{d})=\frac{\tilde{d}}{d},$$

consequently there is a $k\in\mathbb{K}^*$ with

$$\tilde{d}=k\,d$$

and thus we may assume

$$g=k\,d\,(s+h).$$

We have

$$\begin{aligned}\mathbf{c}\,\mathbf{f}&=a_1\,\sigma(g)+a_2\,g=k\,(a_1\,\sigma(d\,(s+h))+a_2\,d\,(s+h))\\&=k\,(a_1\,\sigma(d)(\sigma(s+h))+a_2\,d\,(s+h))\end{aligned}$$

$$\Updownarrow\ (4.23)$$

$$\sigma(s+h)-(s+h)=-\frac{\mathbf{c}\,\mathbf{f}}{k\,a_2\,d}$$

$$\Updownarrow$$

$$\sigma(h) - h = -\beta - \frac{\mathbf{c}}{k}\,\frac{\mathbf{f}}{a_2\,d} \tag{4.25}$$

Define by Lemma 2.4.1 a difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ by

$$\sigma(t) = t + \beta + (\sigma(h) - h).$$

By Proposition 2.3.1 it follows that $(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$ and

$$(\mathbb{F}(s), \sigma) \simeq (\mathbb{F}(t), \sigma). \tag{4.26}$$

By Corollary 2.4.1 there is the -up to a difference field isomorphism- uniquely determined difference field extension $(\mathbb{F}(t)(s_1, \dots, s_n), \sigma)$ of $(\mathbb{F}(t), \sigma)$ with

$$\sigma(s_i) = s_i + \frac{f_i}{a_2\,d}$$

for $1 \leq i \leq n$ and $\mathrm{const}_\sigma \mathbb{F}(t)(s_1, \dots, s_n) = \mathbb{K}$. By (4.25) we get

$$\sum_{i=1}^{n} \frac{f_i}{a_2\,d}\,\frac{c_i}{k} = -\beta - (\sigma(h) - h)$$

and thus

$$\sigma(t) - t = \beta + (\sigma(h) - h) = -\sum_{i=1}^{n} \frac{f_i}{a_2\,d}\,\frac{c_i}{k} = -\sum_{i=1}^{n} \frac{c_i}{k}(\sigma(s_i) - s_i).$$

Consequently

$$\sigma(t + (\frac{c_1}{k}\,s_1 + \dots + \frac{c_n}{k}\,s_n)) = t + (\frac{c_1}{k}\,s_1 + \dots + \frac{c_n}{k}\,s_n)$$

and therefore there is a $c \in \mathbb{K}$ with

$$t = -\frac{c_1}{k}\,s_1 + \dots + \frac{c_n}{k}\,s_n + c. \tag{4.27}$$

Thus by Proposition 2.4.6 we get

$$(\mathbb{F}(s_1, \dots, s_n), \sigma) \overset{(4.27)}{=} (\mathbb{F}(s_1, \dots, s_n)(t), \sigma) \simeq (\mathbb{F}(t)(s_1, \dots, s_n), \sigma) \geq (\mathbb{F}(t), \sigma) \overset{(4.26)}{\simeq} (\mathbb{F}(s), \sigma).$$

$\square$

**Corollary 4.4.1.** *Let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1,\dots,t_e)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$ and let $\mathbf{f}\in\mathbb{E}^n$. Assume there is a proper sum extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ with $g\in\mathbb{E}(s)\setminus\mathbb{E}$ and $\mathbf{0}\neq\mathbf{c}\in\mathbb{K}^n$ such that*

$$\sigma(g)-g=\mathbf{c}\,\mathbf{f}.$$

*Let*

$$\mathrm{V}(\mathbf{a_1},\mathbf{f_1},\mathbb{F}),\dots,\mathrm{V}(\mathbf{a_k},\mathbf{f_k},\mathbb{F})$$

*with*

$$\begin{aligned}\mathbf{a_i}&=(a_{i1},a_{i2})\in(\mathbb{F}^*)^2,\\ \mathbf{f_i}&=(f_{i1},\dots,f_{ir_i})\in\mathbb{F}^{r_i},\ \ r_i>0\end{aligned}$$

*be all sub-problems that have to be solved in the reduction process, given by Theorems 3.2.1 and 3.2.2, for the original problem $\mathrm{V}((1,-1),\mathbf{f},\mathbb{E})$ such that there are $d_1,\dots,d_k\in\mathbb{F}^*$ with*

$$a_{i1}\,\sigma(d_i)-a_{i2}\,d_i=0.$$

*Define the difference field extension $(\mathbb{E}(s_{11},\dots,s_{1k_i},\dots,s_{k1},\dots,s_{ki_k}),\sigma)$ of $(\mathbb{E},\sigma)$ with*

$$\sigma(s_{ij})=s_{ij}+\frac{f_{ij}}{a_{i2}\,d_i}$$

*for $1\leq i\leq k$ and $1\leq j\leq r_i$ and $\mathrm{const}_\sigma\mathbb{E}(s_{11},\dots,s_{kr_k})=\mathbb{K}$. Then there is a proper sum extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ with*

$$(\mathbb{E}(s),\sigma)\simeq(\mathbb{E}(t),\sigma)\leq(\mathbb{E}(s_{11},\dots,s_{kr_k}),\sigma).$$

*Proof.* The corollary is a direct consequence of Theorem 4.4.1 and Theorem 4.4.2. □

**Remark 4.4.1.** Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field over the constant field $\mathbb{K}$, let $(\mathbb{E},\sigma)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1,\dots,t_n)$ and define

$$\mathbb{F}_i:=\mathbb{F}(t_1,\dots,t_i).$$

Let $\mathbf{f}\in\mathbb{E}^n$ and assume that there exists a proper sum extension $(\mathbb{E}(s),\sigma)$, $\mathbf{0}\neq\mathbf{c}\in\mathbb{K}^n$ and $g\in\mathbb{E}[s]\setminus\mathbb{E}$ with

$$\sigma(s)-s\in\mathbb{F}_i$$

and

$$\sigma(g)-g=\mathbf{c}\,\mathbf{f}.$$

Assume that we can solve $\mathrm{V}((1,-1),\mathbf{f},\mathbb{E})$ by the reduction process given by Theorems 3.2.1 and 3.2.2. Then by Corollary 4.4.1 we find a sum extension $(\mathbb{E}(s_1,\dots,s_m),\sigma)$ of $(\mathbb{E},\sigma)$ where all sums $s_j$ are over $\mathbb{F}_i$ such that there is a proper sum extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}_i$ with

$$(\mathbb{E}(s),\sigma)\simeq(\mathbb{E}(t),\sigma)\leq(\mathbb{E}(s_1,\dots,s_m),\sigma).$$

By Proposition 2.4.6 we can permutate the sums $s_j$ in $\mathbb{E}(s_1,\dots,s_m)$ and get an isomorphic sum extension. Therefore we can reorder the sums $s_j$ in such a way that sums involving less complicated expressions are ordered first. Finally we can apply Proposition 2.4.3 to compute a proper sum extension $(\mathbb{E}(u_1,\dots,u_l),\sigma)$ of $(\mathbb{E},\sigma)$ such that

$$(\mathbb{E}(s_1,\dots,s_m),\sigma)\simeq(\mathbb{E}(u_1,\dots,u_l),\sigma).$$

Therefore, whenever there exist a $\mathbf{0}\neq\mathbf{c}\in\mathbb{K}^n$ and a proper sum extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ with $g\in\mathbb{E}(s)\setminus\mathbb{E}$ such that

$$\sigma(g)-g=\mathbf{c}\,\mathbf{f}$$

then

$$(\mathbb{E}(s),\sigma)\simeq(\mathbb{E}(u),\sigma)\leq(\mathbb{E}(u_1,\dots,u_l),\sigma)$$

where $(\mathbb{E}(u),\sigma)$ is some proper sum extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$. $\diamondsuit$

### 4.4.2 Indefinite Summation

Motivating examples for this section can be found in Section 1.2.3. Let $\mathbb{K}$ be field with characteristic 0 and consider the sum

$$\mathrm{Sum}(n)=\sum_{k=0}^{n}F(k) \tag{4.28}$$

with

$$\langle\mathrm{Sum}(0),\mathrm{Sum}(1),\dots\rangle\in\mathcal{S}(\mathbb{K}).$$

Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field over the constant field $\mathbb{K}$, let $(\mathbb{E},\sigma)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{K}(t_1,\dots,t_n)$ and let $\mathbb{F}_i:=\mathbb{F}(t_1,\dots,t_i)$. Assume further we have a difference ring homomorphism[4]

$$h:\left\{\begin{array}{lcl}\mathbb{A}&\to&\mathcal{S}(\mathbb{K})\\ f&\mapsto&\langle\mathrm{ev}(f,0),\mathrm{ev}(f,1),\dots\rangle\end{array}\right.$$

for a homomorphic map $\mathrm{ev}:\mathbb{A}\times\mathbb{N}_0\to\mathbb{K}$ bounded by $L:\mathbb{A}\to\mathbb{N}_0$ where $\mathbb{A}$ is a sub-difference ring of $\mathbb{E}$. Assume further there exists an $f\in\mathbb{A}\cap\mathbb{F}(t_1,\dots,t_\mu)$ where $\mu$ is minimal such that

$$\mathrm{ev}(f,k)=F(k)$$

for some $k\geq\gamma$. Now suppose there does not exist a $g\in\mathbb{E}$ with

$$\sigma(g)-g=f.$$

Then we can construct a proper sum extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ canonically defined by

$$\sigma(t)=t+f.$$

In particular we have

$$\sigma(t)-t=f$$

and thus $t$ is a solution of the given difference equation. In the naive way we could return the result

$$\sum_{k=\gamma}^{n}\mathrm{ev}(f,k)+\sum_{k=0}^{\gamma-1}F(k) \tag{4.29}$$

[4]See Section 2.5 for more details

as the closed form of the sum (4.28).

Instead of this, we try to find a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}_i$ such that $i$ is minimal and there exists a $g \in \mathbb{G}$ with

$$\sigma(g) - g = f. \tag{4.30}$$

For this we use Remark 4.4.1 for $i = 0, 1, \dots$ until we get a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}_i$ such that there exists a $g \in \mathbb{G}$ with (4.30). If we reach the case $i$ without any proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}_i$ and a solution $g \in \mathbb{G}$ then we have proven that there cannot exist a proper sum extension $(\mathbb{E}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ with

$$\sigma(s) - s \in \mathbb{F}_i$$

and $g \in \mathbb{E}(s)$ with (4.30). Therefore going up step by step, the "simplest" solution can be found.

If we reach the point $i = \mu$, without any success, then we know that $(\mathbb{E}(t), \sigma)$ with

$$\sigma(t) = t + f$$

is the only and "simplest" sum over $\mathbb{E}$ which delivers us a solution of the recurrence and we can return the result (4.29).

If we are successful at step $i < \mu$ and find a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}_i$ and a solution $g \in \mathbb{G}$ with (4.30) then by Lemma 4.2.2 and its proof we can construct a proper sum extension $(\mathbb{E}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}_i$ such that

$$g = c\,s + h,$$

with $c \in \mathbb{K}^*$ and $h \in \mathbb{E}$, is a solution of (4.30). In particular, by Proposition 2.3.2 it follows that

$$(\mathbb{E}(s), \sigma) \simeq (\mathbb{E}(t), \sigma).$$

Now we should try to construct a difference ring homomorphism $h : \mathbb{A}[s] \to \mathcal{S}(\mathbb{K})$. If we succeed in this and $g \in \mathbb{A}[s]$ then we can find a $\delta \geq \gamma$ such that for all $k \geq \delta$ we have

$$\mathrm{ev}(g, n) = \sum_{k=\delta}^{n} F(k)$$

for all $n \geq \delta$.

**Implementation Note 4.4.1.** Setting the option

$$\texttt{SimplifyByExt}- > \texttt{Depth}$$

in functions like `SigmaReduce` and `SolveRecurrence`, introduced in Chapter 1, expressions in terms of sums and products are tried to be simplified by appropriate sum extensions as described above.

### 4.4.3 Definite Summation

Motivating examples for this section can be found in Section 1.3.5. Let $\mathbb{K}$ be a field with characteristic 0, $n$ be transcendental over $\mathbb{K}$ and consider the sum

$$\mathrm{Sum}(n) = \sum_{k=0}^{n} F(n,k)$$

with

$$\langle \mathrm{Sum}(0), \mathrm{Sum}(1), \dots \rangle \in \mathcal{S}(\mathbb{K}).$$

Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field over the constant field $\mathbb{K}(n)$, let $(\mathbb{E},\sigma)$ be a reduced product-sum extension of $(\mathbb{F},\sigma)$ with $\mathbb{E} = \mathbb{F}(t_1,\dots,t_n)$ and let $\mathbb{F}_i := \mathbb{F}(t_1,\dots,t_i)$.

