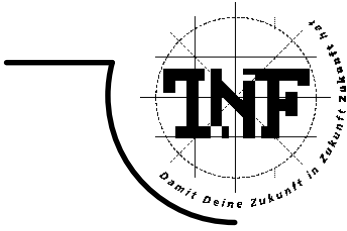




JOHANNES KEPLER  
UNIVERSITÄT LINZ  
Netzwerk für Forschung, Lehre und Praxis



# Power Series Solutions to Holonomic Differential Equations and the General Algebraic Equation

DIPLOMARBEIT

zur Erlangung des akademischen Grades

DIPLOMINGENIEURIN

in der Studienrichtung

INDUSTRIEMATHEMATIK

Angefertigt am *Institut für Symbolisches Rechnen (RISC)*

Betreuung:

*Univ.-Prof. Dr. Peter Paule*

Eingereicht von:

*Karin Nachbagauer*

*Matr.-Nr. 0355569*

*Wagnerberg 9*

*4490 St. Florian*

*Linz, Juni 2009*

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**Johannes Kepler Universität**

A-4040 Linz · Altenbergerstraße 69 · Internet: <http://www.uni-linz.ac.at> · DVR 0093696

Die Arbeit wurde unterstützt durch eine Forschungsbeihilfe aus dem Projekt SFB F013 des österreichischen Forschungsfonds FWF.

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# Abstract

This diploma thesis consists of two parts: the analysis and description of power series methods for solving differential equations and the application to algebraic equations.

In *Chapter 1* and *2* we set the stage by introducing algebraic and analytic theorems, definitions and notations.

The next three chapters deal with solving second order differential equations by the Frobenius Method. We set in describing the method in *Chapter 3*, next we have a closer look at examples for the Frobenius Method calculated with the help of Mathematica in *Chapter 4*, and go on with the Existence Theory in *Chapter 5* to justify and validate the derived solutions.

As a major application domain, we consider the problem of solving algebraic equations like  $y(x) = y^5 + xy^2 - 1 = 0$ . In *Chapter 6* we give a summary about how to solve a general algebraic equation up to degree 4, and briefly consider approximate solutions for higher degrees. In addition, we analyse the behavior of solutions of the example above in an experimental way.

Solving algebraic equations with the powerful power series method expands the range of application of the Frobenius Method. So we transform a general algebraic equation into a differential equation with the help of the algorithm *AEtoDE*, which is described in *Chapter 7*.

Finally, in *Chapter 8* we describe Mellin Series as another solution method for algebraic equations, again depending on power series manipulations.

## **Acknowledgements.**

I want to thank Peter Paule for supervising my diploma thesis, especially giving me the chance to dip into the fascinating world of combinatorics. He supported me finding helpful literature and ideas.

I also would like to thank the participants of the combinatoric seminars in Hagenberg for the helpful discussions and hints.

My special thanks go to Christoph Koutschan, who gave me the formative hints for my implementation and always was willing to listen to my problems.

I want to thank my family members for their support.

Most of all I want to thank my parents, who made my dream to become a mathematician possible.

## **Support Notice.**

This diploma thesis was supported by SFB grant F1305 of the Austrian Science Funds FWF.

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## Chapter 1

# Algebraic and Analytic Preliminaries

We use the notation  $\mathbb{N}$  for  $\{0, 1, 2, 3, \dots\}$ , and  $\mathbb{N}^*$  for  $\mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}$

The ring of polynomials in the variable  $z$  over the coefficient ring  $R$  is denoted as usual by  $R[z]$ .

For the degree of  $p(z) \in R[z]$  we use the notation  $\deg(p(z))$ .

### Definition 1

Given  $p(z) \in \mathbb{C}[z]$ . If  $z_0 \in \mathbb{C}$  is such that  $p(z_0) = 0$ , then  $z_0$  is called a root of the polynomial  $p(z)$ .

### Theorem 1 (Fundamental Theorem of Algebra)

Let  $p(z) \in \mathbb{C}[z]$  with  $\deg(p(z)) = n \in \mathbb{N}$ . Then  $p(z)$  has exactly  $n$  complex roots where each root is counted with its multiplicity.

Informally this means, each algebraic equation of the form

$$z^n + a_{n-1}z^{n-1} + \dots + a_0 = 0,$$

$a_i \in \mathbb{C}$ , has at most  $n$  different solutions.

### Theorem 2 (Theorem of Abel-Galois-Ruffini) <sup>1</sup>

The solutions of an equation of degree  $n \leq 4$  can be given in terms of a formula

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<sup>1</sup>[19] G. Piltz, 1984.

using arithmetic operations and (nested) radicals only. Equations of an degree  $n \geq 5$  cannot be solved in terms of such expressions.

Each algebraic equation with coefficients  $X_1, \dots, X_n \in \mathbb{C}$

$$y^n + X_1 y^{n-1} + X_2 y^{n-2} + \dots + X_{n-1} y + X_n = 0,$$

can be transformed into the algebraic equation

$$y^n + x_1 y^{n-1} + x_2 y^{n-2} + \dots + x_{n-1} y - 1 = 0,$$

with coefficients  $x_1, \dots, x_{n-1} \in \mathbb{C}$ , by the simple transformation  $y \rightarrow \sqrt[n]{-X_n} y$ .

### Definition 2

For  $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ :

$$z! := \Gamma(z + 1),$$

where  $\Gamma$  denotes the Gamma function.

### Definition 3

For  $a \in \mathbb{C}$ ,  $n \in \mathbb{N}$ :

$$(a)_n := a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

where  $(a)_n$  denotes the Pochhammer symbol (also called rising factorials).

### Definition 4

The falling factorials are defined as usual by

$$[x]_m := x(x-1) \cdots (x-m+1), \quad m \geq 1,$$

and  $[x]_0 := 1$ .

### Theorem 3 (Multinomial Theorem) <sup>2</sup>

For  $k, n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_k \in \mathbb{C}$  the following equation holds:

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{\substack{n_1 + \dots + n_k = n \\ n_i \in \mathbb{N}}} \frac{n!}{n_1! n_2! \dots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

---

<sup>2</sup>[9], A. Kemnitz, p. 377, 2002.



## Chapter 2

# Formal Power and Laurent Series

### 2.1 Basic Concepts

Most of the notations and propositions in this chapter are taken from P. Paule [15].

Throughout this chapter,  $\mathbb{F}$  denotes a field containing the rational numbers as a subfield.

#### Notation 1 (Formal Power Series)

The ring of formal power series in  $z$  over  $\mathbb{F}$  is denoted by  $(\mathbb{F}[[z]], +, \cdot)$  where

$$\mathbb{F}[[z]] = \left\{ \sum_{m=0}^{\infty} a_m z^m \mid a_m \in \mathbb{F} \right\},$$

and where  $+$  is defined componentwise

$$\sum_{m=0}^{\infty} a_m z^m \pm \sum_{m=0}^{\infty} b_m z^m = \sum_{m=0}^{\infty} (a_m \pm b_m) z^m,$$

and where  $\cdot$  is the usual Cauchy product

$$\sum_{m=0}^{\infty} a_m z^m \cdot \sum_{m=0}^{\infty} b_m z^m = \sum_{m=0}^{\infty} c_m z^m,$$

with

$$c_m := \sum_{k=0}^{\infty} a_k b_{m-k}.$$

**Note:** A priori the term  $\sum_{m=0}^{\infty} a_m z^m$  is just another notation for the sequence  $(a_m)_{m \geq 0}$ .

**Definition 5 (Convergence in the ring of formal power series)**

For the sequence  $(f_k(z))_{k \geq 0}$  with elements in  $\mathbb{F}[[z]]$  and  $f(z) = \sum_{m=0}^{\infty} a_m z^m \in \mathbb{F}[[z]]$  we define

$$\lim_{k \rightarrow \infty} f_k(z) = f(z)$$

if and only if for all  $m \in \mathbb{N}$  there exists a  $\delta(m) \in \mathbb{N}$  such that the coefficients of  $z^m$  of the entries  $f_k(z)$  of the sequence  $(f_k(z))_{k \geq 0}$  are equal to  $a_m$  for all  $k \geq \delta(m)$ .

**Note:** If we impose the definition of a limit of a sequence of formal power series, we can interpret  $\sum_{m=0}^{\infty} a_m z^m$  as follows:

$$\sum_{m=0}^{\infty} a_m z^m = \lim_{N \rightarrow \infty} \sum_{m=0}^N a_m z^m.$$

**Proposition 1 (Limit Rules in the ring of formal power series)**

For the sequences  $(f_k(z))_{k \geq 0}$  and  $(g_k(z))_{k \geq 0}$  with elements in  $\mathbb{F}[[z]]$  and such that  $\lim_{k \rightarrow \infty} f_k(z) = f(z)$  and  $\lim_{k \rightarrow \infty} g_k(z) = g(z)$  we have

$$\lim_{k \rightarrow \infty} \left( f_k(z) \cdot^{\dagger} g_k(z) \right) = f(z) \cdot^{\dagger} g(z).$$

**Theorem 4 (Abel's Theorem) <sup>1</sup>**

If a power series  $f(w) = \sum_{k \geq 0} f_k w^k$  converges at a point  $w_0$  on its circle of convergence, then

$$\sum_{k=0}^{\infty} f_k w_0^k = \lim_{w \rightarrow w_0} f(w),$$

where  $w$  tends to  $w_0$  along a radius inside the circle of convergence.

**Lemma 1 (Multiplicative Inverse)**

The formal power series  $f(z) = \sum_{m=0}^{\infty} a_m z^m \in \mathbb{F}[[z]]$  has a multiplicative inverse  $f^{-1}(z)$  if and only if  $a_0 \neq 0$ .

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<sup>1</sup>[11], T.M. MacRobert, 1947.

**Note:** For  $f(z) \cdot g(z) = 1$  with  $f(z) = \sum_{m=0}^{\infty} a_m z^m$  and  $g(z) = \sum_{m=0}^{\infty} b_m z^m \in \mathbb{F}[[z]]$  one writes

$$f^{-1}(z) := \frac{1}{f(z)} := g(z).$$

### Notation 2 (Formal Laurent Series)

The quotient field  $\mathbb{F}((z))$  of the ring of formal power series  $\mathbb{F}[[z]]$  is called the field of formal Laurent series, i. e.,

$$\mathbb{F}((z)) = \left\{ \frac{a(z)}{b(z)} \mid a(z), b(z) \in \mathbb{F}[[z]], b(z) \neq 0 \right\}.$$

Or, in other words,

$$\mathbb{F}((z)) = \left\{ \sum_{n=-n_0}^{\infty} a_n z^n \mid n_0 \in \mathbb{Z}, a_n \in \mathbb{F} \right\} = \left\{ z^m \sum_{n=0}^{\infty} a_n z^n \mid m \in \mathbb{Z}, a_n \in \mathbb{F} \right\}.$$

**Note:** For  $f(z) \in \mathbb{F}((z))$  we sometimes write  $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$  with the understanding that there is a  $n_0 \in \mathbb{Z}$  such that  $a_n = 0$  for all  $n < n_0$ .

### Theorem 5 (Transfer Principle)

Let  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  be sequences in  $\mathbb{C}$ . If  $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$  holds for all  $z \in ]-\epsilon, \epsilon[$  for some  $\epsilon > 0$ , then  $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$  holds in  $\mathbb{C}[[z]]$ .

### Definition 6 (Coefficient Functional)

Let  $f(z) = \sum_{m=-\infty}^{\infty} a_m z^m \in \mathbb{F}((z))$ . For  $n \in \mathbb{Z}$  we define

$$[z^n]f(z) := a_n.$$

**Note:**  $[z^n]$  can be considered as a coefficient functional on  $\mathbb{F}[[z]]$  viewed as a vector space over  $\mathbb{F}$ . For picking the constant term, we use the symbol  $L := [z^0]$

$$f(0) = [z^0]f(z) = L f(z) := a_0.$$

**Note:**  $f(z) = g(z)$  holds in  $\mathbb{F}[[z]]$  if and only if  $n \in \mathbb{N}_0$  applies

$$[z^n]f(z) = [z^n]g(z)$$

for all  $n \in \mathbb{N}_0$ .

**Definition 7 (Order of Power Series)**

For a power series  $f \in \mathbb{F}[[z]]$ , we define its order by

$$\text{ord } f = \min \{m : [z^m]f(z) \neq 0 \text{ and } m \in \mathbb{N}\}.$$

**Definition 8 (Differential Ring)**

Let  $R$  be a ring, respectively field. A derivation  $D$  on  $R$  is a map  $D : R \rightarrow R$  such that for all  $f$  and  $g \in R$  one has:

$$D(f + g) = Df + Dg,$$

and

$$D(f \cdot g) = (Df) \cdot g + f \cdot (Dg).$$

$(R, D)$  is called a differential ring, respectively field.

**Definition 9 (Derivation in the ring of formal power series)**

The standard derivation  $\frac{d}{dz}$  on  $\mathbb{F}[[z]]$  is defined as the map

$$\begin{aligned} \frac{d}{dz} : \mathbb{F}[[z]] &\rightarrow \mathbb{F}[[z]], \\ \sum_{n=0}^{\infty} a_n z^n &\mapsto \sum_{n=0}^{\infty} a_{n+1} (n+1) z^n \\ &= \sum_{n=1}^{\infty} a_n n z^{n-1}. \end{aligned}$$

This turns  $\mathbb{F}[[z]]$  into a differential ring  $(\mathbb{F}[[z]], \frac{d}{dz})$ .

**Definition 10 (Derivation in the field of formal Laurent series)**

The standard derivation  $\frac{d}{dz}$  on  $\mathbb{F}((z))$  is defined as the map

$$\begin{aligned} \frac{d}{dz} : \mathbb{F}((z)) &\rightarrow \mathbb{F}((z)), \\ z^m \sum_{n=0}^{\infty} a_n z^n &\mapsto m z^{m-1} \sum_{n=0}^{\infty} a_n z^n + z^m \sum_{n=1}^{\infty} a_n n z^{n-1} \end{aligned}$$

This turns  $\mathbb{F}((z))$  into a differential field  $(\mathbb{F}((z)), \frac{d}{dz})$ .

**Notation:** For convenience, we denote  $\frac{d}{dz}$  by  $D$ , or  $\frac{d}{dz}f(z)$  by  $f'(z)$ . We also introduce the abbreviated notation

$$\begin{aligned} Df(z) &= f'(z), \\ D^k f(z) &= f^{(k)}(z), \end{aligned}$$

for  $k \in \mathbb{N}$ .

### Proposition 2

Let  $f(z)$  and  $g(z) \in \mathbb{F}[[z]]$  such that  $g(0) = [z^0]g(z) = 0$ . Then

$$f(g(z))' = f'(g(z)) \cdot g'(z).$$

### Proposition 3

Let  $a(z)$  and  $b(z) \in \mathbb{F}[[z]]$ . The differential equation

$$y'(z) = a(z) \cdot y(z) + b(z)$$

always has a solution  $y(z) \in \mathbb{F}[[z]]$ . The solution  $y(z)$  is uniquely determined by the choice of  $y(0)$ .

### Definition 11 <sup>2</sup>

$f(x) = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$  is called holonomic, if there exist

$$p_0(x), p_1(x), \dots, p_m(x) \in K[x],$$

not all zero, such that

$$p_m(x)f^{(m)}(x) + \dots + p_1(x)f'(x) + p_0(x)f(x) = 0.$$

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<sup>2</sup>[15], P.Paule, 2007.

## 2.2 Lagrange Inversion

### Definition 12 (Composition in the ring of formal power series)

For  $f(z) = \sum_{m=0}^{\infty} a_m z^m$  and  $g(z) = \sum_{m=0}^{\infty} b_m z^m \in \mathbb{F}[[z]]$  with  $g(0) = 0$  the composition of  $f(z)$  and  $g(z)$  in  $\mathbb{F}[[z]]$  is defined in the following way

$$(f \circ g)(z) := f(g(z)) := \sum_{m=0}^{\infty} a_m g(z)^m.$$

### Theorem 6 (Compositional Insertion Homomorphism)

For  $h(z) = h_1 z^1 + h_2 z^2 + \dots \in \mathbb{F}[[z]]$  we define the mapping

$$\begin{aligned} \varphi_h : \mathbb{F}[[z]] &\rightarrow \mathbb{F}[[z]], \\ f(z) &\mapsto \varphi_h(f(z)) := f(h(z)). \end{aligned}$$

Then  $\varphi_h$  is a ringhomomorphism.

### Notation 3 (Residue Symbol)

Acting on formal Laurent series the coefficient functional  $[z^{-1}]$  is called the residue and written as  $M$ .

### Lemma 2

For any  $f(z) \in \mathbb{F}((z))$  and any  $k, l \in \mathbb{Z}$ , we have

$$[z^l]f(z) = [z^{k+l}]z^k f(z).$$

### Theorem 7 (Lagrange Inversion - first version) <sup>3</sup>

Let  $f, g \in \mathbb{C}[[z]]$  with  $\text{ord } g = 1$ . Then there exists an  $h \in \mathbb{C}[[z]]$  such that  $h \circ g = f$ , and its coefficients  $c_m$  can be found as

$$\begin{aligned} c_0 &:= Lf(z), \\ [\forall m > 0] c_m &:= \frac{1}{m} Mf'(z)g(z)^{-m} = Mf(z)g'(z)g(z)^{-m-1}. \end{aligned}$$

For the proof of the Lagrange inversion formula we need the following Lemma.

<sup>3</sup>[22] M. Rosenkranz, 1997.

**Lemma 3**

Let  $g \in \mathbb{C}[[z]]$  with  $\text{ord } g = 1$ . Then the residues of the formal Laurent series  $g'/g^n$  are

$$M \frac{g'(z)}{g(z)^{n+1}} = \delta_{n0}$$

for all  $n \in \mathbb{Z}$ .

**Proof (Lemma 3)**

For the proof of the Lemma we need a case distinction.

For  $n \neq 0$  we have

$$g'(z)g(z)^{-n-1} = -\frac{1}{n}(g(z)^{-n})'.$$

If we apply the residue functional  $M$  on both sides, we get

$$Mg'(z)g(z)^{-n-1} = M\left(-\frac{1}{n}(g(z)^{-n})'\right).$$

The residue of a derivative of a formal Laurent series is always zero, because the  $[z^{-1}]$ -coefficient does not appear. Hence we proved the statement for  $n \neq 0$ .

For the case  $n = 0$  we write  $g(z) = \frac{z}{G(z)}$  with  $\text{ord } G = 0$ . With this setting we get

$$\frac{g'(z)}{g(z)} = \frac{G(z) - G'(z) \cdot z}{G^2(z)} \cdot \frac{G(z)}{z} = \frac{1}{z} - \frac{G'(z)}{G(z)}.$$

If we now apply the residue functional on both sides, we get

$$M\left(\frac{g'(z)}{g(z)}\right) = M\left(\frac{1}{z}\right) - M\left(\frac{G'(z)}{G(z)}\right) = 1 - 0,$$

because  $\frac{1}{G(z)}$  and  $G'(z)$  are formal power series and the residue of a formal Laurent series is zero. □

**Proof (Theorem 7)**

Our goal is to find the coefficients  $c_m$  of a power series  $h$  with  $h \circ g = f$ .

So in the first step we take the derivative of

$$f(z) = h(z) \circ g(z) = \sum_{n \geq 0} c_n g(z)^n,$$

and get (by Proposition 2)

$$f'(z) = \sum_{n \geq 0} n c_n g(z)^{n-1} g'(z).$$

If we divide both sides by  $g(z)^m$  with  $m > 0$ , we get

$$f'(z)g(z)^{-m} = \sum_{n \geq 0} n c_n \frac{g'(z)}{g(z)^{m-n+1}}.$$

Now we apply the residue functional and derive

$$M f'(z)g(z)^{-m} = M \sum_{n \geq 0} n c_n \frac{g'(z)}{g(z)^{m-n+1}}.$$

Because we know the following property

$$M \frac{g'(z)}{g(z)^{m-n+1}} = \delta_{m-n,0} = \delta_{m,n},$$

the only term which remains in  $M \sum_{n \geq 0} n c_n \frac{g'(z)}{g(z)^{m-n+1}}$  is  $m c_m$ .

Hence, we get

$$\frac{1}{m} M f'(z)g(z)^{-m} = c_m.$$

With the following auxiliary calculation

$$\begin{aligned} 0 &= M(f(z)g(z)^{-m})' \\ &= M f'(z)g(z)^{-m} - m M f(z)g(z)^{-m-1}g'(z), \end{aligned}$$

we derive the demanded equation

$$[\forall m > 0] c_m = \frac{1}{m} M f'(z)g(z)^{-m} = M f(z)g'(z)g(z)^{-m-1}.$$

Since  $Lf(z)$  is the constant term coefficient  $c_0g(z)^0$ , we get

$$Lf(z) = c_0.$$

□

For some applications, it is convenient to write  $g(z)$  as a fraction  $\frac{z}{G(z)}$  like in the Lemma before. For that we can apply the following equivalent variant of Lagrange's Inversion.



**Theorem 8 (Lagrange Inversion - second version)** <sup>4</sup>

Let  $f, G \in \mathbb{C}[[z]]$  with  $\text{ord} G = 0$  and set  $g(z) = \frac{z}{G(z)}$ . Then there exists an  $h \in \mathbb{C}[[z]]$  such that  $h \circ g = f$ , and its coefficients  $c_m$  can be found as

$$\begin{aligned} c_0 &:= [z^0]f(z), \\ [\forall m > 0] c_m &:= \frac{1}{m}[z^{m-1}]f'(z)G(z)^m = [z^m]f(z)g'(z)G(z)^{m+1}. \end{aligned}$$

**Proof (Theorem 8)**

For the proof of the fraction variant of the Lagrange inversion formula, we can utilize the first version.

Our goal is to find the coefficients  $c_m$  of a power series  $h$  with  $h \circ g = f$ , where  $g(z) = \frac{z}{G(z)}$ .

Constituting  $g(z) = \frac{z}{G(z)}$  in  $c_m = Mf(z)g'(z)g(z)^{-m-1}$  we derive

$$c_m = [z^{-1}]f(z)g'(z)\left(\frac{z}{G(z)}\right)^{-m-1} = [z^{-1}]z^{-m-1}f(z)g'(z)G(z)^{m+1},$$

which is exactly

$$c_m = [z^m]f(z)g'(z)G(z)^{m+1}.$$

With the auxiliary calculation

$$\begin{aligned} 0 &= M(f(z)g(z)^{-m})' \\ &= Mf'(z)g(z)^{-m} - mMf(z)g(z)^{-m-1}g'(z), \end{aligned}$$

we get the equivalent equations

$$\begin{aligned} \frac{1}{m}Mf'(z)g(z)^{-m} &= Mf(z)g(z)^{-m-1}g'(z) \\ \frac{1}{m}[z^{-1}]f'(z)G(z)^m\left(\frac{1}{z}\right)^m &= [z^{-1}]f(z)G(z)^{m+1}g'(z)\left(\frac{1}{z}\right)^{m+1} \\ \frac{1}{m}[z^{m-1}]f'(z)G(z)^m &= [z^{m-1}]f(z)G(z)^{m+1}g'(z)\frac{1}{z} \\ &= [z^m]f(z)G(z)^{m+1}g'(z). \end{aligned}$$

Hence, we derived the demanded equation

$$c_m = \frac{1}{m}[z^{m-1}]f'(z)G(z)^m = [z^m]f(z)g'(z)G(z)^{m+1}.$$

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<sup>4</sup>[22] M. Rosenkranz, 1997.

Since  $Lf(z)$  is the constant term coefficient  $[z^0]f(z)$ , we get

$$c_0 = [z^0]f(z).$$

□

## 2.3 Useful Lemmas

It will be useful to consider the following situation.<sup>5</sup>

Suppose that

$$u = u_1 u_2 u_3 \cdots u_n,$$

each of the  $u$ 's being a function of the parameter  $z$ . Let differentiation with respect to  $z$  be indicated by primes. Then from

$$\ln u = \ln u_1 + \ln u_2 + \ln u_3 + \cdots + \ln u_n$$

it follows that

$$\frac{u'}{u} = \frac{u'_1}{u_1} + \frac{u'_2}{u_2} + \frac{u'_3}{u_3} + \cdots + \frac{u'_n}{u_n}.$$

Hence

$$u' = u \cdot \left( \frac{u'_1}{u_1} + \frac{u'_2}{u_2} + \frac{u'_3}{u_3} + \cdots + \frac{u'_n}{u_n} \right). \quad (2.1)$$

Next we consider an analogous situation in the context of formal power series.

### Definition 13

$$L(z) := \ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} \in \mathbb{F}[[z]].$$

**Note:** The following rule for differentiation is valid:

$$L'(z) = D \ln(1+z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}.$$

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<sup>5</sup>[20] E.D. Rainville, S. 324, 1964.

Let  $f(z) \in \mathbb{F}[[z]]$  with  $\text{ord } f > 0$ . For  $K(z) \in \mathbb{F}[[z]]$  defined as

$$\begin{aligned} K(z) &:= \ln(1 + f(z)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (f(z))^{n+1} \\ &= (L \circ f)(z) \\ &= L(f(z)) \end{aligned}$$

the following rule for differentiation is valid:

$$\begin{aligned} K'(z) &= D \ln(1 + f(z)) \\ &= D(L \circ f)(z) \\ &\stackrel{\star}{=} L'(f(z))f'(z) \\ &= \frac{1}{1 + f(z)} \cdot f'(z). \end{aligned}$$

The equation  $\star$  is satisfied by Proposition 2.

Now consider a formal power series analog of (2.1). Given  $u(z), f_1(z), f_2(z) \in \mathbb{F}[[z]]$ . Suppose that

$$1 + u(z) = u_1(z) \cdot u_2(z),$$

where  $u_i(z) := 1 + f_i(z)$ . Let differentiation with respect to  $z$  be indicated by primes.

#### Lemma 4

For  $f_i(z) \in \mathbb{C}[[z]]$  with  $\text{ord}(f_i) > 0$ :

$$\ln(1 + u(z)) = \ln(1 + f_1(z)) + \ln(1 + f_2(z)). \quad (2.2)$$

#### Proof (Lemma 4)

We show that the left side of (2.2), denoted by  $a(z)$ , fulfills the same differential equation of order one as the right hand side of (2.2), denoted by  $b(z)$ .

In view of Proposition 3, to complete the proof, we then have to show that the initial values coincide, i. e.,  $a(0) = b(0)$ .