Assume we have a difference ring homomorphism[5]

$$h : \left\{ \begin{array}{lcl} \mathbb{A} & \rightarrow & \mathcal{S}(\mathbb{K}) \\ f & \mapsto & \langle \mathrm{ev}(f,0), \mathrm{ev}(f,1), \dots \rangle \end{array} \right.$$

for a homomorphic map $\mathrm{ev} : \mathbb{A} \times \mathbb{N}_0 \to \mathbb{K}$ bounded by $L : \mathbb{A} \to \mathbb{N}_0$ where $\mathbb{A}$ is a sub-difference ring of $\mathbb{E}$. Furthermore assume that there exists an $\mathbf{f} = (f_1,\dots,f_m) \in \mathbb{A}^m$ with

$$\mathrm{ev}(f_i,k) = F(n+i-1,k)$$

for some $k \geq \gamma$. Additionally, suppose that there do not exist a $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}(n)^m$ and a $g \in \mathbb{F}$ such that

$$\sigma(g) - g = \mathbf{c}\,\mathbf{f} = c_1\,f_1 + \dots + c_m\,f_m. \tag{4.31}$$

Then we could proceed like in Section 4.3.1 with step two. Instead of this we apply Remark 4.4.1 for the case $i = 0,1,2,\dots$ and try to find a proper sum extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}_i$ such that there exist a $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^n$ and a $g \in \mathbb{G}$ with (4.31).

If we fail to find a proper sum extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}_i$ such that there exist a $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^m$ and a $g \in \mathbb{G}$ with (4.31) then we have proven that there does not exist a sum extension in terms of $\mathbb{F}_i$ to solve the problem (4.31).

Let $\mu$ be minimal such that there is not a $j$ with $1 \leq j \leq m$ such that

$$f_j \notin \mathbb{F}_\mu.$$

If we reach the point $i = \mu$ then any found sum extension is as similar complex as the sums $s_i$ in the proper sum extension $(\mathbb{F}(s_1,\dots,s_m),\sigma)$ canonically defined by (4.18). If one finds a proper sum extension over $\mathbb{F}_\mu \setminus \mathbb{F}_{\mu-1}$ then I observed that one can usual find a recurrence of order $m+1$. This means our proper sum extension does not deliver any interesting recurrence.

Now assume we succeed in finding a proper sum extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}_i$ with $i < \mu$ and in finding a $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^m$ and $g \in \mathbb{G}$ with (4.31). Then by Lemma 4.2.2 and its proof we can construct a proper sum extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{F},\sigma)$ and a solution

$$g = k\,s + h$$

with $k \in \mathbb{K}^*$ and $h \in \mathbb{E}$ and $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^m$ with (4.31). If we are able to construct a difference ring homomorphism $h : \mathbb{A}[s] \to \mathcal{S}(\mathbb{K})$ with $g \in \mathbb{A}[\mathrm{s}]$ then we are able to find a recurrence of order $m-1$ for $\mathrm{Sum}[n]$ by using the additional sum $s$.

[5]See Section 2.5 for more details

**Implementation Note 4.4.2.** Setting the option

$$\texttt{SimplifyByExt}->\texttt{Depth}$$

in the function `GenerateRecurrence`, one invokes the strategy described above in order to find a recurrence with an appropriate sum extension.

## 4.5 Difference Field Extensions For Difference Equations

In the following we consider the problem to find appropriate sum extensions in which there exist solutions for a given difference equation of any order. Additionally we sketch how one can find d'Alembertian extension to obtain further solutions for a given difference equation.

This section is inspired by S. A. Abramov's and M. Petkovšek's article [AP94] and P. A. Hendriks's and M. F. Singer's work [HS99]. Let $(\mathbb{F}, \sigma)$ be a difference field, $(\mathbb{A}, \sigma)$ be a difference ring extension of $(\mathbb{F}, \sigma)$ and $\mathcal{L} \in \mathbb{F}[\sigma]^*$ be factored by

$$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \dots \mathcal{L}_n$$

where the $\mathcal{L}_i$ are linear factors in $\sigma$ and $\mathcal{R}$ has no linear right factor in $\sigma$. In [AP94] Abramov and Petkovšek define $y \in \mathbb{A}^*$ with

$$\mathcal{L}(y) = 0$$

as a *d'Alembertian solution* of $\mathcal{L}$, if

$$\mathcal{L}_1 \dots \mathcal{L}_n(y) = 0. \tag{4.32}$$

They developed an algorithm to find all d'Alembertian solutions of $\mathcal{L}$; in order to achieve this, one has to assume that there exists an algorithm which takes an operator $\mathcal{L} \in \mathbb{F}[\sigma]^*$ as input and returns a hyperexponential element $g$ over $\mathbb{F}$ in a difference ring extension of $(\mathbb{F}, \sigma)$ such that $\mathcal{L}(g) = 0$, if it exists. Otherwise the algorithm must tell that there does not exist such an element in any difference ring extension.

Let $(\mathbb{K}(x), \sigma)$ be a difference field canonically defined by

$$\sigma(x) = x + 1$$

where the constant field $\mathbb{K}$ is a subfield of the complex numbers. Clearly, by Section 2.5, the difference field $(\mathbb{K}(x), \sigma)$ can be embedded in the ring of sequences $(\mathcal{S}(\mathbb{K}), S)$. Hendriks and Singer developed a theory based on Galois theory of difference equations to find all *Liouvillian solutions* of $\mathcal{L} \in \mathbb{K}(x)[\sigma]^*$ in $\mathcal{S}(\mathbb{K})$. The main difference between d'Alembertian solutions and Liouvillian solutions over the difference field $(\mathbb{K}(x), \sigma)$ is loosely speaking that one not only factorize $\mathcal{L}$ by hypergeometric sequences $g$ over $\mathbb{K}(x)$ with $\mathcal{L}(g) = 0$ but one may also use the interlacing sequence $g$ of $n$ hypergeometric sequences with $\mathcal{L}(g) = 0$ to factorize $\mathcal{L}$. In particular d'Alembertian solutions are included in Liouvillian solutions for the case $\mathbb{K}(x)$. Moreover, if one restricts to find only d'Alembertian solutions, both algorithms are essentially the same.

Remarkable in the result of [HS99] is the following: if one finds a difference ring extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ which contains all d'Alembertian solutions[6] of $\mathcal{L}$ then any element $g$ of a d'Alembertian ring extension[7] of $(\mathbb{K}(x), \sigma)$ with $\mathcal{L}(g) = 0$ is already contained in the set of d'Alembertian solutions. This means that one has found all solutions of nested sum and product extensions where the product extensions must be over the difference field $(\mathbb{K}(x), \sigma)$.

In the following we will not focus on Liouvillian solutions, since I do not see how one can represent the interlacing $x$ of hypergeometric elements in $\mathcal{S}(\mathbb{K})$ in the form

$$\sigma(x) = \alpha\, x + \beta$$

[6] Actually they are even able to prove this result for Liouvillian solutions.

[7] A difference ring extension $(\mathbb{K}(x)[t_1, \dots, t_n], \sigma)$ of $(\mathbb{K}(x), \sigma)$ is called d'Alembertian, if $t_i$ is a product extension over $\mathbb{K}(x)$ or a sum extension over $\mathbb{K}(x)[t_1, \dots, t_{i-1}]$.

which is the basic way how I consider difference rings and fields in this thesis.

The theory given in [AP94] and [HS99] delivers algorithms with which one can compute difference ring extensions in order to represent all d'Alembertian solutions. But the extensions represented in these difference rings are highly algebraic. Besides finding such extensions and computing the solutions of a difference equation in these extensions, I consider it as an essential step to eliminate these algebraic relations. In order to achieve this, I restrict myself not only to d'Alembertian solutions of a difference ring but I mainly focus on sum solutions; this means one can work in difference fields. In order to eliminate algebraic relations in those sum extensions, we can therefore use the indefinite summation algorithm described in the previous chapter which is based on $\Pi\Sigma$-fields.

Furthermore the sums delivered by the following algorithms are highly nested defined. With the results of Section 4.4 we can reduce depth of the sum extensions, if possible, and can therefore simplify the result further. These last remarks are illustrated in Section 1.3.4.2.

In addition, in Section 4.5.4 I describe how one can avoid algebraic relations in that sum-extensions during its construction.

### 4.5.1 Right Division of First Order Linear Shift Operators

**Lemma 4.5.1.** *Let $(\mathbb{F},\sigma)$ be a difference field, $\mathcal{L}\in\mathbb{F}[\sigma]^*$ and let $(\mathbb{E},\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ in which we have $d\in\mathbb{E}^*$ hyperexponential over $\mathbb{F}$. Then:*

$$\mathcal{L}(d)=0$$
$$\Updownarrow$$
$$\exists\mathcal{R}\in\mathbb{F}[\sigma]^*:\mathcal{L}=\mathcal{R}(\sigma-\frac{\sigma(d)}{d}).$$

*Proof.* Let $d\in\mathbb{E}^*$ hyperexponential over $\mathbb{F}$ with

$$\mathcal{L}(d)=0.$$

Since

$$(\sigma-\frac{\sigma(d)}{d})(d)=0$$

and $\sigma-\frac{\sigma(d)}{d}\in\mathbb{F}[\sigma]$, it follows by the Euclidean right division that there is an $\mathcal{R}\in\mathbb{F}[\sigma]^*$ with

$$\mathcal{L}=\mathcal{R}(\sigma-\frac{\sigma(d)}{d}).$$

Contrary, if there is an $\mathcal{R}\in\mathbb{F}[\sigma]^*$ with

$$\mathcal{L}=\mathcal{R}(\sigma-\frac{\sigma(d)}{d})$$

then

$$\mathcal{L}(d)=\mathcal{R}(\sigma-\frac{\sigma(d)}{d})(d)=\mathcal{R}(0)=0.$$

□

**Lemma 4.5.2.** *Let $(\mathbb{F},\sigma)$ be a difference field and $\mathcal{L}\in\mathbb{F}[\sigma]^*$. Then there exists a factorization*

$$\mathcal{L}=\mathcal{R}\,\mathcal{L}_1\cdots\mathcal{L}_n$$

*where $\mathcal{L}_i=\sigma-\frac{\sigma(d_i)}{d_i}$ with $d_i\in\mathbb{F}^*$ and $\mathcal{R}\in\mathbb{F}[\sigma]^*$ such that there does not exist a $d\in\mathbb{F}^*$ with*

$$\sigma-\frac{\sigma(d)}{d}\mid\mathcal{R}.$$

*Proof.* We will do the proof by induction on the order of $\mathcal{L}$. If the order is 0, i.e. $\mathcal{L}\in\mathbb{F}^*$ then the lemma clearly holds. Now assume the lemma holds for all difference operators of order $m\geq 0$ and let $\mathcal{L}\in\mathbb{F}[\sigma]^*$ be of order $m+1$. If there does not exist a $d\in\mathbb{F}$ with

$$\sigma-\frac{\sigma(d)}{d}\mid\mathcal{L}$$

then the lemma also holds. Otherwise, assume there is such a $d$. Then by Lemma 4.5.1 there is an $\mathcal{R}\in\mathbb{F}[\sigma]^*$ of order $m$ with

$$\mathcal{L}=\mathcal{R}\,(\sigma-\frac{\sigma(d)}{d})$$

and therefore by the induction assumption the lemma follows. □

## 4.5.2 Sum Extensions

### 4.5.2.1 A Criterion for Existence of Sum Extensions

In this section we will find a criterion which tells us if there cannot exist a sum extension which yields to additional solutions of a given difference equation.

**Lemma 4.5.3.** *Let* $(\mathbb{E}, \sigma)$ *be a reduced sum extension of* $(\mathbb{F}, \sigma)$*,* $f \in \mathbb{F}$ *and* $\mathcal{L} \in \mathbb{F}[\sigma]^*$*. Then*

$$\exists g \in \mathbb{E} \setminus \mathbb{F} : \mathcal{L}(g) = f$$

$$\Downarrow$$

$$\exists g \in \mathbb{F}^* : \mathcal{L}(g) = 0.$$

*Proof.* We will do the proof by induction on the number n of extensions in the difference field $(\mathbb{E}, \sigma)$. For $n = 0$ the theorem clearly holds. Now assume that the theorem holds for $n$ extensions and consider the proper sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $n + 1$ extensions, i.e. $\mathbb{E} = \mathbb{F}(s_1, \dots, s_{n+1})$. Let $g \in \mathbb{E} \setminus \mathbb{F}$ with

$$\mathcal{L}(g) = f.$$

If $g \in \mathbb{F}(s_1, \dots, s_n)$ then by the induction assumption the theorem follows. Otherwise, $g \in \mathbb{E} \setminus \mathbb{G}$ where $\mathbb{G} := \mathbb{F}(s_1, \dots, s_n)$. By Proposition 4.1.2 it follows that

$$g \in \mathbb{G}[s_{n+1}]$$

with $\deg(g) > 0$. Thus there are a $k > 0$, $h \in \mathbb{G}^*$ and a $u \in \mathbb{G}[s_{n+1}]$ with $\deg(u) < k$ such that

$$g = s_{n+1}^k h + u.$$

Since $\mathcal{L}(g) \in \mathbb{F}$ and $\mathcal{L} \in \mathbb{F}[\sigma]^*$, it follows that

$$\mathcal{L}(h) = 0$$

and therefore by the induction assumption we may conclude that there is an $h' \in \mathbb{F}$ with

$$\mathcal{L}(h') = 0.$$

□

**Theorem 4.5.1.** *Let* $(\mathbb{E}, \sigma)$ *be a sum extension of* $(\mathbb{F}, \sigma)$*. Let* $f \in \mathbb{F}$ *and* $\mathcal{L} \in \mathbb{F}[\sigma]^*$*. Then*

$$\exists g \in \mathbb{E} \setminus \mathbb{F} : \mathcal{L}(g) = f$$

$$\Downarrow$$

$$\exists g \in \mathbb{F}^* : \mathcal{L}(g) = 0.$$

*Proof.* By Proposition 2.4.3 there exists a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$(\mathbb{G}, \sigma) \simeq (\mathbb{E}, \sigma)$$

and thus by Lemma 4.5.3 the theorem is proven. □

**Remark 4.5.1.** This theorem is the essential result to find sum extensions which deliver additional solutions of a given recurrence.

- In particular, this theorem says that there can exist only a sum extension which delivers further solutions if there exists already a solution in the underlying difference field. Contrary, if there does not exist a solution in the underlying difference field, there cannot exist a sum extension which gives a solution. In this case one has only a chance to find a solution, if one extends the underlying difference field by extensions where products are involved as will be discussed in Section 4.5.3.

- On one side, if one specializes my theorem to the difference field case $(\mathbb{K}(x), \sigma)$ canonically defined by
$$\sigma(x) = x + 1$$
with constant field $\mathbb{K}$ and considers only the homogeneous case $f = 0$, one obtains a special case of [HS99, Theorem 5.1]. That theorem deals with the more general case of Liouvillian solutions of a given difference equation; my sum solutions are included in Liouvillian solutions.

  On the other side, I consider also the inhomogeneous case of difference equations. Furthermore, my theorem can be applied for any difference field and not only for the simple case $(\mathbb{K}(x), \sigma)$.

$\diamondsuit$

#### 4.5.2.2 Complete Sum Extensions for Linear Difference Equations

**Definition 4.5.1.** Let $(\mathbb{E}, \sigma)$ be a sum extension of $(\mathbb{F}, \sigma)$ with $\mathrm{const}_\sigma\mathbb{E} = \mathrm{const}_\sigma\mathbb{F} =: \mathbb{K}$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}$. $(\mathbb{E}, \sigma)$ is called *complete* w.r.t. $(\mathcal{L}, f)$, if whenever there is a sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ with $\mathrm{const}_\sigma\mathbb{G} = \mathbb{K}$ such that there is a $g \in \mathbb{G}$ with
$$\mathcal{L}(g) \in \{f, 0\}$$
then we have $g \in \mathbb{E}$. $\diamondsuit$

Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}$ and assume we have a sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ which is complete w.r.t. $(\mathcal{L}, 0)$. Now let $(\mathbb{G}, \sigma)$ be any sum extension of $(\mathbb{F}, \sigma)$ with $\mathbb{G} = \mathbb{F}(t_1, \dots, t_n)$ and $\mathrm{const}_\sigma\mathbb{G} = \mathbb{K}$ such that we get a solution $g \in \mathbb{G} \setminus \mathbb{F}$ with
$$\mathcal{L}(g) \in \{f, 0\}.$$
Then we can construct by Corollary 2.4.1 a sum extension $(\mathbb{E}(t_1, \dots, t_n), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\mathrm{const}_\sigma\mathbb{E}(t_1, \dots, t_n) = \mathbb{K}$ and clearly we have
$$g \in \mathbb{E}(t_1, \dots, t_n).$$
But since $(\mathbb{E}, \sigma)$ is complete w.r.t. $(\mathcal{L}, f)$, it follows that $g \in \mathbb{E}$. Therefore if there exists a complete sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ w.r.t. $(\mathcal{L}, f)$ and we can construct it, we will get all solutions which live in a sum extension.

**Theorem 4.5.2.** *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}$. Suppose there is a sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ which is complete w.r.t. $(\mathcal{L}, f)$. Assume there is a $g \in \mathbb{G} \setminus \mathbb{F}$ such that*
$$\mathcal{L}(g) = f.$$

*Then there are a $d \in \mathbb{F}^*$ and a $\mathcal{R} \in \mathbb{F}[\sigma]^*$ such that*

$$\mathcal{L} = \mathcal{R}(\sigma - \frac{\sigma(d)}{d}).$$

*Furthermore, there is a $w \in \mathbb{G}^*$ such that*

$$\mathcal{R}(w) = f.$$

*Additionally, there is a $t \in \mathbb{G}$ with*

$$\mathcal{L}(d\,t) = f$$

*and $\sigma(t) = t + \frac{w}{\sigma(d)}$.*

*Proof.* Assume there is a $g \in \mathbb{G} \setminus \mathbb{F}$ such that

$$\mathcal{L}(g) = f.$$

Then by Theorem 4.5.1 we find a $d \in \mathbb{F}^*$ such that

$$\mathcal{L}(d) = 0.$$

Thus by Lemma 4.5.1 we can find an $\mathcal{R} \in \mathbb{F}[\sigma]^*$ with

$$\mathcal{L} = \mathcal{R}(\sigma - \frac{\sigma(d)}{d}).$$

For

$$w := (\sigma - \frac{\sigma(d)}{d})(g) \in \mathbb{G}$$

it follows immediately that

$$\mathcal{R}(w) = f.$$

If $w = 0$ then

$$\sigma(g) - \frac{\sigma d}{d}\,g = 0 \;\Leftrightarrow\; \sigma(\frac{g}{d}) = \frac{g}{d}$$

and hence $\frac{g}{d} \in \mathbb{K}$, a contradiction to $g \notin \mathbb{F}$. Hence $w \neq 0$. By Proposition 2.4.1 there is a difference field $(\mathbb{G}(t), \sigma)$ of $(\mathbb{G}, \sigma)$ with

$$\sigma(t) = t + \frac{w}{\sigma(d)}$$

and $\mathrm{const}_\sigma \mathbb{G}(t) = \mathbb{K}$. We have

$$\mathcal{L}(d\,t) = \mathcal{R}(\sigma(d\,t) - \sigma(d)\,t) = \mathcal{R}(w) = f$$

and thus $t \in \mathbb{G}$. $\square$

**Algorithm 4.5.1.** Find a sum solution of an inhomogeneous recurrence

**$((\mathbb{E}, \sigma)$,$p)$=FindOneSumSolutionsForInhomEqu$((\mathbb{F}, \sigma), \mathcal{L}, f)$**

- Input: A difference field $(\mathbb{F}, \sigma)$ with $\mathbb{K} = \mathrm{const}_\sigma \mathbb{F}$ in which one can solve linear difference equations, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}^*$.
- Output: A sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathrm{const}_\sigma \mathbb{E} = \mathbb{K}$ in which there is a $p \in \mathbb{E}^*$ with $\mathcal{L}(p) = f$. If there does not exist such a sum extension then $((\mathbb{E}, \sigma)$,$p)$:=$((\mathbb{F}, \sigma)$,$\perp)$

(1) Compute for $\mathbf{a} := \text{vect}(\mathcal{L})$ the solution space $\text{V}(\mathbf{a}, (f), \mathbb{F})$ where $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^{\text{order}(\mathcal{L})+1}$
(2) IF there exists a $p \in \mathbb{F}^*$ with $\mathcal{L}(p) = f$
  THEN Return($(\mathbb{F}, \sigma)$,$p$)
(3) IF there does not exist a $d \in \mathbb{F}^*$ with $\mathcal{L}(d) = 0$
  THEN RETURN($(\mathbb{F}, \sigma), \perp$)
  ELSE TAKE such a $d$.
(4) COMPUTE $\mathcal{R} \in \mathbb{F}[\sigma]^*$ such that $\mathcal{L} = \mathcal{R}(\sigma - \frac{\sigma(d)}{d})$
(5) $((\mathbb{G}, \sigma), p)$=FindOneSumSolutionForInhomEqu($(\mathbb{F}, \sigma), \mathcal{R}, f$)
(6) IF $p = \perp$ THEN RETURN $((\mathbb{F}, \sigma), \perp)$
(7) CONSTRUCT the sum extension $(\mathbb{G}(s), \sigma)$ of $(\mathbb{G}, \sigma)$ with
  $\sigma(s) = s + \frac{p}{\sigma(d)}$ and
  $\text{const}_\sigma \mathbb{G}(s) = \mathbb{K}$
(8) RETURN $((\mathbb{G}(s), \sigma), d\,s)$

**Proposition 4.5.1.** *Let $(\mathbb{F}, \sigma)$ be a difference field in which one can solve linear difference equations, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}^*$. Then Algorithm 4.5.1 with input $((\mathbb{F}, \sigma), \mathcal{L}, f)$ terminates. If there does not exist a sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with the same constant field and with $g \in \mathbb{G}$ such that $\mathcal{L}(g) = f$, then the output is $((\mathbb{F}, \sigma), \perp)$. Otherwise, the output is $((\mathbb{G}, \sigma), g)$ with a sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ and $g \in \mathbb{G}$ with $\mathcal{L}(g) = f$.*

*Proof.* **Termination:** If the termination condition (2) is not fulfilled in the first $n$ recursion steps then one factors $\mathcal{L} \in \mathbb{F}[\sigma]^*$ with order $m$ to

$$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_n$$

where the $\mathcal{L}_i \in \mathbb{F}[\sigma]$ have order one and $\mathcal{R} \in \mathbb{F}[\sigma]^*$. Thus after at most $m$ recursion steps condition (3) is satisfied and the algorithm terminates.

**Correctness:** If there is a $p \in \mathbb{F}$ with

$$\mathcal{L}(p) = f$$

in Step (2) then the output $((\mathbb{F}, \sigma), p)$ for a solution $\mathcal{L}(p) = f$ is correct.

Otherwise assume there does not exist a $p \in \mathbb{F}^*$ with $\mathcal{L}(p) = f$. If there does not exist a $d \in \mathbb{F}^*$ with $\mathcal{L}(d) = 0$ then by Lemma 4.5.1 there is not a $d \in \mathbb{F}^*$ with $\sigma - \frac{\sigma(d)}{d} \mid \mathcal{L}$ and thus by Theorem 4.5.2 there is no sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with the same constant field and with $g \in \mathbb{G}$ such that $\mathcal{L}(g) = f$. Therefore the output $((\mathbb{F}, \sigma), \perp)$ in Step (3) is correct.

Otherwise assume there is a $d \in \mathbb{F}^*$ with $\mathcal{L}(d) = 0$ and let $\mathcal{R} \in \mathbb{F}[\sigma]^*$ be defined by

$$\mathcal{L} = \mathcal{R}\,(\sigma - \frac{\sigma(d)}{d}).$$

Now assume that in step (5) of Algorithm 4.5.1 the output $((\mathbb{E}, \sigma), p)$ is correct. If $p = \perp$ then by Theorem 4.5.2 there does not exist a sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with the same constant field and $g \in \mathbb{G}$ with $\mathcal{L}(g) = f$ and therefore the output $((\mathbb{F}, \sigma), \perp)$ is correct in Step (6).

Otherwise, if $p \in \mathbb{E}^*$ then by Theorem 4.5.2 it follows immediately that

$$\mathcal{L}(d\,s) = f$$

and consequently the output $((\mathbb{E}(s), \sigma), d\,s)$ is correct in Step (8). $\square$

**Remark 4.5.2.** Please note that in Algorithm 4.5.1 one obtains a sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ which is **not proper**. But by Proposition 2.4.3 there exists a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$(\mathbb{E}, \sigma) \overset{\tau}{\simeq} (\mathbb{G}, \sigma).$$

In particular, by indefinite summation, i.e. by our indefinite summation algorithm, the difference field $(\mathbb{G}, \sigma)$ and the isomorphism $\tau$ can be computed and therefore one can transform the solution $g \in \mathbb{E}$ of Algorithm 4.5.1 to $\tau(g) \in \mathbb{G}$ where the element $\tau(g)$ is expressed by proper sums.

Finally this result $\tau(g)$ in a proper sum extension is **highly recursively** defined. Using Section 4.4 one can simplify this proper sum extension by representing it by a proper sum extension with a lower recursion depth. These observations are illustrated in Section 1.3.4.2.

$\Diamond$

#### 4.5.2.3 Sum Extensions and Factorization of Difference Operators in Linear Right Factors

The following theorem emphasizes the link between sum extensions which are complete w.r.t. $(\mathcal{L}, f)$ and the factorization of a difference operator into linear right factors.