We compute the derivatives of  $a(z)$  and  $b(z)$ , compare the results:

$$\begin{aligned}
a'(z) &= \frac{u'(z)}{1+u(z)}, \\
b'(z) &= \frac{1}{1+f_1(z)} \cdot f_1'(z) + \frac{1}{1+f_2(z)} \cdot f_2'(z) \\
&= \frac{(1+f_2(z))f_1'(z) + (1+f_1(z))f_2'(z)}{(1+f_1(z))(1+f_2(z))} \\
&= \frac{u'(z)}{1+u(z)}.
\end{aligned}$$

So we found out that both sides fulfill the same differential equation. Finally we verify that the initial values match:

$$\begin{aligned}
[z^0] a(z) &= \ln(1 + [z^0] u(z)), \\
[z^0] b(z) &= \ln(1 + [z^0] f_1(z)) + \ln(1 + [z^0] f_2(z)) \\
&= \ln \underbrace{[(1 + [z^0] f_1(z)) \cdot (1 + [z^0] f_2(z))]}_{(1+[z^0]u(z))} \\
&= \ln(1 + [z^0] u(z))
\end{aligned}$$

and observe that the initial values coincide.

□

Then from Lemma 4 it follows that

$$\frac{u'}{1+u} = \frac{u'_1}{u_1} + \frac{u'_2}{u_2}.$$

Hence

$$u' = (1+u) \cdot \left( \frac{u'_1}{u_1} + \frac{u'_2}{u_2} \right). \quad (2.3)$$

Given  $u(z), f_i(z) \in \mathbb{F}[[z]]$ . Suppose that

$$1 + u(z) = u_1(z) \cdot u_2(z) \cdots u_n(z),$$

where  $u_i(z) := 1 + f_i(z)$ . Let differentiation with respect to  $z$  again be indicated by primes. Then from

$$\begin{aligned}
\ln(1 + u(z)) &= \ln(u_1(z) \cdot u_2(z) \cdots u_n(z)) \\
&= \ln(1 + f_1(z)) + \ln(1 + f_2(z)) + \cdots + \ln(1 + f_n(z))
\end{aligned}$$

it follows that

$$\frac{u'}{1+u} = \frac{u'_1}{u_1} + \frac{u'_2}{u_2} + \cdots + \frac{u'_n}{u_n}.$$

Hence

$$u' = (1+u) \cdot \left( \frac{u'_1}{u_1} + \frac{u'_2}{u_2} + \cdots + \frac{u'_n}{u_n} \right).$$



## Chapter 3

# Frobenius Method

We want to solve holonomic differential equations, more precise we want to find a basis for the solution space in form of power series and eventually with logarithms.

### Definition 14

Let  $b_0(x), b_1(x), \dots, b_n(x)$  be meromorphic functions defined in some regions of the complex plane. The differential equation

$$b_0(x)y^{(n)}(x) + b_1(x)y^{(n-1)} + \dots + b_n(x)y = 0$$

is called **holonomic**, if all the  $b_i(x)$  are polynomials from  $\mathbb{C}[x]$ .<sup>1</sup>

We will achieve this goal with the help of the Frobenius Method, which can be found in [20] and [21], and which also forms the basis for our presentation. Restricting to the holonomic paradigm, the differential equations treated here will be confined to those with polynomial coefficients, but the results remain essentially unchanged when the coefficients are analytic functions, or rather the functions have power series expansions valid about some point.

In addition, we restrict the observation to differential equations of second order, in fact of the recurrent appearance and importance.

The principle of the method is very simple. We plug the ansatz  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  into our differential equation and obtain a recurrence for the coefficient  $a_n$ . The closed form for  $a_n$  induces the solution. We have to look at the ordinary and

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<sup>1</sup>For a detailed account of holonomic functions (i. e., solutions of holonomic differential equations) and sequences see [12].

singular points of the differential equation to choose which variant of this principle we have to use to get the right solution.

### Definition 15 (Ordinary and singular points)

For the homogeneous linear differential equation

$$b_0(x)y^{(n)}(x) + b_1(x)y^{(n-1)}(x) + \cdots + b_n(x)y(x) = 0 \quad (3.1)$$

with polynomial coefficients  $b_0(x), \dots, b_n(x) \in \mathbb{C}[x]$ , with  $b_0(x) \neq 0$ , the point  $x = x_0$  is an **ordinary point** of the equation if  $b_0(x_0) \neq 0$ . Any point  $x = x_1$  for which  $b_0(x_1) = 0$ , is a **singular point** of the equation.

### Definition 16 (Regular singular points for order 2)

Assume that the point  $x = x_0$  is a singular point of the second order differential equation

$$b_0(x)y''(x) + b_1(x)y'(x) + b_2(x)y(x) = 0 \quad (3.2)$$

with polynomial coefficients  $b_i(x) \in \mathbb{C}[x]$ , which do not have common roots. Due to  $b_0(x_0) = 0$ ,  $b_0(x)$  has a factor  $(x - x_0)$  to some power. By dividing by  $b_0(x) \neq 0$  we transform our equation (3.2) into

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where  $p(x)$  and  $q(x)$  are rational functions of  $x$ . Since  $x = x_0$  is a singular point, at least one or maybe both of  $p(x)$  and  $q(x)$  has a denominator which contains the factor  $(x - x_0)$ .

We call the singular point  $x = x_0$  a **regular singular point** of the equation, if both of the following conditions are satisfied:

- The denominator of  $p(x)$  does not contain the factor  $(x - x_0)$  to a power higher than one.
- The denominator of  $q(x)$  does not contain the factor  $(x - x_0)$  to a power higher than two.

We call the singular point  $x = x_0$  a **irregular singular point** of the equation, if it is not a regular singular point.



Singular points of a linear equation of a higher order than 2 are classified in much the same way.

**Example:**

The singular point  $x = x_0$  of the equation

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0$$

is called a regular singular point if the factor  $(x - x_0)$  does not appear in the denominator of  $p(x)$  to a power higher than one, of  $q(x)$  to a power higher than two and of  $r(x)$  to a power higher than three. If these conditions are not fulfilled, then the singular point is called an irregular singular point of the equation.

We will explain the different variants of the Frobenius Method by showing examples for every case.

We concentrate on solutions valid about the origin  $x = 0$ . If we solve an equation about the point  $x = x_0$ , we obtain solutions valid in a region surrounding this point and the solutions are expressed in powers of  $(x - x_0)$ . So for simplicity, whenever we want to obtain solutions about a point other than the origin, we translate the point  $x = 0$  to that point and proceed with the usual techniques.

The important step is translating the axes, putting  $x - x_0 = v$  and setting  $\frac{dy}{dx} = \frac{dy}{dv}$ .

**Example:**

We want to solve the equation

$$y''(x) + (x - 1)^2 y'(x) - 4(x - 1)y(x) = 0 \tag{3.3}$$

about the ordinary point  $x = 1$ . This means, we want to find a series solution of the form

$$y(x) = \sum_{k=0}^{\infty} y_k (x - 1)^k = \sum_{k=0}^{\infty} y_k v^k,$$

where  $v = x - 1$ . The differential equation (3.3) is then transformed into a differential equation with respect to  $v$  as follows.

Consider  $x$  as a function in  $v$ , i. e.,

$$x = x(v) = v + 1,$$

and define

$$Y(v) := y(x(v)).$$

Observe that due to  $x'(v) = \frac{dx}{dv}(v) = 1$ :

$$\frac{dY}{dv}(v) = D_v y(x(v)) = y'(x(v)) \cdot x'(v) = y'(x),$$

and

$$\frac{d^2Y}{dv^2}(v) = D_v y'(x(v)) = y''(x(v)) \cdot x'(v) = y''(x).$$

Hence, equation (3.3) turns into

$$\frac{d^2Y}{dv^2}(v) + (x(v) - 1)^2 \frac{dY}{dv}(v) - 4(x(v) - 1)Y(v) = 0,$$

or equivalently:

$$Y''(v) + v^2 Y'(v) - 4vY(v) = 0.$$

In the following section we obtain power series solutions valid near an ordinary point of a linear equation.

### 3.1 Solutions near an ordinary point

We want to solve the following differential equation (see [20], Section 103, Exercice 16) of third order near an ordinary point

$$y'''(x) + x^2 y''(x) + 5x y'(x) + 3y(x) = 0. \quad (3.4)$$

The coefficient of  $y'''(x)$  is one and therefore never gets zero. Hence, the differential equation (3.4) has no singular point. We solve near the ordinary point  $x = 0$ . We know from Theorem 9 in Chapter 5 that there is a solution

$$y = \sum_{n=0}^{\infty} a_n x^n \quad (3.5)$$

where three arbitrary complex constants pop up, namely,  $a_0$ ,  $a_1$  and  $a_2$ .

**Note:** In general, for order  $d$  one can freely choose the first  $d$  coefficients of the series, i. e.,  $a_0, \dots, a_{d-1}$ .

This power series converges inside a circle with center at  $x = 0$  which spread out to the singular point nearest to  $x = 0$ . If there exists no singular point to (3.5), the power series solution is valid for all  $x \in \mathbb{C}$ .

Hence, we know in advance that there is a solution (3.5) valid in the complex

plane with  $a_0$ ,  $a_1$  and  $a_2$  arbitrary constants.

In the first step we substitute (3.5) into our differential equation (3.4) and derive

$$\sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^{n-3} + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 5na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0.$$

Since there is the same power of  $x$  in the last three sums, we can combine these series to

$$\sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^{n-3} + \sum_{n=0}^{\infty} \underbrace{(n(n-1) + 5n + 3)}_{=n^2+4n+3} a_n x^n = 0.$$

$n^2 + 4n + 3$  can be factored to  $(n+1)(n+3)$  and so we get:

$$\sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^{n-3} + \sum_{n=0}^{\infty} (n+1)(n+3)a_n x^n = 0.$$

We know that for a power series, which vanishes identically over any interval, each coefficient in the series must be zero. So in the next step we write the two series in a form in which the exponents on  $x$  will be the same. When we want to determine  $a_n$ , we can easily pick off the coefficient of each power of  $x$ .

We shift the index in the second series, replacing  $n$  by  $(n-3)$ .

$$\sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^{n-3} + \sum_{n=3}^{\infty} (n-2)na_{n-3} x^{n-3} = 0.$$

If this should be zero the coefficient of each separate power of  $x$  must be zero. For  $n = 0$ ,  $n = 1$  and  $n = 2$  the second series has not yet started, so we get contributions from the first series only.

$$\begin{aligned} n = 0 : & \quad 0 \cdot a_0 = 0, \\ n = 1 : & \quad 0 \cdot a_1 = 0, \\ n = 2 : & \quad 0 \cdot a_2 = 0, \\ n \geq 3 : & \quad n(n-1)(n-2)a_n + (n-2)na_{n-3} = 0. \end{aligned}$$

Hence,  $a_0$ ,  $a_1$  and  $a_2$  can be chosen arbitrarily. We use the relation for  $n \geq 3$  to determine the other  $a$ 's, that means we get a relation for  $a_n$  in terms of  $a_0$ ,  $a_1$  and  $a_2$ . Since we can divide

$$\underbrace{n(n-1)(n-2)}_{\neq 0 \text{ for } n \geq 3} a_n + (n-2)na_{n-3} = 0$$

by  $n(n-1)(n-2)$ , we derive a recurrence relation for  $a_n$ :

$$a_n = -\frac{1}{(n-1)}a_{n-3}, \quad n \geq 3.$$

We can see that each  $a$  is determined by the  $a$  with subscript three lower than its own. We arrange the iterated instances of the relation in three rows to derive a general shape.

$$\begin{aligned} a_3 &= -\frac{1}{2}a_0, & a_6 &= -\frac{1}{5}a_3, & a_9 &= -\frac{1}{8}a_6, \dots, & a_{3k} &= -\frac{1}{3k-1}a_{3k-3} \\ a_4 &= -\frac{1}{3}a_1, & a_7 &= -\frac{1}{6}a_4, & a_{10} &= -\frac{1}{9}a_7, \dots, & a_{3k+1} &= -\frac{1}{3k}a_{3k-2} \\ a_5 &= -\frac{1}{4}a_2, & a_8 &= -\frac{1}{7}a_5, & a_{11} &= -\frac{1}{10}a_8, \dots, & a_{3k+2} &= -\frac{1}{3k+1}a_{3k-1} \end{aligned}$$

We obtain the product of corresponding members of the equation from each row. The result,

$$\begin{aligned} a_{3k} &= \underbrace{\left(-\frac{1}{3k-1}\right)\left(-\frac{1}{3k-4}\right)\left(-\frac{1}{3k-7}\right)\cdots\left(-\frac{1}{2}\right)}_{k \text{ terms}} a_0, \\ a_{3k+1} &= \underbrace{\left(-\frac{1}{3k}\right)\left(-\frac{1}{3k-3}\right)\left(-\frac{1}{3k-6}\right)\cdots\left(-\frac{1}{3}\right)}_{k \text{ terms}} a_1, \\ a_{3k+2} &= \underbrace{\left(-\frac{1}{3k+1}\right)\left(-\frac{1}{3k-2}\right)\left(-\frac{1}{3k-5}\right)\cdots\left(-\frac{1}{4}\right)}_{k \text{ terms}} a_2, \end{aligned}$$

simplifies for  $k \geq 1$  to

$$\begin{aligned} a_{3k} &= (-1)^k \frac{1}{2 \cdot 5 \cdot 8 \cdots (3k-1)} a_0, \\ a_{3k+1} &= (-1)^k \frac{1}{3 \cdot 6 \cdot 9 \cdots (3k)} a_1, \\ a_{3k+2} &= (-1)^k \frac{1}{4 \cdot 7 \cdot 10 \cdots (3k+1)} a_2. \end{aligned}$$

We substitute the expressions for the  $a$ 's into the series (3.5) for  $y$ . The nature of our obtained expressions claims to split the series into the three series

$$y = \left[ a_0 + \sum_{k=1}^{\infty} a_{3k} x^{3k} \right] + \left[ a_1 x + \sum_{k=1}^{\infty} a_{3k+1} x^{3k+1} \right] + \left[ a_2 x^2 + \sum_{k=1}^{\infty} a_{3k+2} x^{3k+2} \right].$$

Now we use the known results for  $a_{3k}$ ,  $a_{3k+1}$  and  $a_{3k+2}$  to obtain the general solution dependent on the arbitrary constants  $a_0$ ,  $a_1$  and  $a_2$

$$y = a_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k}}{2 \cdot 5 \cdot 8 \cdots (3k-1)} \right] + a_1 \left[ x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k+1}}{3 \cdot 6 \cdot 9 \cdots (3k)} \right] + a_2 \left[ x^2 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k+2}}{4 \cdot 7 \cdot 10 \cdots (3k+1)} \right].$$

We know from theory, e.g., ratio test, that these series converge at least for  $|x| < 1$ .