**Theorem 4.5.3.** *Let $(\mathbb{F}, \sigma)$ be a difference field, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}$. Let $(\mathbb{G}, \sigma)$ be a sum extension of $(\mathbb{F}, \sigma)$ which is complete w.r.t. $(\mathcal{L}, f)$. Assume there is a $g \in \mathbb{G} \setminus \mathbb{F}$ such that*

$$\mathcal{L}(g) = f.$$

*Then one can find $\mathcal{L}_i = \sigma - \frac{\sigma(d_i)}{d_i}$ with $d_i \in \mathbb{F}^*$ and $\mathcal{R} \in \mathbb{F}[\sigma]^*$ such that*

1. *$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_n$ with $n \geq 1$,*
2. *$\mathcal{R}$ has no first order linear right factor of the type $\mathcal{L}_i$ and*
3. *$\forall g \in \mathbb{G} : (\mathcal{L}(g) = f \Rightarrow \mathcal{L}_1 \cdots \mathcal{L}_n(g) \in \mathbb{F})$.*

*Proof.* By Lemma 4.5.2 we find a factorization

$$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_n$$

with $n \geq 0$, $\mathcal{L}_i = \sigma - \frac{\sigma(d_i)}{d_i}$ for some $d_i \in \mathbb{F}^*$ and

$$\nexists d \in \mathbb{F}^* : \ \sigma - \frac{\sigma(d)}{d} \mid \mathcal{R}. \tag{4.33}$$

By Theorem 4.5.2 there are a $d \in \mathbb{F}^*$ and an $\mathcal{R} \in \mathbb{F}[\sigma]^*$ with

$$\mathcal{L} = \mathcal{R}(\sigma - \frac{\sigma(d)}{d}),$$

consequently $n \geq 1$ and hence the first two statements of the theorem are proven. We will prove the third statement by induction on $n$. Assume we have

$$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1$$

with (4.33). Thus by Lemma 4.5.1, there is no $d \in \mathbb{F}^*$ with

$$\mathcal{R}(d) = 0$$

and thus by Theorem 4.5.1 it follows that

$$\nexists u \in \mathbb{G} \setminus \mathbb{F} : \mathcal{R}(u) = f. \tag{4.34}$$

Let $h \in \mathbb{G}$ with

$$\mathcal{L}(h) = f.$$

Then for $w := \mathcal{L}_1(h) \in \mathbb{G}$ we have

$$\mathcal{R}(w) = f$$

and by (4.34) it follows that $w \in \mathbb{F}$. Therefore the induction base holds. Now assume the statement of the theorem holds for a factorization of length $n - 1$ and consider

$$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_{n-1}\mathcal{L}_n$$

with (4.33) and $\mathcal{L}_i = \sigma - \frac{\sigma(d_i)}{d_i}$ for some $d_i \in \mathbb{F}^*$. Let $h \in \mathbb{G}$ with

$$\mathcal{L}(h) = f$$

and take

$$w := \mathcal{L}_n(h) \in \mathbb{G}.$$

We have

$$\mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_{n-1}(w) = f,$$

thus by the induction assumption

$$\mathcal{L}_1 \cdots \mathcal{L}_{n-1}(w) \in \mathbb{F}$$

and hence

$$\mathcal{L}_1 \cdots \mathcal{L}_n(h) \in \mathbb{F}.$$

□

Specializing this theorem to the case $f = 0$ with the difference field $(\mathbb{K}(x), \sigma)$ canonically induced by $\sigma(x) = x + 1$ yields to a result which is included in [HS99, Theorem 5.5].

Furthermore, one can see that sum solutions are a special case of d'Alembertian solutions defined in [AP94], if one looks at (4.32) which defines d'Alembertian solutions.

**Corollary 4.5.1.** *Let $(\mathbb{F}, \sigma)$ be a difference field, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and let $(\mathbb{G}, \sigma)$ be a reduced sum extension of $(\mathbb{F}, \sigma)$ which is complete w.r.t. $(\mathcal{L}, 0)$. Assume there is a $g \in \mathbb{G} \setminus \mathbb{F}$ such that*

$$\mathcal{L}(g) = 0.$$

*Then one can find $\mathcal{L}_i = \sigma - \frac{\sigma(d_i)}{d_i}$ with $d_i \in \mathbb{F}^*$ and $\mathcal{R} \in \mathbb{F}[\sigma]^*$ such that*

1. *$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_n$ with $n \geq 1$,*
2. *$\mathcal{R}$ has no first order linear right factor of type $\mathcal{L}_i$ and*
3. *$\forall g \in \mathbb{G} : (\mathcal{L}(g) = 0 \Rightarrow \mathcal{L}_1 \cdots \mathcal{L}_n(g) = 0)$*

*Proof.* By Theorem 4.5.3 there are $\mathcal{L}_i = \sigma - \frac{\sigma(d_i)}{d_i}$ with $d_i \in \mathbb{F}^*$ and $\mathcal{R} \in \mathbb{F}[\sigma]^*$ such that

1. $\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_n$ with $n \geq 1$,
2. $\mathcal{R}$ has no first order linear right factor of type $\mathcal{L}_i$ and
3. $\forall g \in \mathbb{G} : (\mathcal{L}(g) = 0 \Rightarrow \mathcal{L}_1 \cdots \mathcal{L}_n(g) \in \mathbb{F})$.

Let $g \in \mathbb{G}$ with

$$\mathcal{L}(g) = 0$$

and assume

$$w := \mathcal{L}_1 \cdots \mathcal{L}_n(g) \neq 0.$$

Thus

$$\mathcal{R}(w) = 0$$

and therefore by Lemma 4.5.1 $\mathcal{R}$ has the right factor $\sigma - \frac{\sigma(w)}{w}$, a contradiction. □

#### 4.5.2.4 Finding Complete Sum Extensions

The following theorem delivers an algorithm to find sum extensions which are complete w.r.t. $(\mathcal{L}_1, 0)$.

**Theorem 4.5.4.** *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\mathcal{L}_1 \in \mathbb{F}[\sigma]^*$, $d \in \mathbb{F}^*$ and $\mathcal{L}_2 := \sigma - \frac{\sigma(d)}{d}$. Assume there exists a sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathrm{const}_\sigma\mathbb{E} = \mathbb{K}$ which is complete w.r.t. $(\mathcal{L}_1, 0)$ and let*

$$\{f_1, \dots, f_k\} \subseteq \mathbb{E}^*$$

*be a basis of $\ker \mathcal{L}_1$. Then there is a sum extension $(\mathbb{E}(t_1, \dots, t_k), \sigma)$ of $(\mathbb{F}, \sigma)$ with*

$$\sigma(t_i) = t_i + \frac{f_i}{\sigma(d)}$$

*for $1 \leq i \leq k$ and $\mathrm{const}_\sigma\mathbb{E}(t_1, \dots, t_k) = \mathbb{K}$. Furthermore $(\mathbb{E}(t_1, \dots, t_k), \sigma)$ is complete w.r.t. $(\mathcal{L}_1 \mathcal{L}_2, 0)$ and*

$$\{d, d\,t_1, \dots, d\,t_k\} \subseteq \mathbb{E}(t_1, \dots, t_k)$$

*is a basis of $\ker \mathcal{L}_1 \mathcal{L}_2$.*

*Proof.* Let $(\mathbb{H}, \sigma)$ with $\mathbb{H} := \mathbb{E}(t_1, \dots, t_k)$ be as stated in the theorem. First we consider the case $k = 0$, i.e. there does not exist a $p \in \mathbb{F}^*$ with $\mathcal{L}_1(p) = 0$. By Lemma 4.5.1 it follows that there does not exist a $p \in \mathbb{F}^*$ with $\sigma - \frac{\sigma(p)}{p} \mid \mathcal{L}_1$. Let $(\mathbb{G}, \sigma)$ be a sum extension of $(\mathbb{H}, \sigma)$ with constant field $\mathbb{K}$ and $g \in \mathbb{G}^*$ with

$$\mathcal{L}_1 \mathcal{L}_2(g) = 0.$$

Then by Corollary 4.5.1 it follows that

$$0 = \mathcal{L}_2(g) = \sigma(g) - \frac{\sigma(d)}{d} g,$$

therefore

$$\sigma(\frac{g}{d}) = \frac{g}{d}$$

and consequently $g = c\,d$ for some $c \in \mathbb{K}$. It follows that $(\mathbb{E}, \sigma)$ is already complete w.r.t. $(\mathcal{L}_1 \mathcal{L}_2, 0)$ and $\{d\}$ is a basis of $\ker \mathcal{L}_1 \mathcal{L}_2$. Consequently the induction base holds. Now consider the case $n > 0$. We have for all i:

$$\mathcal{L}_2(d\,t_i) = \sigma(d\,t_i) - \sigma(d)\,t_i = \sigma(d)(\sigma(t_i) - t_i) = \sigma(d)\frac{f_i}{\sigma(d)} = f_i$$

and thus

$$\mathcal{L}_2(d\,t_i) = f_i. \tag{4.35}$$

Consequently

$$\mathcal{L}_1 \mathcal{L}_2(d\,t_i) = 0$$

and therefore

$$\langle d, d\,t_1, \dots, d\,t_k \rangle_{\mathbb{K}} \subseteq \mathrm{Ker}_{\mathbb{H}}\, \mathcal{L}_1 \mathcal{L}_2.$$

Let $(\mathbb{G}, \sigma)$ be a sum extension of $(\mathbb{H}, \sigma)$ with constant field $\mathbb{K}$ and $g \in \mathbb{G}$ with

$$\mathcal{L}_1 \mathcal{L}_2(g) = 0.$$

Take
$$w := \mathcal{L}_2(g).$$
As $(\mathbb{H}, \sigma)$ is complete w.r.t. $(\mathcal{L}_1, 0)$ and $\mathcal{L}_1(w) = 0$, we have $w \in \mathbb{E}$. Thus there are $c_i \in \mathbb{K}$ with
$$\mathcal{L}_2(g) = w = \sum_{i=1}^{k} c_i\, f_i$$
for some $c_i \in \mathbb{K}$. By Equation (4.35) it follows for
$$h := d \sum_{j=1}^{k} c_j\, t_j$$
that
$$\mathcal{L}_2(h) = \mathcal{L}_2(d \sum_{j=1}^{k} c_j\, t_j) = \sum_{j=1}^{k} c_j\, \mathcal{L}_1(d\, t_j) = \sum_{j=1}^{k} c_j\, f_j.$$
Thus by
$$\begin{aligned}
\mathcal{L}_2(g) = \mathcal{L}_2(h) &\Leftrightarrow \mathcal{L}_2(g-h) = 0\\
&\Leftrightarrow (\sigma - \frac{\sigma(d)}{d})(g-h) = 0\\
&\Leftrightarrow \sigma(g-h) = \frac{\sigma(d)}{d}(g-h)\\
&\Leftrightarrow \frac{\sigma(g-h)}{g-h} = \frac{\sigma(d)}{d}\\
&\Leftrightarrow \sigma(\frac{g-h}{d}) = \frac{g-h}{d}
\end{aligned}$$
there is a $x \in \mathbb{K}$ such that
$$g - h = x\, d$$
and therefore
$$g = d \sum_{j=1}^{k} c_j\, t_j + x\, d \in \mathbb{E}(t_1, \dots, t_n).$$
Consequently $\mathbb{E}(t_1, \dots, t_k)$ is already complete w.r.t. $(\mathcal{L}_1\, \mathcal{L}_2, 0)$ and
$$\langle d, d\, t_1, \dots, d\, t_k\rangle_{\mathbb{K}} = \mathrm{Ker}_H\, \mathcal{L}_1\, \mathcal{L}_2.$$
Finally we will prove that $\{d, d\, t_1, \dots, d\, t_k\}$ is linearly independent over $\mathbb{K}$. Assume there are $c_i \in \mathbb{K}$ with
$$c_1\, d\, t_1 + \dots + c_k\, d\, t_k + c_{k+1}\, d = 0.$$
Hence by applying $\mathcal{L}_2$ on the equation we get
$$c_1\, f_1 + \dots + c_k\, f_k = 0$$
and thus $c_1, \dots, c_k = 0$ because $\{f_1, \dots, f_k\}$ is linearly independent over $\mathbb{K}$. But then also $c_{k+1} = 0$ and therefore $\{d, d\, t_1, \dots, d\, t_k\}$ is linearly independent over $\mathbb{K}$.

$\square$

Given a difference equation $\mathcal{L}(g) = 0$ in a difference field $(\mathbb{F}, \sigma)$, by Theorem 4.5.4 we obtain the following algorithm in order to compute a sum extension $(\mathbb{E}, \sigma)$ which is complete w.r.t. $(\mathcal{L}_1, 0)$ and to find a basis of all solutions in $\mathbb{E}$. Please note that this algorithm is the same as introduced in [AP94], if one restricts only to sum extensions.

**Algorithm 4.5.2.** Find all sum solutions for a homogeneous recurrence

**$((\mathbb{E}, \sigma)$,$B$)=FindHomSumSolutions($(\mathbb{F}, \sigma), \mathcal{L}$)**

Input: A difference field $(\mathbb{F}, \sigma)$ with $\mathbb{K} = \text{const}_\sigma \mathbb{F}$ in which one can solve homogeneous linear difference equations and $\mathcal{L} \in \mathbb{F}[\sigma]^*$.

Output: A sum extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\text{const}_\sigma \mathbb{E} = \mathbb{K}$ which is complete w.r.t. $(\mathcal{L}, 0)$.
A basis $B$ of $\text{Ker}_{\mathcal{L}}\, \mathbb{E}$.

(1) Compute for $\mathbf{a} := \text{vect}(\mathcal{L})$ the solution space $\text{V}(\mathbf{a}, (0), (\mathbb{F}, \sigma))$ where $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^{\text{order}(\mathcal{L})+1}$

(2) IF there is no $d \in \mathbb{F}^*$ with $\mathcal{L}(d) = 0$
THEN RETURN($(\mathbb{F}, \sigma), \emptyset$)
ELSE TAKE such a $d$.