A nonhomogeneous equation can be handled in much the same way, if the nonhomogeneous part has a power series expansion. In this case the coefficients have to be equated in two power series.

**General Remark:** From now on we restrict to differential equations of order 2; however the whole method can be extended to the case of higher orders.

## 3.2 Solutions near Regular Singular Points

In this section we deal with the solution of linear equations near regular singular points. In the case of solutions near a regular singular point we have to look at the indicial equation of the differential equation. The roots of this special equation give us the idea how to proceed in the method. Now we want to deduce this indicial equation for our problem.

Suppose  $x = 0$  is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

where  $p$  and  $q$  are rational functions in  $x$ . The denominator of  $p(x)$  has no factor  $x$  to a power higher than one. Therefore we can write  $p(x)$  in terms of a rational function  $r(x) \in \mathbb{C}(x)$

$$p(x) = \frac{r(x)}{x},$$

with  $r(0) \in \mathbb{C}$ . Consequently,  $r(x)$  has a power series expansion about  $x = 0$  of the form

$$p(x) = \frac{p_0}{x} + p_1 + p_2x + p_3x^2 + \cdots .$$

With the same argument we get an expansion about  $x = 0$  for  $q(x)$  of the form

$$q(x) = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3x + q_4x^2 + \dots,$$

valid in some region surrounding the origin.

This motivates the following ansatz for a power series solution to our equation:

$$y = \sum_{n=0}^{\infty} a_n x^{n+c} = a_0 x^c + a_1 x^{c+1} + a_2 x^{c+2} + \dots \quad (3.6)$$

with appropriate chosen  $c$  and coefficients  $a_n \in \mathbb{C}$ . The series for  $y$ ,  $p$  and  $q$  are inserted into our differential equation  $y'' + p(x)y' + q(x)y = 0$  and we only consider the first few terms.

$$\begin{aligned} & c(c-1)a_0x^{c-2} + (c+1)ca_1x^{c-1} + (c+2)(c+1)a_2x^c + \dots \\ & + \left[ \frac{p_0}{x} + p_1 + p_2x + \dots \right] [ca_0x^{c-1} + (c+1)a_1x^c + (c+2)a_2x^{c+1} + \dots] \\ & + \left[ \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3x + \dots \right] [a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \dots] = 0 \end{aligned}$$

multiplied out gives us

$$\begin{aligned} & c(c-1)a_0x^{c-2} + (c+1)ca_1x^{c-1} + (c+2)(c+1)a_2x^c + \dots \\ & + p_0ca_0x^{c-2} + [p_0(c+1)a_1 + p_1ca_0]x^{c-1} + \dots \\ & + q_0a_0x^{c-2} + [q_0a_1 + q_1a_0]x^{c-1} + \dots = 0. \end{aligned}$$

To derive the indicial equation, we set the coefficient of  $x^{c-2}$  to zero and obtain

$$[c(c-1) + p_0c + q_0]a_0 = 0.$$

We assume  $a_0 \neq 0$ , because  $a_0$  is the coefficient of the lowest power of  $x$  in the solution. Dividing by  $a_0$  delivers the indicial equation at  $x = 0$ :

$$c(c-1) + p_0c + q_0 = 0, \quad (3.7)$$

where  $p_0$  and  $q_0$  are known constants. Since the equation (3.7) is a quadratic equation, it gives us two roots  $c = c_1$  and  $c = c_2$ . We call the 'larger' root  $c_1$ , if we have  $c_1 \geq c_2$  in case the roots are real; otherwise we call  $c_1$  the 'larger' root if  $Re(c_1) \geq Re(c_2)$ .

We call the difference of the two roots  $c = c_1 - c_2$ . The further steps of the technique depend on  $c$ . If the difference  $c$  is not integral, we get a solution for each of the two values  $c_1$  and  $c_2$  of the form (3.6), where  $a_0$  should be arbitrary.

On the contrary, if the difference  $c$  is integral, a logarithmic term may be involved in the solution.

**Proposition 4 (Form of the solution near a regular singular point)** <sup>2</sup>

Let  $x = 0$  be a regular singular point of the equation

$$y'' + p(x)y' + q(x)y = 0. \quad (3.8)$$

Let  $c_1, c_2 \in \mathbb{C}$  be roots of the indicial equation (3.7).

It can be proved (see Theorem 10 and 11 in Section 5.2) that equation (3.8) always has a general solution either of the form

$$y = A \sum_{n=0}^{\infty} a_n x^{n+c_1} + B \sum_{n=0}^{\infty} b_n x^{n+c_2}$$

or of the form

$$y = (A + B \ln x) \sum_{n=0}^{\infty} a_n x^{n+c_1} + B \sum_{n=0}^{\infty} b_n x^{n+c_2}$$

in which  $A$  and  $B$  are arbitrary constants. Furthermore, it can be proved that the infinite series which occur in the above forms of solution converge in at least the annular region bounded by two circles centered at  $x = 0$ , one of arbitrarily small radius, the other extending to the singular point (of the equation) nearest  $x = 0$ .

We have to consider the different cases for the difference  $c$  of the roots  $c_1$  and  $c_2$  to get an idea which solution is possible or rather which is the correct one for our problem. Step by step we examine the cases  $c$  is nonintegral,  $c$  is zero and  $c$  is a positive integer. Again we explain the method by showing an example.

### 3.2.1 Difference $c$ of roots is nonintegral

We consider the differential equation (see [20], Section 107, (1))

$$2xy''(x) + (1+x)y'(x) - 2y(x) = 0 \quad (3.9)$$

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<sup>2</sup>[20] E.D. Rainville, S. 317, 1964.

with the only regular singular point  $x = 0$ . After the transformation

$$y''(x) + \frac{x+1}{2x}y'(x) - \frac{1}{2x}y = 0,$$

we have that

$$p(x) = \frac{x+1}{2x} = \frac{1/2}{x} + \frac{1}{2}$$

and

$$q(x) = -\frac{1}{2x} = \frac{0}{x^2} + \frac{-1/2}{x}.$$

Hence the indicial equation (3.7) takes on the form

$$c(c-1) + \frac{1}{2}c + 0 = 0,$$

or equivalently,

$$c^2 - \frac{1}{2}c = 0.$$

Let us assume that there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+c},$$

which we directly plug into our equation (3.9) and derive

$$\sum_{n=0}^{\infty} 2(n+c)(n+c-1)a_n x^{n+c-1} + \sum_{n=0}^{\infty} (n+c)a_n x^{n+c-1} + \sum_{n=0}^{\infty} (n+c)a_n x^{n+c} - 2 \sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

In the next step we collect terms with the same exponent of  $x$  and get

$$\sum_{n=0}^{\infty} (n+c)(2n+2c-1)a_n x^{n+c-1} + \sum_{n=0}^{\infty} (n+c-2)a_n x^{n+c} = 0.$$

We want to derive a recurrence relation for  $a_n$ . Hence we shift the index from  $n$  to  $n-1$  in the second sum to get the same exponent of  $x$  in both sums:

$$\sum_{n=0}^{\infty} (n+c)(2n+2c-1)a_n x^{n+c-1} + \sum_{n=1}^{\infty} (n+c-3)a_{n-1} x^{n+c-1} = 0.$$

Since the total coefficient of each power of  $x$  must vanish, we derive for the determination of  $c$  and the  $a_n$ 's the following equations

$$\begin{aligned} n = 0 : & \quad c(2c-1)a_0 = 0, \\ n \geq 1 : & \quad (n+c)(2n+2c-1)a_n + (n+c-3)a_{n-1} = 0. \end{aligned}$$



Without loss of generality we assume  $a_0 \neq 0$ . We note that the case  $n = 0$  delivers the indicial equation

$$c(2c - 1) = 0,$$

with the two roots  $c_1 = \frac{1}{2}$  and  $c_2 = 0$ . The difference of the roots is  $c = c_1 - c_2 = \frac{1}{2}$ , which is nonintegral. In this case the Frobenius method always gives (see (5.9) and (5.10) in Chapter 5) the two linearly independent solutions

$$y = \sum_{n=0}^{\infty} a_n x^{n+c_1} \quad \text{and} \quad y = \sum_{n=0}^{\infty} a_n x^{n+c_2}.$$

To determine the recurrence relation we first substitute  $c_1 = \frac{1}{2}$  for the value  $c$

$$\begin{aligned} n \geq 1 : \quad & (n + \frac{1}{2})(2n + 1 - 1)a_n + (n + \frac{1}{2} - 3)a_{n-1} = 0, \\ & (n + \frac{1}{2})2na_n + (n - \frac{5}{2})a_{n-1} = 0, \\ n \geq 1 : \quad & a_n = -\frac{(2n - 5)a_{n-1}}{(2n + 1)2n}. \end{aligned}$$

We collect the first terms in a vertical array to get a general formula for  $a_n$ :

$$\begin{aligned} a_0 & \text{ is arbitrary,} \\ a_1 & = -\frac{(-3)a_0}{2 \cdot 3}, \\ a_2 & = -\frac{(-1)a_1}{4 \cdot 5}, \\ a_3 & = -\frac{(+1)a_2}{6 \cdot 7}, \\ & \vdots \\ a_n & = -\frac{(2n - 5)a_{n-1}}{2n(2n + 1)}. \end{aligned}$$

Iteration brings for  $n \geq 1$ :

$$\begin{aligned} a_n & = (-1)^n \frac{(-3)(-1)(+1)(+3) \cdots (2n - 5)a_0}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots 2n(2n + 1)} \\ & = (-1)^n \frac{3a_0}{2^n n! (2n - 3)(2n - 1)(2n + 1)}. \end{aligned}$$

We use  $a_0 = 1$  to write the particular solution  $y_1$  corresponding to  $c_1 = \frac{1}{2}$

$$y_1 = x^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{(-1)^n 3x^{n+\frac{1}{2}}}{2^n n! (2n - 3)(2n - 1)(2n + 1)}.$$

In the next step we want to find also a particular solution  $y_2$  corresponding to the second root  $c_2 = 0$ , i. e., we consider the analogous recurrence with  $c_2 = 0$ . To avoid confusion we write  $b_n$  instead of  $a_n$  for this second case

$$n \geq 1 : (n + c)(2n + 2c - 1)b_n + (n + c - 3)b_{n-1} = 0.$$

Again we substitute  $c_2 = 0$  for the value  $c$  and derive

$$n(2n - 1)b_n + (n - 3)b_{n-1} = 0.$$

We obtain the following recurrence relation

$$b_n = -\frac{(n - 3)b_{n-1}}{n(2n - 1)}.$$

As usual we use a vertical array for the coefficients:

$$\begin{aligned} b_0 & \text{ is arbitrary,} \\ b_1 & = -\frac{(-2)b_0}{1 \cdot 1}, \\ b_2 & = -\frac{(-1)b_1}{2 \cdot 3}, \\ b_3 & = -\frac{0 \cdot b_2}{3 \cdot 5}, \\ & \vdots \\ b_n & = 0 \text{ for } n \geq 3. \end{aligned}$$

With the setting  $b_0 = 1$  we calculate the only remaining coefficients  $b_1 = 2$  and  $b_2 = \frac{1}{3}$ . Therefore we found the second particular solution  $y_2$ , corresponding to the smaller root  $c_2 = 0$ , to be

$$y_2 = 1 + 2x + \frac{1}{3}x^2.$$

Since  $x = 0$  is the only singular point from  $\mathbb{C}$ , the two solutions  $y_1$  and  $y_2$  are valid at least for all  $x \in \mathbb{C}^*(= \mathbb{C} \setminus \{0\})$ .

The two solutions  $y_1$  and  $y_2$  are linearly independent because of the following observation:

$x^c$ , where  $c \notin \mathbb{Z}$ , is not analytic in an open disc around 0.

We reformulate the solution  $y_1$ :

$$\begin{aligned} y_1 &= x^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{(-1)^n 3x^{n+\frac{1}{2}}}{2^n n! (2n-3)(2n-1)(2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+\frac{1}{2}}}{2^n n! (2n-3)(2n-1)(2n+1)} \\ &= x^{\frac{1}{2}} \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^n n! (2n-3)(2n-1)(2n+1)}}_{\text{analytic}}. \end{aligned}$$

$x^{\frac{1}{2}}$  is not analytic in an open disc around 0, because it can be represented as

$$x^{\frac{1}{2}} = e^{\frac{1}{2} \cdot \ln(x)},$$

and the function  $\ln(x)$  is not analytic.

The solution

$$y_2 = 1 + 2x + \frac{1}{3}x^2$$

is analytic for all  $x$ .

To test for linear independence we set up the following equation:

$$\alpha \cdot y_1(x) = \beta \cdot y_2(x),$$

where  $\alpha, \beta \in \mathbb{C}$ . The equation is only fulfilled for the case  $\alpha = 0$  and  $\beta = 0$ .

We cannot get equality of something not-analytic and something analytic by multiplication of a complex number. Hence  $y_1$  and  $y_2$  are linearly independent.

The general solution can be written as

$$y = Ay_1 + By_2,$$

with arbitrary constants  $A$  and  $B$ .

### 3.2.2 Difference $c$ of roots is zero - equal roots

We cannot obtain two linearly independent solutions, if the indicial equation has equal roots. In this section we present a method to get two linearly independent solutions in this case.

Consider the problem of solving the differential equation (see [20], Section 109, Exercise 1)

$$x^2 y''(x) - x(x+1)y'(x) + y(x) = 0 \quad (3.10)$$

about the regular singular point  $x = 0$ . We can set up the indicial equation as developed in the theory above by looking at  $p(x)$  and  $q(x)$  in

$$y''(x) + \underbrace{\frac{-x(x+1)}{x^2}}_{p(x)} y'(x) + \underbrace{\frac{1}{x^2}}_{q(x)} y(x) = 0.$$

From  $p(x) = \frac{-x(x+1)}{x^2} = \frac{-x-x^2}{x^2} = -\frac{1}{x} - 1$  and  $q(x) = \frac{1}{x^2}$  we derive  $p_0 = -1$  and  $q_0 = 1$ .

The indicial equation

$$c(c-1) + p_0 c + q_0 = 0$$

becomes

$$\begin{aligned} c(c-1) - c + 1 &= 0, \\ c^2 - 2c + 1 &= 0, \end{aligned}$$

with the roots  $c_1 = c_2 = 1$ . Substituting  $c_1 (= c_2)$  for  $c$  as before in

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c} \quad (3.11)$$

yields only one solution. From now on we call the left member of our differential equation  $L(y)$ , i. e.,

$$L(y) := x^2 y''(x) - x(x+1)y'(x) + y(x).$$

The strategy is to put (3.11) into  $L(y)$  and try to come as close as possible to make  $L(y)$  zero without choosing  $c$ .

$$\begin{aligned} L(y) &= x^2 \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} - (x^2+x) \sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} \\ &= \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c} - \sum_{n=0}^{\infty} a_n (n+c) x^{n+c} - \sum_{n=0}^{\infty} a_n (n+c) x^{n+c+1} + \sum_{n=0}^{\infty} a_n x^{n+c} \\ &= \sum_{n=0}^{\infty} a_n [(n+c)(n+c-1) - (n+c) + 1] x^{n+c} - \sum_{n=0}^{\infty} a_n (n+c) x^{n+c+1} \end{aligned}$$

With the index shift  $n$  to  $n - 1$  in the second sum we attain the same exponent of  $x$  in both sums

$$L(y) = \sum_{n=0}^{\infty} a_n [(n+c)(n+c-1) - (n+c) + 1] x^{n+c} \quad (3.12)$$

$$- \sum_{n=1}^{\infty} a_{n-1} (n+c-1) x^{n+c}. \quad (3.13)$$

We normally derive the indicial equation, if the coefficient of  $x^{n+c}$ , where  $n = 0$ , is set to 0. Now we leave  $c$  as a parameter and try to make every term except the  $n = 0$ -term in  $L(y)$  zero. More precisely, we set the coefficients to zero for all  $n \geq 1$ :

$$[(n+c)(n+c-1) - (n+c) + 1]a_n - (n+c-1)a_{n-1} = 0.$$

By simplifications we derive a recurrence relation for our coefficient  $a_n$ :

$$a_n = \frac{(n+c-1)a_{n-1}}{[(n+c)(n+c-1) - (n+c-1)]} = \frac{a_{n-1}}{n+c-1}.$$

We use a vertical array to list the coefficients iterated back to  $a_0$ :

$$\begin{aligned} a_0 & \text{ is arbitrary,} \\ a_1 & = \frac{a_0}{c}, \\ a_2 & = \frac{a_1}{c+1} = \frac{a_0}{c(c+1)}, \\ a_3 & = \frac{a_2}{c+2} = \frac{a_0}{c(c+1)(c+2)}, \\ & \vdots \\ a_n & = \frac{a_{n-1}}{n+c-1} = \frac{a_0}{c(c+1)(c+2)\dots(c+n-1)} \end{aligned}$$

We choose  $a_0 = 1$  and write the  $n$ -th coefficient in reliance only on  $c$ :

$$a_n(c) = \frac{1}{c(c+1)(c+2)\dots(c+n-1)}.$$

This way we computed an approximation to a solution that is now depending on  $c$  and  $x$ ; namely,

$$y(x, c) = x^c + \sum_{n=1}^{\infty} a_n(c) x^{n+c},$$

in which for  $n \geq 1$ :

$$a_n(c) = \frac{1}{c(c+1)(c+2)\dots(c+n-1)}.$$

By construction,  $y(x, c)$  has the property that  $L(y(x, c))$  is just the summand for  $n = 0$  in the sum (3.12), i. e.,

$$L[y(x, c)] = [c(c-1) - c + 1]a_0x^c = (c-1)^2a_0x^c.$$

The choice  $c = 1$  makes the right member of  $L[y(x, c)]$  zero, hence  $c = 1$  yields a solution. The parameter  $c$  occurs with the power two in  $L[y(x, c)]$ . We know from elementary calculus, that the derivation of a function with an exponent contains this exponent decremented by one. Differentiating  $L[y(x, c)]$  with respect to  $c$  yields

$$\frac{\partial L[y(x, c)]}{\partial c} = 2(c-1)a_0x^c + (c-1)^2a_0x^c \ln x.$$

If we take in account that we set  $a_0 = 1$  we obtain

$$\frac{\partial L[y(x, c)]}{\partial c} = 2(c-1)x^c + (c-1)^2x^c \ln x.$$

The choice  $c = 1$  makes the right member of  $\frac{\partial L[y(x, c)]}{\partial c}$  zero. But the crucial point now is that

$$\frac{\partial L[y(x, c)]}{\partial c} = L\left(\frac{\partial y(x, c)}{\partial c}\right). \quad (3.14)$$

Hence two solutions of  $L(y) = 0$  are

$$y_1 = y(x, c) |_{c=1} = y(x, 1) = x + \sum_{n=1}^{\infty} a_n(1)x^{n+1},$$

and

$$\begin{aligned} y_2 &= \frac{\partial y(x, c)}{\partial c} |_{c=1} = x^c \ln x + \sum_{n=1}^{\infty} a_n(c)x^{n+c} \ln x + \sum_{n=1}^{\infty} a'_n(c)x^{n+c} |_{c=1} \\ &= y_1 \ln x + \sum_{n=1}^{\infty} a'_n(1)x^{n+1}. \end{aligned}$$

**Note:** Essentially one needs to prove that (3.14), i. e., that interchanging the order of differentiation (like in  $D_c D_y y(x, c) = D_y D_c y(x, c)$ ) is indeed allowed.

But we omit this step, since we use this rule only for finding (3.15) and (3.16) below. That they are indeed two linearly independent solutions, can be verified in a direct fashion.

Before we can calculate the solutions (3.15) and (3.16), we differentiate  $a_n(c)$  by help of the formula (2.3) and get

$$a'_n(c) = a(n) \cdot \left( \frac{1}{c} + \frac{1}{c+1} + \frac{1}{c+2} + \cdots + \frac{1}{c+n-1} \right).$$

In the following calculation we need the commonly used expression  $H_n$  for a partial sum of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^n \frac{1}{k} = H_n.$$

We need to figure out  $a_n(1)$  and  $a'_n(1)$

$$\begin{aligned} a_n(1) &= \frac{1}{\underbrace{1 \cdot 2 \cdot 3 \cdots n}_{n!}} \\ a'_n(1) &= a_n(1) \cdot \underbrace{\left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right)}_{:=H_n} \\ &= \frac{1}{n!} H_n \end{aligned}$$

We plug the calculated coefficients into our solutions  $y_1$  and  $y_2$ , and obtain after some simplifications that

$$y_1 = x + \sum_{n=1}^{\infty} \frac{1}{n!} x^{n+1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1} = x e^x, \quad (3.15)$$

$$y_2 = x e^x \ln x + \sum_{n=1}^{\infty} \frac{1}{n!} H_n x^{n+1}. \quad (3.16)$$

With the desired solutions we can set up the general solution valid for all finite  $x \in \mathbb{C}^*$ :

$$y = A y_1 + B y_2,$$

with  $A$  and  $B$  arbitrary constants.  $y_1$  and  $y_2$  are linearly independent because of the presence of  $\ln(x)$  in  $y_2$ .  $\ln(x)$  is not analytic in an open disc around 0, therefore

$y_2$  is not analytic.  $y_1$  is analytic for all  $x$ . To test for linear independence we set up the following equation:

$$\alpha \cdot y_1(x) = \beta \cdot y_2(x),$$

where  $\alpha, \beta \in \mathbb{C}$ . The equation is only fulfilled for the case  $\alpha = 0$  and  $\beta = 0$ . We cannot get equality of something not-analytic and something analytic by multiplication of a complex number. Hence  $y_1$  and  $y_2$  are linearly independent.

Summarizing we can post the following statement:

If the indicial equation has two equal roots  $c_1 = c_2$ , then the two linearly independent solutions will always have the form

$$y_1 = x^{c_1} + \sum_{n=1}^{\infty} a_n x^{n+c_1},$$

and

$$y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+c_1},$$

where  $c_1$ ,  $a_n$  and  $b_n$  are dependent upon the coefficients in the particular equation being solved.

### 3.2.3 Difference $c$ of roots is a positive integer - nonlogarithmic case

We consider the differential equation (see [20], Section 110, (1))

$$xy''(x) - (4+x)y'(x) + 2y(x) = 0. \quad (3.17)$$

Again we denote the left side as  $L(y)$  and plug in our ansatz for the general solution  $y = \sum_{n=0}^{\infty} a_n x^{n+c}$  and derive

$$L(y) = \sum_{n=0}^{\infty} [(n+c)(n+c-1) - 4(n+c)] a_n x^{n+c-1} - \sum_{n=0}^{\infty} (n+c-2) a_n x^{n+c}.$$



With an index shift in the second sum from  $n$  to  $n - 1$  we get the same exponent of  $x$

$$L(y) = \sum_{n=0}^{\infty} (n+c)(n+c-5)a_n x^{n+c-1} - \sum_{n=1}^{\infty} (n+c-3)a_{n-1} x^{n+c-1}.$$

We derive the indicial equation, if we take the summand for  $n = 0$  in the sum above. Hence, we get with  $a_0 \neq 0$

$$c(c-5) = 0.$$

We get the two roots  $c_1 = 5$  and  $c_2 = 0$ , so the difference  $c = c_1 - c_2 = 5$ . Consequently, we will get two power series solutions, one starting with a  $x^0$ -term and the other starting with a  $x^5$ -term. Or, in other words, if we first use the larger root  $c_1 = 5$ , or rather use the ansatz series

$$y = \sum_{n=0}^{\infty} a_n x^{n+5}$$

we cannot get two solutions, because the  $x^0$ -term would never enter our solution. But if we use the smaller root  $c_2 = 0$ , our solution ansatz has the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and the  $x^5$ -term appears too. Plugging into the left member  $L(y)$  gives

$$L(y) = \sum_{n=0}^{\infty} n(n-5)a_n x^{n-1} - \sum_{n=1}^{\infty} (n-3)a_{n-1} x^{n-1}.$$

To get  $L(y) = 0$  each coefficient in the series has to vanish

$$\begin{aligned} n = 0 : & \quad 0 \cdot a_0 = 0, \\ n \geq 1 : & \quad n(n-5)a_n - (n-3)a_{n-1} = 0. \end{aligned}$$

The initial value  $a_0$  can be chosen arbitrarily and since we cannot divide by  $(n-5)$  until  $n > 5$ , we write the relations separately in a vertical array:

$$\begin{aligned}
n = 0 : & \quad a_0 \text{ is arbitrary,} \\
n = 1 : & \quad -4a_1 + 2a_0 = 0, \\
n = 2 : & \quad -6a_2 + a_1 = 0, \\
n = 3 : & \quad -6a_3 + 0 \cdot a_2 = 0, \\
n = 4 : & \quad -4a_4 - a_3 = 0, \\
n = 5 : & \quad 0 \cdot a_5 - 2a_4 = 0, \\
n \geq 6 : & \quad a_n = \frac{(n-3)a_{n-1}}{n(n-5)}.
\end{aligned}$$

From these relations we can conclude that:

$$\begin{aligned}
a_1 &= \frac{1}{2}a_0, \\
a_2 &= \frac{1}{6}a_1 = \frac{1}{12}a_0, \\
a_3 &= 0, \\
a_4 &= 0, \\
0 \cdot a_5 &= 0.
\end{aligned}$$

Since  $a_5$  again can be chosen arbitrarily, each  $a_n$ , where  $n > 5$ , will be calculated with  $a_5$  as:

$$\begin{aligned}
a_6 &= \frac{3a_5}{6 \cdot 1}, \\
a_7 &= \frac{4a_6}{7 \cdot 2}, \\
&\vdots \\
a_n &= \frac{(n-3)a_{n-1}}{n(n-5)}.
\end{aligned}$$

By iterating back we derive a general formula for  $a_n$  with  $a_5$  arbitrary:

$$a_n = \frac{3 \cdot 4 \cdot 5 \cdots (n-3)a_5}{(6 \cdot 7 \cdot 8 \cdots n)(n-5)!} = \frac{3 \cdot 4 \cdot 5a_5}{(n-2)(n-1)n(n-5)!}.$$

Summarizing, the general solution is

$$y = a_0\left(1 + \frac{1}{2}x + \frac{1}{12}x^2\right) + a_5 \left[ x^5 + \sum_{n=6}^{\infty} \frac{60x^n}{(n-2)(n-1)n(n-5)!} \right], \quad (3.18)$$

with  $a_0$  and  $a_5$  arbitrary.