(3) COMPUTE $\mathcal{R} \in \mathbb{F}[\sigma]^*$ such that $\mathcal{L} = \mathcal{R}(\sigma - \frac{\sigma(d)}{d})$

(4) $((\mathbb{G}, \sigma), \{g_1, \dots, g_k\})$=FindHomSumSolutions($(\mathbb{F}, \sigma), \mathcal{R}$)

(5) CONSTRUCT the sum extension $(\mathbb{G}(t_1, \dots, t_k), \sigma)$ of $(\mathbb{G}, \sigma)$ with
$\sigma(t_i) = t_i + \frac{g_i}{\sigma(d)}$ for $(1 \leq i \leq k)$ and
$\text{const}_\sigma \mathbb{G}(t_1, \dots, t_k) = \mathbb{K}$

(6) RETURN $((\mathbb{G}(t_1, \dots, t_k), \sigma), \{d, d\,t_1, \dots, d\,t_k\})$

**Proposition 4.5.2.** *Let $(\mathbb{F}, \sigma)$ be a difference field in which one can solve homogeneous linear difference equations and $\mathcal{L} \in \mathbb{F}[\sigma]^*$. Then Algorithm 4.5.2 with input $((\mathbb{F}, \sigma), \mathcal{L})$ terminates. Let $((\mathbb{E}, \sigma)$,$B)$ be the output. Then $(\mathbb{E}, \sigma)$ is a sum extension of $(\mathbb{F}, \sigma)$ with $\text{const}_\sigma \mathbb{E} = \text{const}_\sigma \mathbb{F}$ which is complete w.r.t. $(\mathcal{L}, 0)$ and $B$ is a basis of $\text{Ker}_{\mathbb{E}}\, \mathcal{L}$*

*Proof.* **Termination:** In the algorithm one factors $\mathcal{L} \in \mathbb{F}[\sigma]^*$ with order $m$ to

$$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_n$$

where the $\mathcal{L}_i \in \mathbb{F}[\sigma]$ have order one and $\mathcal{R} \in \mathbb{F}[\sigma]^*$ has no further right factor $\sigma - \frac{\sigma(d)}{d}$ for some $d \in \mathbb{F}^*$. Thus after at most $m$ recursion steps the algorithm terminates.

**Correctness:** If there is not a $d \in \mathbb{F}^*$ with $\mathcal{L}(d) = 0$ then by Theorem 4.5.1 $(\mathbb{F}, \sigma)$ is complete w.r.t. $(\mathcal{L}, 0)$ and 0 is the only solution of $\mathcal{L}$ in $(\mathbb{F}, \sigma)$. Otherwise assume that in step (4) of algorithm 4.5.2 one obtains a sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ that is complete w.r.t. $(\mathcal{R}, 0)$ and a basis $\{g_1, \dots, g_k\}$ for $\text{Ker}_{\mathbb{G}}\, \mathcal{L}$. Then by Theorem 4.5.4 the algorithm delivers the output as stated in the theorem. □

**Remark 4.5.3.** Looking at Remark 4.5.2, we see that one can construct a proper sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{F}, \sigma)$ and a difference field automorphism

$$(\mathbb{E}, \sigma) \overset{\tau}{\simeq} (\mathbb{G}, \sigma)$$

if one can solve first order linear difference equations in $(\mathbb{F}, \sigma)$. Therefore we can transform the solution set $B$ to solutions expressed by proper sums.

Again note that one gets a proper sum extension which is **highly recursively** defined. Using Section 4.4 one can simplify this proper sum extension by representing it by a proper sum extension with a lower recursion depth. ◊

Combining Algorithms 4.5.1 and 4.5.2 one gets the following algorithm:

**Algorithm 4.5.3.** Find all sum solutions for a inhomogeneous recurrence

**$((\mathbb{E},\sigma)$,$B$,$p$)=FindSumSolutionsForSolutionSpace(**$(\mathbb{F},\sigma),\mathcal{L},f$**)**

Input: A difference field $(\mathbb{F},\sigma)$ with $\mathbb{K}=\text{const}_\sigma\mathbb{F}$ in which one can solve linear difference equations, $\mathcal{L}\in\mathbb{F}[\sigma]^*$ and $f\in\mathbb{F}^*$.

Output: A sum extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\text{const}_\sigma\mathbb{E}=\mathbb{K}$ which is complete w.r.t. $(\mathcal{L},f)$.
A basis $B$ of $\text{Ker}_\mathcal{L}\,\mathbb{E}$.
$p:=\begin{cases} q\in\mathbb{E}^*:\mathcal{L}(q)=f & \text{if such a } q \text{ exists}\\ \perp & \text{otherwise}\end{cases}$

(1) Compute for $\mathbf{a}:=\text{vect}(\mathcal{L})$ the solution space $\text{V}(\mathbf{a},(f),(\mathbb{F},\sigma))$ where $\mathbf{0}\neq\mathbf{a}\in\mathbb{F}^{\text{order}(\mathcal{L})+1}$

(2) Let
$$p:=\begin{cases} g\in\mathbb{F}:\mathcal{L}(g)=f & \text{if such a } g \text{ exists}\\ \perp & \text{otherwise}\end{cases}$$

(3) IF there is not a $d\in\mathbb{F}^*$ with $\mathcal{L}(d)=0$
THEN RETURN($(\mathbb{F},\sigma),\emptyset,p$)
ELSE TAKE such a $d$.

(4) COMPUTE $\mathcal{R}\in\mathbb{F}[\sigma]^*$ such that $\mathcal{L}=\mathcal{R}(\sigma-\frac{\sigma(d)}{d})$

(5) IF $p\neq\perp$ THEN

(6) $((\mathbb{G},\sigma),\{g_1,\dots,g_k\})$=FindHomSumSolutions($(\mathbb{F},\sigma),\mathcal{R}$)

(7) CONSTRUCT the sum extension $\mathbb{G}(t_1,\dots,t_k)$ of $(\mathbb{G},\sigma)$ with
$\sigma(t_i)=t_i+\frac{g_i}{\sigma(d)}$ for $(1\leq i\leq k)$ and
$\text{const}_\sigma\mathbb{G}(t_1,\dots,t_k)=\mathbb{K}$

(8) RETURN $((\mathbb{G}(t_1,\dots,t_k),\sigma),\{d,d\,t_1,\dots,d\,t_k\},p)$

(9) ELSE

(10) $((\mathbb{G},\sigma),\{g_1,\dots,g_k\},p)$=FindSumSolutionsForSolutionSpace($(\mathbb{F},\sigma),\mathcal{R},f$)

(11) CONSTRUCT the sum extension $\mathbb{G}(t_1,\dots,t_k)$ of $(\mathbb{G},\sigma)$ with
$\sigma(t_i)=t_i+\frac{g_i}{\sigma(d)}$ for $(1\leq i\leq k)$ and
$\text{const}_\sigma\mathbb{G}(t_1,\dots,t_k)=\mathbb{K}$

(12) IF $p=\perp$ RETURN $((\mathbb{G}(t_1,\dots,t_k),\sigma),\{d,d\,t_1,\dots,d\,t_k\},\perp)$

(13) CONSTRUCT the sum extension $(\mathbb{G}(t_1,\dots,t_k)(s),\sigma)$ of $(\mathbb{G}(t_1,\dots,t_k),\sigma)$ with
$\sigma(s)=s+\frac{p}{\sigma(d)}$ and
$\text{const}_\sigma\mathbb{G}(t_1,\dots,t_k)(s)=\mathbb{K}$

(14) RETURN $(\mathbb{G}(t_1,\dots,t_k)(s),\sigma),\{d,d\,t_1,\dots,d\,t_k\},d\,s)$

**Corollary 4.5.2.** *Let $(\mathbb{F},\sigma)$ be a difference field in which one can solve linear difference equations, $\mathcal{L}\in\mathbb{F}[\sigma]^*$ and $f\in\mathbb{F}^*$. Then Algorithm 4.5.3 with input $((\mathbb{F},\sigma),\mathcal{L},f)$ terminates. Let $((\mathbb{E},\sigma)$,$B$,$p$) be the output of the algorithm. Then $(\mathbb{E},\sigma)$ is a sum extension of $(\mathbb{F},\sigma)$ with $\text{const}_\sigma\mathbb{E}=\text{const}_\sigma\mathbb{F}$ which is complete w.r.t. $(\mathcal{L},f)$, $B$ is a basis of $\text{Ker}_\mathbb{E}\,\mathcal{L}$ and $p$ is a solution of $\mathcal{L}(p)=f$ if such a $p\in\mathbb{E}$ exists, otherwise $p=\perp$.*

*Proof.* This is a consequence of Propositions 4.5.1 and 4.5.2. □

**Implementation Note 4.5.1.** Besides an important modification (Implementation Note 4.5.3), Algorithm 4.5.3 is invoked by setting the option

$$\texttt{NestedSumExt}- > \texttt{Infinity}$$

in the function `SolveRecurrence`. Using the function `FindSumSolutions` Algorithm 4.5.3 is automatically used without any option. In particular, that function will play a major role to find d'Alembertian extensions as described in the next section (Implementation Note 4.5.2).

### 4.5.3 Dealing with d'Alembertian Extensions

In this section we will sketch how one can find appropriate sum and **product** extensions, more precisely d'Alembertian Extensions, in terms of difference fields.

#### 4.5.3.1 A Criterion for Existence of d'Alembertian Extensions

**Lemma 4.5.4.** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}$. We have*

$$\exists g \in \mathbb{F}(t) \setminus \mathbb{F} : \mathcal{L}(g) = f$$

$$\Downarrow$$

$$\exists g \in \mathbb{F}^*, d \in \mathbb{Z}^* : \mathcal{L}(g\, t^d) = 0.$$

*Proof.* Let $g \in \mathbb{F}(t) \setminus \mathbb{F}$ with

$$\mathcal{L}(g) = f.$$

By Proposition 4.1.1 it follows that $g = \sum_i g_i\, t^i \in \mathbb{F}[t, \frac{1}{t}]$ and

$$\mathcal{L}(g_i\, t^i) = f_i\, t^i$$

for all $i \in \mathbb{Z}$. Since $g \notin \mathbb{F}$, there is at least one $g_d \neq 0$ with $d \neq 0$ such that

$$\mathcal{L}(g_d\, t^d) = f_d\, t^d$$

for some $f_d \in \mathbb{F}$. Since $f \in \mathbb{F}$, it follows that $f_d = 0$ which proves the lemma. ☐

**Lemma 4.5.5.** *Let $(\mathbb{E}, \sigma)$ be a $\Pi$-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} := \mathbb{F}(t_1, \ldots, t_l)$ and $t_i$ hyperexponential over $\mathbb{F}$ for $1 \leq i \leq l$. Then for any $\mathcal{L} \in \mathbb{F}[\sigma]^*$ we have*

$$\exists g \in \mathbb{E}^* : \mathcal{L}(g) = 0$$

$$\Downarrow$$

$$\exists \text{ a hyperexponential } g \in \mathbb{E}^* \text{ over } \mathbb{F} : \mathcal{L}(g) = 0.$$

*Proof.* The proof is done by induction on the number $l$ of $\Pi$-extensions. For the induction base $\mathbb{E} = \mathbb{F}$, $l = 0$, nothing has to be shown. Now consider the $\Pi$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{E} := \mathbb{F}(t_1, \ldots, t_l)$ and $t_i$ hyperexponential over $\mathbb{F}$ and assume that for any $\tilde{\mathcal{L}} \in \mathbb{F}[\sigma]$ the theorem holds in $(\mathbb{E}, \sigma)$. Let $(\mathbb{E}(t), \sigma)$ be a $\Pi$-extension of $(\mathbb{E}, \sigma)$ with $t$ hyperexponential over $\mathbb{F}$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $g \in \mathbb{E}(t)^*$ with

$$\mathcal{L}(g) = 0.$$

If $g \in \mathbb{E}$, nothing has to be proven by the induction assumption. Otherwise, assume $g \notin \mathbb{E}$. Then by Lemma 4.5.4 there are a $u \in \mathbb{E}^*$ and a $d \in \mathbb{Z}^*$ such that

$$\mathcal{L}(u\, t^d) = 0.$$

If $u \in \mathbb{F}$ then we are done. Otherwise, assume $u \notin \mathbb{F}$ and take

$$\tilde{\mathcal{L}} := \frac{1}{t^d} \mathcal{L}\, t^d \in \mathbb{F}[\sigma].$$

Since $\frac{\sigma(t)}{t} \in \mathbb{F}^*$, we have

$$\tilde{\mathcal{L}}(u) = \frac{1}{t^d} \mathcal{L}\, t^d(u) = \frac{1}{t^d} \mathcal{L}(t^d\, u) = 0.$$

Therefore by the induction assumption we may assume that there is a hyperexponential $v \in \mathbb{E}^*$ over $\mathbb{F}$ such that

$$\tilde{\mathcal{L}}(v) = 0$$

thus

$$0 = \tilde{\mathcal{L}}(v) = \frac{1}{t^d}\mathcal{L}\, t^d(v) = \frac{1}{t^d}\mathcal{L}(t^d\, v).$$

and consequently $\mathcal{L}(t^d\, v) = 0$. □

This is the corresponding result of Theorem 4.5.1. Please note that in the assumption we restricted already to the situation that there exists a **reduced** d'Alembertian extension, whereas in Theorem 4.5.1 we were able to deal with the more general case of sum extensions and not only with **proper** sum extensions.