Summarizing if the difference  $c$  is a positive integer, we try a series ansatz of the form  $y = \sum_{n=0}^{\infty} a_n x^{n+c}$  using the smaller root  $c = c_2$ . If  $a_0$  and  $a_c$  both can be chosen arbitrarily, we can go straight forward and obtain the general solution like above. Otherwise we get a relation for  $a_c$  that is impossible, while our usual assumption  $a_0 \neq 0$  stays valid. In this case the general solution will involve a logarithmic term as in the case of equal roots, described in Section 3.2.2. This case will be treated now.

### 3.2.4 Difference $c$ of roots is a positive integer - logarithmic case

For the differential equation (see [20], Section 110, (4), and Section 111, (1))

$$x^2 y''(x) + x(1-x)y'(x) - (1+3x)y(x) = 0. \quad (3.19)$$

it will turn out that  $a_0$  and  $a_c$  both cannot be chosen arbitrarily. Plugging in our ansatz for the general solution  $y = \sum_{n=0}^{\infty} a_n x^{n+c}$  into the left hand side denoted by  $L(y)$ , and executing the usual simplifications and index shifts leads to

$$L(y) = \sum_{n=0}^{\infty} (n+c+1)(n+c-1)a_n x^{n+c} - \sum_{n=1}^{\infty} (n+c+2)a_{n-1} x^{n+c}.$$

The indicial equation  $(c+1)(c-1) = 0$  gives us the two roots  $c_1 = 1$  and  $c_2 = -1$ . The difference  $c = c_1 - c_2 = 2$  is a positive integer. Using  $c = -1$  we find and write the relations

$$n \geq 1 : \quad n(n-2)a_n - (n+1)a_{n-1} = 0,$$

with  $a_0$  kept arbitrarily, in a vertical array:

$$\begin{aligned} n = 1 : & \quad -a_1 - 2a_0 = 0, \\ n = 2 : & \quad 0 \cdot a_2 - 3a_1 = 0, \\ n \geq 3 : & \quad a_n = \frac{(n+1)a_{n-1}}{n(n-2)}. \end{aligned}$$

These relations get simplified to

$$\begin{aligned} a_1 &= -2a_0, \\ 0 \cdot a_2 &= 3a_1 = -6a_0. \end{aligned}$$

These relations are only satisfied if we choose  $a_0 = 0$ , but then  $a_2$  will be the only arbitrary constant and we can only derive one solution, which coincides for the solution corresponding to the larger root  $c_1 = 1$ . This is an instance where a logarithmic term will enter the solution. The further proceeding is exactly the same procedure like in the Section 3.2.2, where the indicial equation had equal roots. Only one step is different than before, namely the choice for  $a_0$ .

As in Section 3.2.2 we want to get  $L(y) = 0$  without choosing  $c$ . For  $n \geq 1$  we have:

$$a_n = \frac{(n+c+2)a_{n-1}}{(n+c+1)(n+c-1)},$$

which can be iterated further to derive:

$$\begin{aligned} a_n &= \frac{(c+3)(c+4) \cdots (c+n+2)a_0}{(c+2)(c+3)(c+4) \cdots (c+n+1) \cdot c(c+1) \cdots (c+n-1)} \\ &= \frac{(c+n+2)a_0}{(c+2) \cdot c(c+1) \cdots (c+n-1)}. \end{aligned}$$

Consequently, for

$$y = a_0 x^c + \sum_{n=1}^{\infty} \frac{(c+n+2)a_0 x^{n+c}}{(c+2) \cdot c(c+1)(c+2) \cdots (c+n-1)}$$

we have that

$$L(y) = (c+1)(c-1)a_0 x^c.$$

We have seen that for the choice  $c = 1$  we get only one solution coinciding with that for  $c = -1$ . However, for the choice  $c = -1$  we in principle would be able - by following the same strategy as in Section 3.2.2 - to derive two solutions by using  $y(x, c)$  and  $\frac{\partial y(x, c)}{\partial c}$ , see Section 3.2.2 for more details.

But if we choose  $c = -1$ , a problem arises because of the term  $(c+1)$  in the denominator. To eliminate this term and therefore the troubles, we set  $a_0 = (c+1)$  and derive:

$$\begin{aligned} y(x, c) &= (c+1)x^c + \sum_{n=1}^{\infty} \frac{(c+1)(c+n+2)x^{n+c}}{(c+2) \cdot c(c+1)(c+2) \cdots (c+n-1)} \\ &= (c+1)x^c + \sum_{n=1}^{\infty} \frac{(c+n+2)x^{n+c}}{(c+2) \cdot c(c+2) \cdots (c+n-1)}, \end{aligned}$$

with the corresponding

$$L(y(x, c)) = (c + 1)^2(c - 1)x^c.$$

Then, as in Section 3.2.2 we obtain two linearly independent solutions:

$$\begin{aligned} y_1 &= y(x, c) |_{c=-1} = y(x, -1), \\ y_2 &= \left( \frac{\partial y(x, c)}{\partial c} \right) |_{c=-1}. \end{aligned}$$

The example in Section 3.19 shows more details.

In general the essential point for this case is to choose  $a_0 = (c - c_2)$ , where  $c_2$  is the smaller root of the indicial equation.

### 3.3 Solutions near Irregular Singular Points

This chapter is a brief introduction to the topic of solutions near an irregular singular point. The solutions are not obtained in a direct way like above, but even though we will find a solution for the specific differential equation (see [20], Section 126, (1))

$$x^2 y''(x) + (3x - 1)y'(x) + y = 0, \quad (3.20)$$

which has a irregular singular point at the origin  $x = 0$ . As above we call the left side of (3.20)  $L(y)$ . The standard proceeding, i. e., to plug the ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}$$

into the differential equation (3.20), leads to

$$\begin{aligned} L(y) &= \sum_{n=0}^{\infty} (n+c)(n+c-1)a_n x^{n+c} + (3x-1) \sum_{n=0}^{\infty} (n+c)a_n x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} \\ &= \sum_{n=0}^{\infty} a_n x^{n+c} \underbrace{((n+c)(n+c-1) + 3(n+c) + 1)}_{\substack{=n^2+2nc+c^2+2n+2c+1 \\ =(n+c+1)^2}} - \sum_{n=0}^{\infty} a_n x^{n+c-1} \end{aligned}$$

$$\begin{aligned}
L(y) &= \sum_{n=0}^{\infty} (n+c+1)^2 a_n x^{n+c} - \sum_{n=0}^{\infty} (n+c) a_n x^{n+c-1} \\
&= \sum_{n=1}^{\infty} (n+c)^2 a_n x^{n+c-1} - \sum_{n=0}^{\infty} (n+c) a_n x^{n+c-1}.
\end{aligned}$$

The term for  $n = 0$  delivers the indicial equation

$$c = 0,$$

which gives one root, i. e.,  $c = 0$ .

The indicial equation has degenerated from a quadratic form in the case of a regular singular point to a linear form in the case of an irregular one.

For  $n \geq 1$  we derive with  $c = 0$  the following recurrence relation:

$$n^2 a_{n-1} - n a_n = 0.$$

With a simple transformation we derive

$$a_n = n a_{n-1} = n(n-1) a_{n-2} = n(n-1)(n-2) a_{n-3} \cdots = n! a_0,$$

which gives the tentative solution

$$y_1(x) = 1 + \sum_{n=1}^{\infty} n! x^n,$$

where  $a_0 = 1$ . The ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \cdot \frac{x^{n+1}}{x^n} \right| < 1,$$

together with the side calculation

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1,$$

leads to the following condition for  $x$  for sufficiently large  $n$ :

$$|x| < \frac{1}{n+1}.$$

Hence the series is divergent for  $x \neq 0$ .

Because we only found a divergent solution to (3.20), following [20], we try another approach. We represent  $n!$  in the integral gamma function form:

$$n! = \int_0^{\infty} e^{-t} t^n dt,$$

which can be found, e. g., in [20]. The solution  $y_1$  then gets - by formal (!) manipulation - modified to

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} \left( \int_0^{\infty} e^{-t} t^n dt \right) x^n \\ &= \int_0^{\infty} e^{-t} \left( \sum_{n=0}^{\infty} (xt)^n \right) dt \\ &= \int_0^{\infty} \frac{e^{-t}}{1-xt} dt. \end{aligned}$$

Let us assume that  $x \leq 0$ . From  $-xt \geq 0$  we get  $1-xt \geq 1$  and therefore

$$0 \leq \frac{1}{1-xt} \leq 1. \quad (3.21)$$

Because of the observation

$$\left| \int_0^{\infty} \frac{e^{-t}}{1-xt} dt \right| \leq \int_0^{\infty} e^{-t} dt = 1$$

we found out that the integral

$$Y_1(x) := \int_0^{\infty} \frac{e^{-t}}{1-xt} dt$$

is absolutely and uniformly convergent on any closed interval  $\subseteq ]-\infty, 0]$ . The inequality (3.21) leads us also to the same conclusion for

$$Y_1'(x) := \int_0^{\infty} \frac{te^{-t}}{(1-xt)^2} dt$$

because of

$$\left| \int_0^{\infty} \frac{te^{-t}}{(1-xt)^2} dt \right| \leq \int_0^{\infty} te^{-t} dt = 1.$$

A similar result holds for  $Y_1''(x)$ .

Finally we show that  $Y_1(x)$  indeed is a particular solution of (3.20).

First we multiply  $Y_1$  by  $x$  and get

$$xY_1 = \int_0^{\infty} \frac{xe^{-t}}{1-xt} dt.$$

We differentiate the equation with respect to  $x$  and obtain

$$\begin{aligned} xY_1' + Y_1 &= \int_0^\infty \frac{xt e^{-t}}{(1-xt)^2} + \frac{e^{-t}}{1-xt} dt \\ &= \int_0^\infty \frac{(1-xt)e^{-t} + xt e^{-t}}{(1-xt)^2} dt \\ &= \int_0^\infty \frac{e^{-t}}{(1-xt)^2} dt. \end{aligned}$$

Multiplying the equation by  $x$  gives

$$x^2 Y_1' + x Y_1 = \int_0^\infty \frac{x e^{-t}}{(1-xt)^2} dt = \int_0^\infty \underbrace{\frac{x}{(1-xt)^2}}_{=:v'} \cdot \underbrace{e^{-t}}_{=:u} dt.$$

Integration by parts leads to

$$\begin{aligned} x^2 Y_1' + x Y_1 &= \int_0^\infty u \cdot v - [u' \cdot v]_0^\infty \\ &= \int_0^\infty \underbrace{\frac{e^{-t}}{1-xt}}_{=:Y_1} dt + \left[ \frac{e^{-t}}{1-xt} \right]_0^\infty \\ &= Y_1 - 1. \end{aligned}$$

Hence  $Y_1$  satisfies the equation

$$x^2 Y_1' + (x-1)Y_1 = -1,$$

which results after differentiating with respect to  $x$  in

$$\begin{aligned} x^2 Y_1'' + 2x Y_1' + (x-1)Y_1' + Y_1 &= 0 \\ \iff x^2 Y_1'' + (3x-1)Y_1' + Y_1 &= 0. \end{aligned}$$

Summarizing, by rewriting - in a formal way - a divergent series into a well-defined integral representation, we derived a particular solution  $Y_1(x)$ ,  $x \leq 0$ , of (3.20) near an irregular singular point.



## Chapter 4

# Frobenius Method - Examples with Mathematica

In Chapter 3 we described how to derive a solution of a differential equation in form of power series "by hand". Now we want to pass through every instance of the Frobenius Method and solve the differential equations with the assistance of Mathematica. We apply the command `ClearAll["Global' *"]` after each example to delete the values from the previous example.

### 4.1 Solutions near an ordinary point

We want to solve the following differential equation of third order (see (3.4) in Section 3.1)

$$y'''(x) + x^2 y''(x) + 5x y'(x) + 3y(x) = 0 \quad (4.1)$$

near the ordinary point  $x = 0$ . There is a solution

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

where three arbitrary constants pop up, namely,  $a_0$ ,  $a_1$  and  $a_2$ . We try to solve the differential equation with Mathematica's command `DSolve`.

**Example:**

```
In:= dg11 := y'''[x] + x^2 * y''[x] + 5x * y'[x] + 3 * y[x]
```

```
In:= FullSimplify[DSolve[dg11 == 0, y[x], x]]
```

```
Out=
```

$$\left\{ \left\{ y[x] \rightarrow -C[3] - \frac{1}{6} e^{-\frac{x^3}{3}} x (-6C[2] + 2x^2 C[3] \text{ExpIntegralE}\left[\frac{1}{3}, -\frac{x^3}{3}\right] + x C[1] \text{ExpIntegralE}\left[\frac{2}{3}, -\frac{x^3}{3}\right]) \right\} \right\}$$

In spite of using the command *FullSimplify* Mathematica provides a solution, which contains the exponential integral function  $\text{ExpIntegralE}[n, z] = E_n[z]$ . Hence the Frobenius Method for solving differential equations gets more attractive.

We seek for a solution of (4.1) in form of power series. In Section 3.1 we described how to derive a recurrence relation for  $a_n$ . With the help of Mathematica this step gets simplified. For this procedure we use again Mallinger's package *GeneratingFunctions.m*, especially the command  $\text{DE2RE}[dgl == 0, y[x], a[n]]$ , which transforms the differential equation  $dgl$  in  $y[x]$  to a recurrence relation in  $a[n]$ .

### Example:

```
In:= << GeneratingFunctions.m
```

```
In:= dg11 := y'''[x] + x^2 * y''[x] + 5x * y'[x] + 3 * y[x]
```

```
In:= rec = DE2RE[dg11 == 0, y[x], a[n]]
```

```
Out= (1 + n)(3 + n)a[n] + (1 + n)(2 + n)(3 + n)a[3 + n] == 0
```

In this case the command  $\text{RSolve}[rec == 0, a[n], n]$  fails to transform the recurrence relation  $rec$  in  $a[n]$  into a closed form for  $a[n]$  in  $n$ .

Because of the special structure of our recurrence relation we try the command  $\text{RESubsequence}[rec == 0, a[n], 3 * n]$ , which transforms the recurrence relation  $rec$  in  $a[n]$  into a recurrence relation for  $C1[n]$  where  $C1[n] := a[3n]$ ,  $n \geq 0$ . Afterwards the gained recurrence for the subsequence is solved with the command  $\text{RSolve}$  with the choice  $a[0] = 1$ .

```
In:= rec1 = RESubsequence[rec, a[n], 3 * n]
```

```
Out= a[n] + (2 + 3n)a[1 + n] == 0
```

```
In:= C1[n_] := a[n]/.RSolve[{rec1, a[0] == 1}, a[n], n]//FullSimplify
```

```
Out=  $\left\{ \left( -\frac{1}{3} \right)^n \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3} + n\right)} \right\}$ 
```

The same procedure is utilized for the other two subsequences.

We want to obtain a solution in form of three power series with  $3n$ ,  $3n + 1$  and  $3n + 2$  as powers of  $x$ .

```

In:= rec2 = RESubsequence[rec, a[n], 3 * n + 1]
Out= a[n] + 3(1 + n)a[1 + n] == 0

In:= C2[n_] := a[n]/.RSolve[{rec2, a[0] == 1}, a[n], n]//FullSimplify
Out=  $\left\{ \frac{\left(-\frac{1}{3}\right)^n}{\Gamma(n+1)} \right\}$ 

In:= rec3 = RESubsequence[rec, a[n], 3 * n + 2]
Out= a[n] + (4 + 3n)a[1 + n] == 0

In:= C3[n_] := a[n]/.RSolve[{rec3, a[0] == 1}, a[n], n]//FullSimplify
Out=  $\left\{ \left(-\frac{1}{3}\right)^n \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{4}{3}+n\right)} \right\}$ 

```

Because of Definition 2 in Chapter 1,  $C2[n]$  could be represented in the following way:

$$C2[n] := \frac{\left(-\frac{1}{3}\right)^n}{n!}.$$

We replace  $n$  by  $k$  and plug our derived coefficients into our power series solution

$$\begin{aligned}
y = & \left( a[0] + \sum_{k=1}^{\infty} C1[k] * x^{3k} \right) \\
& + \left( a[1] * x + \sum_{k=1}^{\infty} C2[k] * x^{3k+1} \right) \\
& + \left( a[2] * x^2 + \sum_{k=1}^{\infty} C3[k] * x^{3k+2} \right),
\end{aligned}$$

with the arbitrary constants  $a[0]$ ,  $a[1]$  and  $a[2]$ .

## 4.2 Solutions near Regular Singular Points

In the case of solutions near a regular singular point we have a closer look at the indicial equation of the differential equation. The roots of this special equation tell us how to continue the steps of the method. We consider the following cases for the difference of the roots: nonintegral, zero or a positive integer. We start with an example, where the difference of the roots of the indicial equation turns out to be nonintegral.

### 4.2.1 Indicial equation with difference of roots nonintegral

We consider the differential equation (see (3.9) in Section 3.2.1)

$$2xy'' + (x + 1)y' - 2y = 0,$$

with the regular singular point  $x = 0$ . We try to solve the differential equation with Mathematica's command *DSolve*.

**Example:**

```
In:= dg12 := 2x * y''[x] + (x + 1) * y'[x] - 2 * y[x]
```

```
In:= FullSimplify[DSolve[dg12 == 0, y[x], x]]
```

```
Out=
```

$$\left\{ \left\{ y[x] \rightarrow \frac{1}{24} e^{\frac{x}{2}} (2\sqrt{x}(5+x)\mathcal{C}[2] + e^{\frac{x}{2}}(3+x(6+x))(24\mathcal{C}[1] + \sqrt{2\pi}\mathcal{C}[2]\text{Erf}\left[\frac{\sqrt{x}}{\sqrt{2}}\right])) \right\} \right\}$$

In spite of using the command *FullSimplify* Mathematica provides a solution, which contains the error function. We get a better result if we use the Frobenius Method.

Like in the case for solutions near an ordinary point we compute a recurrence relation for the coefficients. As described in Section 3.2.1, we shift  $n$  in the whole recurrence to  $n - 1$ , associated to the index shift and so we derive only contributions from one sum. Afterwards we transform  $n$  to  $n+c$  in the coefficients, but leave the argument of  $a[n]$  unchanged, because of our special ansatz solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}.$$

**Example:**

```
In:= dg12 := 2x * y''[x] + (x + 1) * y'[x] - 2 * y[x]
```

```
In:= rec = DE2RE[dg12 == 0, y[x], a[n]]
```

```
Out= (-2 + n)a[n] + (1 + n)(1 + 2n)a[1 + n] == 0
```

```
In:= rec = rec /. n -> n - 1
```

```
Out= (-3 + n)a[-1 + n] + (1 + 2(-1 + n))na[n] == 0
```

```
In:= rec := (n + c - 3)a[n - 1] + (2n + 2c - 1)(n + c)a[n] == 0
```

We need the recurrence in which  $a[0]$  pops up. For  $n = 0$  the first part  $(n + c - 3)a[n - 1]$  vanishes and we get:

$$(2c - 1)ca[0] = 0.$$

Because  $a[0] \neq 0$ , we divide by  $a[0]$  and derive the indicial equation

$$(2c - 1)c = 0.$$

We proceed with Mathematica to compute the roots of our indicial equation and the difference of the roots.

```
In:= indequ := (2c - 1) * c
In:= roots = c/.Solve[indequ == 0, c]
Out= {0, 1/2}
In:= c1 := roots[[2]], c2 := roots[[1]]
In:= s = c1 - c2
Out= 1/2
```

Since the difference is nonintegral, we will get two linearly independent solutions with each choice for  $c$ :  $c = c_1 = \frac{1}{2}$  and  $c = c_2 = 0$ . In the following we compute a closed form for the coefficient  $a(n)$  for each choice of  $c$  and plug it into the general form for the solution

$$y = x^c + \sum_{n=1}^{\infty} a(n) \cdot x^{n+c},$$

where  $c$  stands for the two choices  $c_1 = 0.5$  and  $c_2 = 0$  from above. We get the following two linearly independent solutions.

```
In:= c := 0.5
In:= coeff1 = a[n]/.RSolve[rec == 0, a[0] == 1, a[n], n]
Out= { 0.282095(-0.5)^n Gamma[-1.5+n] / (Pochhammer[2., -1+n] Pochhammer[2.5, -1+n]) }
In:= sol1 = x^c + Sum[coeff1, {n, 1, Infinity}]
Out= x^0.5 + Sum[ 0.282095(-0.5)^n Gamma[-1.5+n] / (Pochhammer[2., -1+n] Pochhammer[2.5, -1+n]), {n, 1, Infinity} ]

In:= c := 0
In:= coeff2 = a[n]/.RSolve[rec == 0, a[0] == 1, a[n], n]
Out= { 45(-1)^(-2+n) 2^(1-n) sqrt(pi) / (n(2-3n+n^2) Gamma[1/2+n]) }
In:= sol2 = x^c + Sum[coeff2, {n, 1, Infinity}]
```

$$\text{Out}=1 + \sum_{n=1}^{\infty} \left\{ \frac{45(-1)^{-2+n}2^{1-n}\sqrt{\pi}x^n}{n(2-3n+n^2)\Gamma\left[\frac{1}{2}+n\right]} \right\}$$

In spite of the fact that we tried to get a simplified representation with the Mathematica command *FullSimplify*, we did not succeed.

**Note:** *Pochhammer*[ $a, n$ ] gives the Pochhammer symbol  $(a)_n$ , which can be found in Definition 3 in Chapter 1.

The general solution can be written as

$$y = A \cdot \text{sol1} + B \cdot \text{sol2},$$

with arbitrary constants  $A$  and  $B$ .

## 4.2.2 Indicial equation with equal roots

We know from theory that we get two linearly independent solutions also in this case with a variation of the method from the previous instance of a nonintegral difference.

Consider the problem of solving the differential equation (see (3.10) in Section 3.2.2)

$$x^2 y''(x) - x(x+1)y'(x) + y(x) = 0$$

about the regular singular point  $x = 0$ . Again we try to solve the differential equation with Mathematica's command *DSolve*.

**Example:**

```
In:= dg13 := x^2 * y''[x] - x(x + 1) * y'[x] + y[x]
```

```
In:= FullSimplify[DSolve[dg13 == 0, y[x], x]]
```

```
Out=
```

$$\left\{ \left\{ y[x] \rightarrow \frac{2\text{BesselK}[0, 2\sqrt{2}\sqrt{x}]\text{C}[2] - \text{C}[1]\text{HypergeometricOF1Regularized}[1, 2x]}{2x} \right\} \right\}$$

In spite of using the command *FullSimplify* Mathematica provides a solution, which contains the modified Bessel function of the second kind and moreover the regularized confluent hypergeometric function. We get a much better result using the Frobenius Method.

We can set up the indicial equation with Mathematica like in the previous example.

**Example:**

```
In:= dg13 := x^2 * y''[x] - x(x + 1) * y'[x] + y[x]
```

```
In:= rec = DE2RE[dg13 == 0, y[x], a[n]]
```

```
Out= -na[n] + n^2a[1 + n] == 0
```

```
In:= rec = rec /. n -> n - 1
```

```
Out= -(-1 + n)a[-1 + n] + (-1 + n)^2a[n] == 0
```

```
In:= rec := -(n + c - 1)a[-1 + n] + (n + c - 1)^2a[n] == 0
```

We need the recurrence in which  $a[0]$  pops up. For  $n = 0$  the first part  $-(n + c - 1)a[-1 + n]$  vanishes and we get:

$$(c - 1)^2 a[0] = 0.$$

Because  $a[0] \neq 0$ , we divide by  $a[0]$  and derive the indicial equation

$$(c - 1)^2 = 0.$$

We continue to compute the roots of our indicial equation and the difference of the roots.

```
In:= indequ := (c - 1)^2
```

```
In:= roots = c /. Solve[indequ == 0, c]
```

```
Out= {1, 1}
```

```
In:= c1 := roots[[2]], c2 := roots[[1]]
```

```
In:= s = c1 - c2
```

```
Out= 0
```

Since the difference is zero, i. e.  $c = c_1 = c_2$ , we will not get two linearly independent solutions with the ansatz

$$y = x^c + \sum_{n=1}^{\infty} a_n(c) \cdot x^{n+c}.$$

In this case we know from Section 3.2.2 that two linearly independent solutions will always have the form

$$\begin{aligned} sol1 &= x^{c_1} + \sum_{n=1}^{\infty} a_n(c) |_{c=c_1} \cdot x^{n+c_1}, \\ sol2 &= sol1 \cdot \ln x + \sum_{n=1}^{\infty} \frac{\partial a_n(c) |_{c=c_1}}{\partial c} \cdot x^{n+c_1}, \end{aligned}$$

with the root of the indicial equation  $c_1$ . First we compute the solution  $sol1$  with Mathematica.

```
In:= coeff = a[n]/.RSolve[{rec == 0, a[0] == 1}, a[n], n]
```

```
Out= {  $\frac{\Gamma[c]}{\Gamma[c+n]}$  }
```

```
In:= c := 1; coeff
```

```
Out= {  $\frac{1}{\Gamma[1+n]}$  }
```

```
In:= sol1 = x^c + Sum[coeff * x^(n+c), {n, 1, Infinity}]
```

```
Out= x + Sum[ $\frac{x^{1+n}}{\Gamma[1+n]}$ , {n, 1, Infinity}]
```

For the second solution

$$sol2 = sol1 * \ln x + \sum_{n=1}^{\infty} \frac{\partial a_n(c)}{\partial c} \cdot x^{n+c}$$

we need first to calculate  $\frac{\partial a_n(c)}{\partial c}$  and then choose the value for  $c = c_1 = 1$ .

```
In:= ClearAll["Global' * "]
```

```
In:= rec := -(n + c - 1)a[-1 + n] + (n + c - 1)^2 a[n] == 0
```

```
In:= coeff = a[n]/.RSolve[{rec == 0, a[0] == 1}, a[n], n]
```

```
Out= {  $\frac{\Gamma[c]}{\Gamma[c+n]}$  }
```

```
In:= Deriv = D[coeff, c]
```

```
Out= {  $\frac{\Gamma[c] \text{PolyGamma}[0, c]}{\Gamma[c+n]} - \frac{\Gamma[c] \text{PolyGamma}[0, c+n]}{\Gamma[c+n]}$  }
```

```
In:= c := 1
```

```
In:= sol2 = sol1 * Log[x] + Sum[Deriv * x^(n+c), {n, 1, Infinity}]
```

```
Out= sol1 Log[x] + Sum[ $\frac{x^{1+n} \text{HarmonicNumber}[n]}{\Gamma[1+n]}$ , {n, 1, Infinity}]
```



Because of Definition 2 in Chapter 1 the two solutions can be represented in the following way:

$$\begin{aligned} \text{sol1} &= x + \sum_{n=1}^{\infty} \frac{x^{1+n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{1+n}}{n!} = xe^x, \\ \text{sol2} &= xe^x \ln x + \sum_{n=1}^{\infty} \frac{x^{1+n} H_n}{n!}, \end{aligned}$$

according to the results in Section 3.2.2.