**Theorem 4.5.5.** *Let $(\mathbb{E}, \sigma)$ be a reduced d'Alembertian extension of $(\mathbb{F}, \sigma)$. Let $f \in \mathbb{F}$ and $\mathcal{L} \in \mathbb{F}[\sigma]^*$. Then*

$$\exists g \in \mathbb{E} \setminus \mathbb{F} : \mathcal{L}(g) = f$$

$$\Downarrow$$

$$\exists \text{ a hyperexponential } g \in \mathbb{E}^* \text{ over } \mathbb{F} : \mathcal{L}(g) = 0.$$

*Proof.* By Corollary 2.4.3 there is a d'Alembertian extension $(\mathbb{F}(h_1, \dots, h_m)(s_1, \dots, s_n), \sigma)$ of $(\mathbb{F}, \sigma)$ where the $h_i$'s are hyperexponential over $\mathbb{F}$ and the $s_i$'s are sums over $\mathbb{F}(h_1, \dots, h_m)$ such that

$$(\mathbb{E}, \sigma) \simeq (\underbrace{\mathbb{F}(h_1, \dots, h_m)}_{=:\mathbb{H}}(s_1, \dots, s_n), \sigma).$$

If $g \in \mathbb{H}(s_1, \dots, s_n) \setminus \mathbb{H}$ then by Theorem 4.5.1 there is a $p \in \mathbb{H}^*$ such that $\mathcal{L}(p) = 0$. Therefore, in any case, we may assume that there is a $g \in \mathbb{H}^*$ with $\mathcal{L}(g) = 0$ and consequently by Lemma 4.5.5 there is a hyperexponential $q \in \mathbb{E}^*$ over $\mathbb{F}$ such that

$$\mathcal{L}(q) = 0.$$

□

#### 4.5.3.2 Finding d'Alembertian Solutions

Given a difference field $(\mathbb{F}, \sigma)$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and $f \in \mathbb{F}$, we are able to construct a complete sum extension $(\mathbb{E}, \sigma)$ w.r.t. $(\mathcal{L}, f)$ by Algorithm 4.5.3. Now consider the recursion call in Algorithm 4.5.3 or Algorithm 4.5.2 when there does not exist a $d \in \mathbb{F}^*$ such that

$$\mathcal{R}(d) = 0$$

or equivalently $\mathcal{R}$ does not have a right factor of the form

$$\sigma - \frac{\sigma(d)}{d} \tag{4.36}$$

for any $d \in \mathbb{F}^*$. If there exist an $\tilde{\mathcal{R}} \in \mathbb{F}[\sigma]$ and a $g \in \mathbb{F}$ with

$$\mathcal{R} = \tilde{\mathcal{R}}\,(\sigma - g)$$

then we can construct a difference ring extension $(\mathbb{F}[t], \sigma)$ of $(\mathbb{F}, \sigma)$ such that

$$\sigma(t) = g\,t. \tag{4.37}$$

If there does not exist an $n \geq 0$ such that

$$g^n \in \mathrm{H}_{(\mathbb{F},\sigma)} \tag{4.38}$$

then we can even construct a $\Pi$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$\mathcal{R}(t) = \tilde{\mathcal{R}}\,(\sigma(t) - g\,t) = 0.$$

Therefore we can again apply Algorithm 4.5.3 to find a complete sum extension of $(\mathbb{F}(t), \sigma)$ w.r.t. $(\mathcal{L}, f)$.

**Implementation Note 4.5.2.** This idea is realized in my implementation by the function `FindSumSolutions` (Section 1.1.2). Here we return exactly the difference operator $\mathcal{R}$ which does not have any right factor of the form (4.36) for some $d \in \mathbb{F}^*$. In this case the user has to extend the underlying difference field by an appropriate $\Pi$-extension $(\mathbb{F}(t), \sigma)$ canonically defined by (4.37). In order to find that $g$, one needs other algorithms like for instance M. Petkovšek's package Hyper. Further remarks one can find in Section 1.3.4.3.

Doing this, step by step, and extending the difference field by further $\Pi$-extensions over $\mathbb{F}$, we finally might reach the point and find a $\Pi$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{H} := \mathbb{F}(t_1, \dots, t_n)$ and $\frac{\sigma(t_i)}{t_i} \in \mathbb{F}$ such that there does not exist an $\tilde{\mathcal{R}} \in \mathbb{F}[\sigma]$ and a $g \in \mathbb{F}$ with

$$\mathcal{R} = \tilde{\mathcal{R}}\,(\sigma - g). \tag{4.39}$$

During this step-wise extension we might fail to compute a $\Pi$-extension if we run in the situation that there exits an $n > 0$ with (4.38). In this case there is still hope to adapt the underlying difference field to avoid this case. Otherwise this extension is not anymore transcendental over the given difference field and in the worst case we even have to work in difference rings; this case will be excluded in the following.

Now assume we find such a $\Pi$-extension $(\mathbb{H}, \sigma)$ as described above and let $(\mathbb{G}, \sigma)$ be a complete sum extension of w.r.t. $(\mathcal{L}, 0)$. Then one can show that there does not exist any d'Alembertian extension of $(\mathbb{G}, \sigma)$ which includes more solutions for the difference equation

$$\mathcal{L}(g) = 0.$$

Suppose there is a d'Alembertian extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ which contains more solutions than $(\mathbb{G}, \sigma)$. By Corollary 2.4.3 there is a d'Alembertian extension $(\mathbb{F}(h_1, \dots, h_m)(s_1, \dots, s_n), \sigma)$ of $(\mathbb{F}, \sigma)$ where the $h_i$ are hyperexponential over $\mathbb{F}$ and the $s_i$ are sums over $\mathbb{A} := \mathbb{F}(h_1, \dots, h_m)$ such that

$$(\mathbb{E}, \sigma) \simeq (\mathbb{A}(s_1, \dots, s_n), \sigma). \tag{4.40}$$

Since $(\mathbb{G}, \sigma)$ is a complete sum extension of $(\mathbb{H}, \sigma)$, the difference field $(\mathbb{A}, \sigma)$ must be a $\Pi$-extension of $(\mathbb{H}, \sigma)$. Now compute a complete sum extension $(\mathbb{B}, \sigma)$ of $(\mathbb{A}, \sigma)$ w.r.t. $(\mathcal{L}, f)$ by applying Algorithm 4.5.3 where we can force the algorithm to use elements in the difference field $(\mathbb{H}, \sigma)$ in the same order for factoring $\mathcal{L}$ as when we computed the complete sum extension $(\mathbb{G}, \sigma)$ of $(\mathbb{H}, \sigma)$ . Finally we will reach the situation where there is not a $g \in \mathbb{F}^*$ and an $\tilde{\mathcal{R}} \in \mathbb{F}[\sigma]$ with (4.39). Therefore we cannot factor further and consequently we get a complete sum extension $(\mathbb{B}, \sigma)$ of $(\mathbb{A}, \sigma)$ which is equal to $(\mathbb{G}, \sigma)$. Since we have (4.40), $(\mathbb{E}, \sigma)$ cannot contain more solutions than $(\mathbb{G}, \sigma)$.

### 4.5.4 Finding New Sum Extensions

Algorithms 4.5.2 and 4.5.3 deliver sum extensions in which additional solutions of a given difference equation may exist. Given such a sum extension, one is interested in eliminating all algebraic relations of these sums and to transform it to a proper sum extension which consists only of sums which are transcendental over the other sums. This elimination of algebraic relations can be obtained by indefinite summation (Remarks 4.5.2 and 4.5.3) which can be very expensive concerning time and memory aspects.

In this section we will reduce this simplification step by avoiding some algebraic relations in the sum extensions during its construction in Algorithms 4.5.2 and 4.5.3.

**Lemma 4.5.6.** *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and let $\{d_1,\dots,d_n\}\subseteq\mathbb{F}^*$ be linearly independent over $\mathbb{K}$. Let*

$$e_i := \sigma(\frac{d_i}{d_n}) - \frac{d_i}{d_n}$$

*for $1\leq i\leq n$. Then $e_1\cdots e_{n-1}\neq 0$ and $\{e_1,\dots,e_{n-1}\}$ is linearly independent over $\mathbb{K}$.*

*Proof.* Assume there is an $e_i=0$, i.e.

$$\sigma(\frac{d_i}{d_n}) = \frac{d_i}{d_n}$$

for some $i$ with $1\leq i\leq n-1$. Thus $\frac{d_i}{d_n}\in\mathbb{K}$ and therefore there is a $k\in\mathbb{K}$ such that

$$d_i = k\, d_n$$

which is a contradiction to the assumption that $\{d_1,\dots,d_n\}$ is linearly independent over $\mathbb{K}$. Thus

$$e_1\cdots e_{n-1}\neq 0.$$

Assume there are $k_1,\dots,k_{n-1}\in\mathbb{K}$ such that

$$k_1\, e_1 + \dots + k_{n-1}\, e_{n-1} = 0$$

where not all $k_i=0$. Then we have

$$k_1\,(\sigma(\frac{d_1}{d_n}) - \frac{d_1}{d_n}) + \dots + k_{n-1}\,(\sigma(\frac{d_{n-1}}{d_n}) - \frac{d_{n-1}}{d_n}) = 0$$

$$\Updownarrow$$

$$k_1\,(\sigma(d_1)\,d_n - d_1\,\sigma(d_n)) + \dots + k_{n-1}\,(\sigma(d_{n-1})\,d_n - d_{n-1}\,\sigma(d_n)) = 0$$

$$\Updownarrow$$

$$d_n\,\underbrace{(k_1\,\sigma(d_1) + \dots + k_{n-1}\,\sigma(d_{n-1}))}_{\neq 0} + \sigma(d_n)\,\underbrace{(k_1\,d_1 + \dots + k_{n-1}\,d_{n-1})}_{\neq 0} = 0$$

$$\Updownarrow$$

$$\frac{\sigma(d_n)}{d_n} = \frac{\sigma(k_1\,d_1 + \dots + k_{n-1}\,d_{n-1})}{k_1\,d_1 + \dots + k_{n-1}\,d_{n-1}}$$

$$\Updownarrow$$

$$\sigma(\frac{k_1\,d_1 + \cdots + k_{n-1}\,d_{n-1}}{d_n}) = \frac{k_1\,d_1 + \cdots + k_{n-1}\,d_{n-1}}{d_n}.$$

Consequently

$$k\,d_n = k_1\,d_1 + \cdots + k_{n-1}\,d_{n-1}$$

for some $k \in \mathbb{K}$ which contradicts to the assumption that $\{d_1, \dots, d_n\}$ is linearly independent over $\mathbb{K}$. Hence also $\{e_1, \dots, e_{n-1}\}$ is linearly independent over $\mathbb{K}$. $\square$

**Proposition 4.5.3.** *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\mathcal{L} \in \mathbb{F}[\sigma]^*$ and let $\{d_1^{(0)}, \dots, d_n^{(0)}\} \subseteq \mathbb{F}^*$ be linearly independent over $\mathbb{K}$ with*

$$\{d_1^{(0)}, \dots, d_n^{(0)}\} \subseteq \operatorname{Ker}_{\mathbb{F}} \mathcal{L}.$$

*Define*

$$d_i^{(j)} := (\sigma(\frac{d_i^{(j-1)}}{d_{n-j+1}^{(j-1)}}) - \frac{d_i^{(j-1)}}{d_{n-j+1}^{(j-1)}})\sigma(d_{n-j+1}^{(j-1)})$$

*for $1 \leq j < n$ and $1 \leq i \leq n-j$. Then $d_n^{(0)} \cdots d_1^{(n-1)} \neq 0$. Furthermore there is an $\mathcal{R} \in \mathbb{F}[\sigma]^*$ such that*

$$\mathcal{L} = \mathcal{R}\,\mathcal{L}_1 \cdots \mathcal{L}_n$$

*where $\mathcal{L}_i = \sigma - \frac{\sigma(d_i^{(n-i)})}{d_i^{(n-i)}}$.*

*Proof.* We have $d_1^{(0)} \cdots d_n^{(0)} \neq 0$, $\{d_1^{(0)}, \dots, d_n^{(0)}\}$ is linearly independent over $\mathbb{K}$ and $\mathcal{L}(d_i^{(0)}) = 0$ for all $1 \leq i \leq n$ by assumption. Thus there is by Lemma 4.5.1 an $\mathcal{R} \in \mathbb{F}[\sigma]^*$ with

$$\mathcal{L} = \mathcal{R}\,(\sigma - \frac{\sigma(d_n^{(0)})}{d_n^{(0)}})$$

and consequently the induction base holds. Now assume that for $l$ with $1 \leq l < n$ we have $d_1^{(l-1)} \cdots d_{n-l+1}^{(l-1)} \neq 0$, $\{d_1^{(l-1)}, \dots, d_{n-l+1}^{(l-1)}\}$ is linearly independent over $\mathbb{K}$ and that there is an $\mathcal{R} \in \mathbb{F}[\sigma]^*$ with

$$\mathcal{R}\,\mathcal{L}_{n-l+2} \cdots \mathcal{L}_n.$$

Besides this, assume that

$$\mathcal{R}(d_i^{(l-1)}) = 0$$

for all $1 \leq i \leq n-l+1$. Then by Lemma 4.5.6 it follows that $d_1^{(l)} \cdots d_{n-l}^{(l)} \neq 0$ and $\{d_1^{(l)}, \dots, d_{n-l}^{(l)}\}$ is linearly independent over $\mathbb{K}$. Furthermore by Lemma 4.5.1 there is an $\tilde{\mathcal{R}} \in \mathbb{F}[\sigma]$ with

$$\mathcal{R} = \tilde{\mathcal{R}}\underbrace{(\sigma - \frac{\sigma(d_{n-l+1}^{(l-1)})}{d_{n-l+1}^{(l-1)}})}_{=:\mathcal{L}_{n-l+1}}$$

and thus

$$\mathcal{L} = \tilde{\mathcal{R}}\,\mathcal{L}_{n-l+1} \cdots \mathcal{L}_n.$$

Additionally, it follows for all $1 \leq i \leq n-l$ that

$$\begin{aligned}0 &= \mathcal{R}(d_i^{(l-1)}) = \tilde{\mathcal{R}}(\sigma - \frac{\sigma(d_{n-l+1}^{(l-1)})}{d_{n-l+1}^{(l-1)}})(d_i^{(l-1)}) = \tilde{\mathcal{R}}(\sigma(d_i^{(l-1)}) - \frac{\sigma(d_{n-l+1}^{(l-1)})}{d_{n-l+1}^{(l-1)}} d_i^{(l-1)})\\ &= \tilde{\mathcal{R}}((\sigma(\frac{d_i^{(l-1)}}{d_{n-l+1}^{l-1}}) - \frac{d_i^{(l-1)}}{d_{n-l+1}^{(l-1)}})\sigma(d_{n-l+1}^{(l-1)})) = \tilde{\mathcal{R}}(d_i^{(l)}) = 0.\end{aligned}$$

Therefore the induction step $l \to l+1$ holds. Finally for $n = l$ the statement of the theorem is proven. □

**Lemma 4.5.7.** *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\{d_1^{(0)}, \ldots, d_n^{(0)}\} \subseteq \mathbb{F}^*$ be linearly independent over $\mathbb{K}$ and define*

$$d_i^{(j)} := (\sigma(\frac{d_i^{(j-1)}}{d_{n-j+1}^{(j-1)}}) - \frac{d_i^{(j-1)}}{d_{n-j+1}^{(j-1)}})\sigma(d_{n-j+1}^{(j-1)})$$

*for $1 \leq j < n$ and $1 \leq i \leq n-j$. Let $(\mathbb{E}, \sigma)$ with*

$$\begin{array}{lllll}\mathbb{E} := \mathbb{F}( & s_n^{(1)}, & & & \\ & s_n^{(2)}, & s_{n-1}^{(2)}, & & \\ & s_n^{(3)}, & s_{n-1}^{(3)}, & s_{n-2}^{(3)}, & \\ & \vdots & \vdots & \vdots & \\ & s_n^{(n-1)}, & s_{n-1}^{(n-1)}, & s_{n-2}^{(n-1)}, & \cdots \quad , s_2^{(n-1)} \quad )\end{array}$$

*be a sum extension of $(\mathbb{F}, \sigma)$ with*

$$\begin{aligned}\sigma(s_n^{(i)}) &= s_n^{(i)} + \frac{d_{n-i}^{(i)}}{\sigma(d_{n-i+1}^{(i-1)})}\\ \sigma(s_{n-j}^{(i)}) &= s_{n-j}^{(i)} + \frac{d_{n-i+j}^{(i-j)}}{\sigma(d_{n-i+j+1}^{(i-j-1)})} s_{n-j+1}^{(i)}\end{aligned}$$

*for $1 \leq i < n$ and $0 \leq j < i$ and $\mathrm{const}_\sigma\mathbb{E} = \mathbb{K}$. Then for all $1 \leq i < n$ we have*

$$d_n^{(0)} s_{n-i+1}^{(i)} = d_{n-i}^{(0)} + c_1 d_{n-i+1}^{(0)} + \cdots + c_i d_n^{(0)}$$

*for some $c_j \in \mathbb{K}$.*

*Proof.* Let $1 \leq i < n$. We will show for all $k$ with $0 \leq k < i$ that

$$d_{n-i+k+1}^{(i-k-1)} s_{n-k}^{(i)} = d_{n-i}^{(i-k-1)} + c_1 d_{n-i+1}^{(i-k-1)} + \cdots + c_k d_{n-i+k+1}^{(i-k-1)} \tag{4.41}$$

for some $c_j \in \mathbb{K}$. By

$$\sigma(s_n^{(i)}) = s_n^{(i)} + \frac{d_{n-i}^{(i)}}{\sigma(d_{n-i+1}^{(i-1)})} = s_n^{(i)} + \sigma(\frac{d_{n-i}^{(i-1)}}{d_{n-i+1}^{(i-1)}}) - \frac{d_{n-i}^{(i-1)}}{d_{n-i+1}^{(i-1)}}$$

it follows that

$$\sigma(s_n^{(i)} - \frac{d_{n-i}^{(i-1)}}{d_{n-i+1}^{(i-1)}}) = s_n^{(i)} - \frac{d_{n-i}^{(i-1)}}{d_{n-i+1}^{(i-1)}}$$

and thus

$$s_n^{(i)} - \frac{d_{n-i}^{(i-1)}}{d_{n-i+1}^{(i-1)}} \in \mathbb{K}$$

Consequently

$$d_{n-i+1}^{(i-1)}\, s_n^{(i)} = d_{n-i}^{(i-1)} + c_1\, d_{n-i+1}^{(i-1)}$$

for some $c_1 \in \mathbb{K}$ and hence the induction base $k = 0$ holds. Now let $0 \leq k < i-1$ and assume

$$d_{n-i+k+1}^{(i-k-1)}\, s_{n-k}^{(i)} = d_{n-i}^{(i-k-1)} + c_1\, d_{n-i+1}^{(i-k-1)} + \cdots + c_{k+1}\, d_{n-i+k+1}^{(i-k-1)}$$

for some $c_j \in \mathbb{K}$. Then we have

$$\begin{aligned}
\sigma(s_{n-k-1}^{(i)}) &= s_{n-k-1}^{(i)} + d_{n-i}^{(i-k-1)} + c_1\, d_{n-i+1}^{(i-k-1)} + \cdots + c_{k+1}\, d_{n-i+k+1}^{(i-k-1)})/\sigma(d_{n-i+k+2}^{(i-k-2)}) \\
&= s_{n-k-1}^{(i)} + (\sigma(\frac{d_{n-i}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}}) - \frac{d_{n-i}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + c_1\, (\sigma(\frac{d_{n-i+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}}) - \frac{d_{n-i+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}}) \\
&\qquad + \cdots + c_{k+1}\, (\sigma(\frac{d_{n-i+k+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}}) - \frac{d_{n-i+k+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}})
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\sigma(s_{n-k-1}^{(i)} - \frac{d_{n-i}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + c_1\, \frac{d_{n-i+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + \cdots + c_{k+1}\, \frac{d_{n-i+k+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}}) \\
&\qquad = s_{n-k-1}^{(i)} - \frac{d_{n-i}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + c_1\, \frac{d_{n-i+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + \cdots + c_{k+1}\, \frac{d_{n-i+k+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}}.
\end{aligned}$$

Thus there is a $c_{k+2} \in \mathbb{K}$ with

$$s_{n-k-1}^{(i)} = \frac{d_{n-i}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + c_1\, \frac{d_{n-i+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + \cdots + c_{k+1}\, \frac{d_{n-i+k+1}^{(i-k-2)}}{d_{n-i+k+2}^{(i-k-2)}} + c_{k+2},$$

hence the induction hypothesis

$$d_{n-i+k+2}^{(i-k-2)}\, s_{n-k-1}^{(i)} = d_{n-i}^{(i-k-2)} + c_1\, d_{n-i+1}^{(i-k-2)} + \cdots + c_{k+1}\, d_{n-i+k+1}^{(i-k-2)} + c_{k+2}\, d_{n-i+k+2}^{(i-k-2)}$$

holds and consequently (4.41) is proven. Especially for $k = i-1$ we obtain

$$d_n^{(0)}\, s_{n-i+1}^{(i)} = d_{n-i}^{(0)} + c_1\, d_{n-i+1}^{(0)} + \cdots + c_i\, d_n^{(0)}$$

for some $c_j \in \mathbb{K}$ and therefore the lemma is proven. □

**Theorem 4.5.6.** *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$, $\mathcal{L}\in\mathbb{F}[\sigma]^*$ and let $\{d_1^{(0)},\dots,d_n^{(0)}\}\subseteq\mathbb{F}\setminus\{0\}$ be a basis of $\operatorname{Ker}_{\mathbb{F}}\mathcal{L}$. Define*

$$d_i^{(j)} := \Big(\sigma\Big(\frac{d_i^{(j-1)}}{d_{n-j+1}^{(j-1)}}\Big)-\frac{d_i^{(j-1)}}{d_{n-j+1}^{(j-1)}}\Big)\sigma(d_{n-j+1}^{(j-1)})$$

*for $1\leq j<n$ and $1\leq i\leq n-j$ and let $\mathcal{R}\in\mathbb{F}[\sigma]^*$ be defined by*

$$\mathcal{L}=\mathcal{R}\,\mathcal{L}_1\cdots\mathcal{L}_n$$

*where $\mathcal{L}_i=\sigma-\frac{\sigma(d_i^{(n-i)})}{d_i^{(n-i)}}$. Let $(\mathbb{E},\sigma)$ be a sum extension of $(\mathbb{F},\sigma)$ which is complete w.r.t. $(\mathcal{R},0)$ and let $\{f_1,\dots,f_k\}$ be a basis of $\operatorname{Ker}_{\mathbb{E}}\mathcal{R}$.*
*Let $(\mathbb{G},\sigma)$ with $\mathbb{G}=\mathbb{E}(t_n^{(1)},\dots,t_1^{(1)},\dots,t_n^{(k)},\dots,t_1^{(k)})$ be a sum extension of $(\mathbb{E},\sigma)$ with*

$$\sigma(t_n^{(i)})=t_n^{(i)}+\frac{f_i}{d_1^{(n-1)}}$$
$$\sigma(t_j^{(i)})=t_j^{(i)}+\frac{d_{n-j}^{(i-j)}}{d_{n-j+1}^{(j+1)}}\,t_{j+1}^{(i)}$$

*for $1\leq j\leq n-1$ and $1\leq i\leq k$ and $\operatorname{const}_\sigma\mathbb{G}=\mathbb{K}$.*
*Then $(\mathbb{G},\sigma)$ is a complete sum extension of $(\mathbb{F},\sigma)$ w.r.t. $(\mathcal{L},0)$ and*

$$\{d_1^{(0)},\dots,d_n^{(0)},d_n^{(0)}\,t_n^{(1)},\dots,d_n^{(0)}\,t_n^{(k)}\}$$

*is a basis of $\operatorname{Ker}_{\mathbb{G}}\mathcal{L}$. Furthermore $t_n^{(i)}\notin\mathbb{F}$ for all $1\leq i\leq k$.*

*Proof.* Let $(\mathbb{G},\sigma)$ be a sum extension of $(\mathbb{F},\sigma)$ with

$$\begin{array}{llllll}\mathbb{G}=\mathbb{E}( & s_n^{(1)}, & & & & \\ & s_n^{(2)}, & s_{n-1}^{(2)}, & & & \\ & s_n^{(3)}, & s_{n-1}^{(3)}, & s_{n-2}^{(3)}, & & \\ & \vdots & \vdots & \vdots & & \\ & s_n^{(n-1)}, & s_{n-1}^{(n-1)}, & s_{n-2}^{(n-1)}, & \dots & ,s_2^{(n-1)}, \\ \hline & t_n^{(1)}, & t_{n-1}^{(1)}, & t_{n-2}^{(1)}, & \dots & ,t_2^{(1)}, \quad ,t_1^{(1)} \\ & \vdots & \vdots & \vdots & & \vdots \\ & t_n^{(k)}, & t_{n-1}^{(k)}, & t_{n-2}^{(k)}, & \dots & ,t_2^{(k)} \quad ,t_1^{(k)})\end{array} \tag{4.42}$$

where $\operatorname{const}_{\mathbb{F}}\mathbb{G}=\mathbb{K}$ and

1. $\sigma(s_n^{(i)})=s_n^{(i)}+\frac{d_{n-i}^{(i)}}{\sigma(d_{n-i+1}^{(i-1)})}$,

   $\sigma(s_{n-j}^{(i)})=s_{n-j}^{(i)}+\frac{d_{n-i+j}^{(i-j)}}{\sigma(d_{n-i+j+1}^{(i-j-1)})}\,s_{n-j+1}^{(i)}$ for $1\leq i<n$ and $0\leq j<i$ and

2. $\sigma(t_n^{(i)})=t_n^{(i)}+\frac{f_i}{\sigma(d_1^{(n-1)})}$,

$$\sigma(t_j^{(i)}) = t_j^{(i)} + \frac{d_{j+1}^{(n-1-j)}}{\sigma(d_{j+2}^{(n-2-j)})} t_{j+1}^{(i)} \text{ for } 1 \leq i \leq k \text{ and } 0 \leq j < n-1.$$

Then by Proposition 4.5.3 and Theorem 4.5.4 it follows that $(\mathbb{G}, \sigma)$ is a complete sum extension of $(\mathbb{F}, \sigma)$ w.r.t. $(\mathcal{L}, 0)$ and

$$\{d_n^{(0)}, d_n^{(0)} s_n^{(1)}, \dots, d_n^{(0)} s_2^{(n-1)}, d_n^{(0)} t_1^{(1)}, \dots, d_n^{(0)} t_1^{(k)}\}$$

is a basis of $\operatorname{Ker}_{\mathbb{G}} \mathcal{L}$. Furthermore by Lemma 4.5.7 we have that for all $1 \leq i < n$

$$d_n^{(0)} s_{n-i+1}^{(i)} = d_{n-i}^{(0)} + c_1 d_{n-i+1}^{(0)} + \dots + c_i d_n^{(0)}$$

for some $c_j \in \mathbb{K}$ and thus

$$(\mathbb{E}(t_n^{(1)}, t_{n-1}^{(1)}, \dots, t_1^{(k)}), \sigma) = (\mathbb{G}, \sigma)$$

and

$$\{d_1^{(0)}, \dots, d_n^{(0)}, d_n^{(0)} t_1^{(1)}, \dots, d_n^{(0)} t_1^{(k)}\}$$

is a basis of $\operatorname{Ker}_{\mathbb{G}} \mathcal{L}$. But

$$\{d_1^{(0)}, \dots, d_n^{(0)}\}$$

is a basis of $\operatorname{Ker}_{\mathbb{F}} \mathcal{L}$ and thus

$$t_1^{(j)} \notin \mathbb{F}$$

for all $(1 \leq j \leq k)$. $\square$

Applying Theorem 4.5.6 has several advantages.

1. If we solve the solution space $\mathrm{V}(\mathbf{a}, (0), \mathbb{F})$ for $\mathbf{a} := \operatorname{vect}(\mathcal{L})$ then we get a basis

$$\{d_1^{(0)}, \dots, d_n^{(0)}\}$$

of $\operatorname{Ker}_{\mathbb{F}} \mathcal{L}$. Thus applying Theorem 4.5.6 delivers us a **shortcut to factorize**

$$\mathcal{L} = \mathcal{R}\, \mathcal{L}_1 \dots \mathcal{L}_n.$$

Therefore we do not have to find for each step $i$ again a homogeneous solution of

$$\mathcal{R}\, \mathcal{L}_1 \dots \mathcal{L}_i.$$

2. If $\mathcal{R}$ has no homogeneous solution, or in other words, $(\mathbb{F}, \sigma)$ is complete w.r.t. $(\mathcal{R}, 0)$ then also $(\mathbb{F}, \sigma)$ is complete w.r.t. $(\mathcal{L}, 0)$ and

$$\{d_1^{(0)}, \dots, d_n^{(0)}\}$$

are the solutions. Therefore we do not return the sum extension $(\mathbb{G}, \sigma)$ with[8]

$$\begin{array}{lllll}
\mathbb{G} := \mathbb{F}( & s_n^{(1)}, & & & \\
& s_n^{(2)}, & s_{n-1}^{(2)}, & & \\
& s_n^{(3)}, & s_{n-1}^{(3)}, & s_{n-2}^{(3)}, & \\
& \vdots & \vdots & \vdots & \\
& s_n^{(n-1)}, & s_{n-1}^{(n-1)}, & s_{n-2}^{(n-1)}, & \dots, s_2^{(n-1)} \quad )
\end{array}$$

since anyway $(\mathbb{G}, \sigma) = (\mathbb{F}, \sigma)$.

[8]See the sum extension in (4.42).

3. Assume we have a complete sum extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ w.r.t. $(\mathcal{R},0)$ then we can construct the complete sum extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ with $\mathbb{G} := \mathbb{E}(t_n^{(1)}, t_{n-1}^{(1)}, \dots, t_1^{(k)})$ w.r.t. $(\mathcal{L},0)$ and a basis

$$\{d_1^{(0)}, \dots, d_n^{(0)}, d_n^{(0)}\, t_1^{(1)}, \dots, d_n^{(0)}\, t_1^{(k)}\}$$

   of $\operatorname{Ker}_{\mathbb{G}} \mathcal{L}$. We have $d_n^{(0)}\, t_1^{(1)} \notin \mathbb{F}$, which means that **we extend the difference field only if new solutions can be found in it**.

4. In order to get a complete sum extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ w.r.t. $(\mathcal{R},0)$, we can proceed using Theorem 4.5.6.

5. Having a complete sum extension $(\mathbb{G},\sigma)$ of $(\mathbb{F},\sigma)$ w.r.t. $(\mathcal{L},0)$ we are interested in a proper sum representation (see Remark 4.5.3). If we want to compute a proper sum extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ and a difference field isomorphism $\tau$ with

$$(\mathbb{H},\sigma) \overset{\tau}{\simeq} (\mathbb{G},\sigma)$$

   then we have to use expensive operations like solving first order linear difference equations in a sub-difference field of $(\mathbb{G},\sigma)$ in order to check if a sum extension is proper or can be already expressed in the given difference field. Applying Theorem 4.5.6, **we eliminate a priori sum extensions which are not proper** and therefore we avoid expensive operations to solve first order difference equations.

**Implementation Note 4.5.3.** Finally, in the implementation of the functions `FindSumSolutions` and `SolveRecurrence` I realized Algorithm 4.5.3 by exploiting Theorem 4.5.6 as described in the above remarks.

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# Bibliography


[Abr89] S. A. Abramov. Rational solutions of linear differential and difference equations with polynomial coefficients. *U.S.S.R. Comput. Math. Math. Phys.*, 29(6):7–12, 1989.

[Abr95] S. A. Abramov. Rational solutions of linear difference and $q$-difference equations with polynomial coefficients. In T. Levelt, editor, *Proc. ISSAC'95*, pages 285–289. ACM Press, New York, 1995.

[AP94] S. A. Abramov and M. Petkovšek. D'Alembertian solutions of linear differential and difference equations. In J. von zur Gathen, editor, *Proc. ISSAC'94*, pages 169–174. ACM Press, Baltimore, 1994.

[AP99] G. E. Andrews and P. Paule. MacMahon's partition analysis IV: Hypergeometric multisums. *Sém. Lothar. Combin.*, B42i:1–24, 1999.

[APP98] S. A. Abramov, P. Paule, and M. Petkovšek. $q$-Hypergeometric solutions of $q$-difference equations. *Discrete Math.*, 180(1-3):3–22, 1998.

[AU85] G. E. Andrews and K. Uchimura. Identities in combinatorics IV: Differentiation and harmonic numbers. *Util. Math.*, 28:265–269, 1985.

[BP96] M. Bronstein and M. Petkovšek. An introduction to pseudo-linear algebra. *Theoret. Computer Sci.*, 157(1):3–33, 1996.

[Bro99] M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. Technical Report 3797, INRIA Sophia Antipolis, November 1999.

[Bro00] M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, June 2000.

[BS00] M. Bronstein and C. Schneider, 2000. Private Communication.

[Cal94] N. J. Calkin. A curious binomial identity. *Discrete Math.*, 131(1-3):335–337, 1994.

[Coh65] R. M. Cohn. *Difference Algebra*. Interscience Publishers, John Wiley & Sons, 1965.

[Coh89] P. M. Cohn. *Algebra*, volume 2. John Wiley & Sons, 2nd edition, 1989.

[CS98] F. Chyzak and B. Salvy. Non-commutative elimination in ore algebras proves multivariate identities. *J. Symbolic Comput.*, 26(2):187–227, 1998.

[Dil95] K. Dilcher. Some $q$–series identities related to divisor functions. *Discrete Math.*, 145(1-3):83–93, 1995.

[EG95] J. W. Essam and A. J. Guttmann. Vicious walkers and directed polymer networks in general dimension. *Phys. Rev. E.*, 52:5849–5862, 1995.

[FK00] M. Fulmek and C. Krattenthaler. The number of rhombus tilings of a symmetric hexagon which contains a fixed rhombus on the symmetric axis, II. *European J. Combin.*, 21(5):601–640, 2000.

[GKP94] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: a foundation for computer science.* Addison-Wesley Publishing Company, Amsterdam, 2nd edition, 1994.

[Gos78] R. W. Gosper. Decision procedures for indefinite hypergeometric summation. *Proc. Nat. Acad. Sci. U.S.A.*, 75:40–42, 1978.

[HS99] P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.

[Kar81] M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.

[Kar85] M. Karr. Theory of summation in finite terms. *J. Symbolic Comput.*, 1:303–315, 1985.

[Kir96] P. Kirschenhofer. A note on alternating sums. *Electron. J. Combin.*, 3(2), 1996.

[KP98] P. Kirschenhofer and H. Prodinger. Comparisons in Hoare's find algorithm. *Combin. Probab. Comput.*, 7(1):111–120, 1998.

[Lan97] S. Lang. *Algebra.* Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 3rd edition, 1997.

[NP97] I. Nemes and P. Paule. A canonical form guide to symbolic summation. In A. Miola and M. Temperini, editors, *Advances in the Design of Symbolic Computation Systems*, Texts Monogr. Symbol. Comput., pages 84–110. Springer, Wien-New York, 1997.

[Pau95] P. Paule. Greatest factorial factorization and symbolic summation. *J. Symbolic Comput.*, 20(3):235–268, 1995.

[Pet92] M. Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symbolic Comput.*, 14(2-3):243–264, 1992.

[Pet94] M. Petkovšek. A generalization of Gosper's algorithm. *Discrete Math.*, 134(1-3):125–131, 1994.

[PR97] P. Paule and A. Riese. A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated aproach to q-hypergeometric telescoping. In M. Ismail and M. Rahman, editors, *Special Functions, q-Series and Related Topics*, volume 14, pages 179–210. Fields Institute Toronto, AMS, 1997.

[PS95a] P. Paule and M. Schorn. A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. *J. Symbolic Comput.*, 20(5-6):673–698, 1995.

[PS95b] P. Paule and V. Strehl. Symbolic summation - some recent developments. In J. Fleischer et al., editor, *Computer Algebra in Science and Engineering - Algorithms, Systems, and Applications*, pages 138–162. World Scientific, Singapore, 1995.

[PWZ96] M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.

[Rie01] A. Riese. qMultisum - a package for proving q-hypergeometric multiple summation identities. Technical Report SFB 01-7, Johannes Kepler University Linz, January 2001. submitted to J. Symbolic Comput.

[Ris69] R. Risch. The problem of integration in finite terms. *Trans. Amer. Math. Soc.*, 139:167–189, 1969.

[Ris70] R. Risch. The solution to the problem of integration in finite terms. *Bull. Amer. Math. Soc.*, 76:605–608, 1970.

[Sch99] C. Schneider. An implementation of Karr's summation algorithm in Mathematica. Technical Report SFB 99-15, Johannes Kepler University Linz, August 1999.

[Sch00a] C. Schneider. An implementation of Karr's summation algorithm in Mathematica. *Sém. Lothar. Combin.*, S43b:1–10, 2000.

[Sch00b] C. Schneider. Symbolic summation in difference fields. Contributed talk at the "10th SIAM Conference on Discrete Mathematics", Minneapolis, Minnesota, June 2000.

[Sim84] C. C. Sims. *Abstract algebra, A computational approach.* John Wiley & Sons, New York, 1984.

[Sin91] M. F. Singer. Liouvillian solutions of linear differential equations with Liouvillian coefficients. *J. Symbolic Comput.*, 11(3):251–274, 1991.

[vH98] M. van Hoeij. Rational solutions of linear difference equations. In O. Gloor, editor, *Proc. ISSAC'98*, pages 120–123. ACM Press, 1998.

[vH99] M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. *J. Pure Appl. Algebra*, 139(1-3):109–131, 1999.

[Weg97] K. Wegschaider. Computer generated proofs of binomial multi-sum identities. Diploma thesis, RISC Linz, Johannes Kepler University, May 1997.

[Wei01] C. Weixlbaumer. Solutions of difference equations with polynomial coefficients. Diploma thesis, RISC Linz, Johannes Kepler University, January 2001.

[Win96] F. Winkler. *Polynomial Algorithms in Computer Algebra.* Texts and Monographs in Symbolic Computation. Springer, Wien, 1996.

[Zei90] D. Zeilberger. A fast algorithm for proving terminating hypergeometric identities. *Discrete Math.*, 80(2):207–211, 1990.

[Zha99] Z. Zhang. A kind of binomial identity. *Discrete Math.*, 196(1-3):291–298, 1999.

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