Finally we found the two solutions and can set up the general solution valid for all finite  $x \neq 0$ :

$$y = A \cdot \text{sol1} + B \cdot \text{sol2},$$

with  $A$  and  $B$  arbitrary constants.  $\text{sol1}$  and  $\text{sol2}$  are linearly independent because of the presence of  $\ln x$  in  $\text{sol2}$ .

### 4.2.3 Difference $c$ of roots is a positive integer - nonlogarithmic case

We consider the differential equation (see (3.17) in Section 3.2.3)

$$xy''(x) - (4+x)y'(x) + 2y(x) = 0$$

about the regular singular point  $x = 0$ . We try to solve the differential equation with Mathematica's command *DSolve*.

**Example:**

```
In:= dgl4 := x * y''[x] - (4 + x) * y'[x] + 2 * y[x]
```

```
In:= FullSimplify[DSolve[dgl4 == 0, y[x], x]]
```

```
Out=
```

$$\left\{ \left\{ y[x] \rightarrow \frac{\sqrt{x}((12 + x(6 + x))(-iC[1] + C[2]) + e^x(12 + (-6 + x)x)(iC[1] + C[2]))}{\sqrt{\pi}\sqrt{-ix}} \right\} \right\}$$

We get a better result using the Frobenius Method.

We calculate the indicial equation with Mathematica like in the previous examples.

**Example:**

```
In:= dg14 := x * y''[x] - (4 + x) * y'[x] + 2 * y[x]
```

```
In:= rec = DE2RE[dg14 == 0, y[x], a[n]]
```

```
Out= (2 - n)a[n] + (-4 + n)(1 + n)a[1 + n] == 0
```

```
In:= rec = rec /. n -> n - 1
```

```
Out= (3 - n)a[-1 + n] + (-5 + n)na[n] == 0
```

```
In:= rec := (3 - n - c)a[-1 + n] + (-5 + n + c)(n + c)a[n]
```

We need the recurrence in which  $a[0]$  pops up. For  $n = 0$  the first part  $(3 - n - c)a[-1 + n]$  vanishes and we get:

$$(-5 + c)ca[0] = 0.$$

Because  $a[0] \neq 0$ , we divide by  $a[0]$  and derive the indicial equation

$$(c - 5)c = 0.$$

We continue to compute the roots of our indicial equation and the difference of the roots.

```
In:= indequ := (c - 5) * c
```

```
In:= roots = c /. Solve[indequ == 0, c]
```

```
Out= {0, 5}
```

```
In:= c1 := roots[[2]], c2 := roots[[1]]
```

```
In:= s = c1 - c2
```

```
Out= 5
```

We get the two roots  $c1 = 5$  and  $c2 = 0$ , so the difference is  $c = c1 - c2 = 5$ .

We will get two power series solutions, one starting with a  $x^0$ -term and the other starting with a  $x^5$ -term.

We have to single out relations for the first five coefficients

$$\begin{aligned} a_1 &= \frac{1}{2}a_0, \\ a_2 &= \frac{1}{6}a_1 = \frac{1}{12}a_0, \\ a_3 &= 0, \\ a_4 &= 0, \\ 0 \cdot a_5 &= 0 \end{aligned}$$

with  $a_0$  and  $a_5$  arbitrary. Since each  $a_n$ , with  $n > 5$ , will be calculated with  $a_5$  we get a recurrence relation for the coefficients  $a_n$  starting with  $a_5$

$$a_n = \frac{(n-3)a_{n-1}}{n(n-5)}.$$

We transform the recurrence relation to  $(n-3)a[n-1] - n(n-5)a[n] = 0$  and derive the general solution by help of Mathematica.

```
In:=coeff = a[n]/.RSolve[{(n-3)a[n-1] - n(n-5)a[n] == 0, a[0] == 1}, a[n], n]
```

```
Out= { ( (-4+n)(-3+n)C[1] ) / ( 12Gamma[1+n] ) }
```

```
In:=c := 5
```

```
In:=sol1 = x^c + Sum[coeff * x^n, {n, 6, Infinity}]
```

```
Out= x^5 + Sum[ ( (-4+n)(-3+n)x^n ) / ( 12Gamma[1+n] ) , {n, 6, Infinity} ]
```

The general solution is

$$y = a_0\left(1 + \frac{1}{2}x + \frac{1}{12}x^2\right) + a_5 \left[ x^5 + \sum_{n=6}^{\infty} \frac{(-4+n)(-3+n)x^n}{12\Gamma[1+n]} \right],$$

with  $a_0$  arbitrary and  $a_5 = 1$ .

Because of Definition 2 in Chapter 1 the solution can be represented in the following way:

$$y = a_0\left(1 + \frac{1}{2}x + \frac{1}{12}x^2\right) + \left[ x^5 + \sum_{n=6}^{\infty} \frac{x^n}{12(n-2)(n-1)n(n-5)!} \right],$$

according to the results in Section 3.2.3, but with the choice  $a_5 = 1$ .

#### 4.2.4 Difference $c$ of roots is a positive integer - logarithmic case

If we derive a possible relation for  $a_c$ , the general solution will involve a logarithmic term as in the case of equal roots. For the differential equation (see (3.19) in Section 3.2.4)

$$x^2 y''(x) + x(1-x)y'(x) - (1+3x)y(x) = 0.$$

we will detect that  $a_0$  and  $a_c$  are not both arbitrary. Before we start with the usual proceeding of the Frobenius Method, we try to solve the differential equation with

Mathematica's command *DSolve*.

**Example:**

```
In:= dg15 := x2 * y''[x] + x(1 - x) * y'[x] - (1 + 3x) * y[x]
```

```
In:= FullSimplify[DSolve[dg15 == 0, y[x], x]]
```

```
Out=
```

$$\left\{ \left\{ y[x] \rightarrow \frac{(-1 + x(2 + x))C[2] + e^x x^2 (3 + x)(6C[1] + C[2] \text{ExpIntegralEi}[-x])}{6x} \right\} \right\}$$

In spite of using the command *FullSimplify* Mathematica provides a solution, which contains the exponential integral function  $Ei[-x]$ . We try to find a better solution with the Frobenius Method.

**Example:**

```
In:= dg15 := x2 * y''[x] + x(1 - x) * y'[x] - (1 + 3x) * y[x]
```

```
In:= rec = DE2RE[dg15 == 0, y[x], a[n]]
```

```
Out= (-3 - n)a[n] + n(2 + n)a[1 + n] == 0
```

```
In:= rec = rec /. n -> n - 1
```

```
Out= (-2 - n)a[-1 + n] + (-1 + n)(1 + n)a[n] == 0
```

```
In:= rec := (-2 - n - c)a[-1 + n] + (-1 + n + c)(1 + n + c)a[n]
```

We need the recurrence in which  $a[0]$  pops up. For  $n = 0$  the first part  $(-2 - n - c)a[-1 + n]$  vanishes and we get:

$$(-1 + c)(1 + c)a[0] = 0.$$

Because  $a[0] \neq 0$ , we divide by  $a[0]$  and derive the indicial equation

$$(-1 + c)(1 + c) = 0.$$

We continue to compute the roots of our indicial equation and the difference of the roots.

```
In:= indequ := (-1 + c)(1 + c)
```

```
In:= roots = c /. Solve[indequ == 0, c]
```

```
Out= {-1, 1}
```

```
In:= c1 := roots[[2]], c2 := roots[[1]]
```

In:= s = c1 - c2

Out= 2

Inserting the smaller root c1 in

$$rec := (-2 - n - c)a[-1 + n] + (-1 + n + c)(1 + n + c)a[n]$$

gives the relation

$$n \geq 1 : n(n - 2)a_n - (n + 1)a_{n-1} = 0,$$

which results in the relations

$$\begin{aligned} a_1 &= -2a_0, \\ 0 \cdot a_2 &= 3a_1 = -6a_0. \end{aligned}$$

If we choose  $a_0 = 0$ ,  $a_2$  will be the only arbitrary constant and we can only derive one solution corresponding to the larger root  $c_1 = 1$ . Because of this fact a logarithmic term enters the solution. We proceed with the same method like in the case of equal roots. The essential step is the choice for  $a_0$ . We have to choose  $a_0 = (c - c_2)$ , where  $c_2$  is the smaller root of the indicial equation. In our example  $a_0 = (c - (-1)) = (c + 1)$  would be the best choice.

In:= coeff = a[n]/.RSolve[REC == 0, a[0] == (c + 1), a[n], n]

Out=  $\left\{ \frac{(1+c)(2+c+n)\Gamma[c]}{(2+c)\Gamma[c+n]} \right\}$

In:= c := 1; coeff

Out=  $\left\{ \frac{2(3+n)}{3\Gamma[1+n]} \right\}$

In:= sol1 = x<sup>c</sup> +  $\sum_{n=1}^{\infty}$  coeff \* x<sup>n+c</sup>

Out= x +  $\sum_{n=1}^{\infty} \left\{ \frac{2(3+n)x^{n+1}}{3\Gamma[1+n]} \right\}$

For the second solution

$$sol2 = sol1 * \ln x + \sum_{n=1}^{\infty} \frac{\partial a_n(c)}{\partial c} \cdot x^{n+c}$$

we need first to calculate  $\frac{\partial a_n(c)}{\partial c}$  and then choose the value for  $c = c_1 = 1$ .

In:= ClearAll["Global' \*"]

In:= rec := (-2 - n - c)a[-1 + n] + (-1 + n + c)(1 + n + c)a[n] == 0

In:= coeff = a[n]/.RSolve[{rec == 0, a[0] == 1}, a[n], n]

```

Out= { (1+c)(2+c+n)Gamma[c] }
      { (2+c)Gamma[c+n] }
In:= Deriv = D[coeff, c]
Out= { Gamma[c]((2+c)^2+n+(1+c)(2+c)(2+c+n)PolyGamma[0,c]-(1+c)(2+c)(2+c+n)PolyGamma[0,c+n]) }
      { (2+c)^2Gamma[c+n] }
In:= c := 1
In:= Deriv = FullSimplify[Deriv]
In:= sol2 = sol1 * Log[x] + Sum_{n=1}^{\infty} Deriv * x^{n+c}
Out= sol1 Log[x] + Sum_{n=1}^{\infty} { x^{1+n}(9+n-6(3+n)HarmonicNumber[n]) }
      { 9Gamma[1+n] }

```

Because of Definition 2 in Chapter 1 the two solutions can be represented in the following way:

$$\begin{aligned}
 sol1 &= x + \sum_{n=1}^{\infty} \frac{2(3+n)x^{n+1}}{3n!}, \\
 sol2 &= sol1 \ln x + \sum_{n=1}^{\infty} \frac{x^{n+1}(9+n-6(3+n)H_n)}{9n!}.
 \end{aligned}$$

## Chapter 5

# Frobenius Method - Existence Theory

In this chapter we will prove the existence of the solutions of holonomic differential equations, which are obtained by the Frobenius Method. We execute the proofs for linear differential equations of order two, but the analog definitions, proofs and theorems also apply with corresponding modifications to linear differential equations of higher order.

Our description is based on the presentation in [21].

### 5.1 Solutions near an ordinary point

#### Theorem 9 <sup>1</sup>

Let  $p(z)$ ,  $q(z)$  be meromorphic functions defined in a neighborhood of  $z_0 \in \mathbb{C}$ .  
If  $z_0$  is an ordinary point of the equation

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0, \quad (5.1)$$

then there exists a series

$$w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (5.2)$$

- with a nonzero radius of convergence,

---

<sup>1</sup>[21], E. D. Rainville, p. 71, Theorem 16, 1972.

- containing two arbitrary constants  $a_0$  and  $a_1$ ,
- satisfying the differential equation (5.1) within the circle of convergence of the series.

**Proof (Theorem 9):**

Given the linear differential equation of order two (5.1) with the ordinary point  $z_0$ . From this we know that  $p(z)$  and  $q(z)$  are analytic at  $z_0$ . Hence there exist the following expansions of  $p(z)$  and  $q(z)$

$$p(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \text{ with } |z - z_0| < R_1,$$

$$q(z) = \sum_{n=0}^{\infty} d_n (z - z_0)^n \text{ with } |z - z_0| < R_2,$$

where  $R_1$  is the radius of convergence of the series  $p(z)$  and  $R_2$  the radius of convergence of the series  $q(z)$ , respectively.  $R_1$  and  $R_2$  are both greater than zero, and  $b_n, d_n \in \mathbb{C}$ .

**Idea of the proof:**

We prove the existence of the constants  $a_n$  in (5.2), such that this  $w$  solves our differential equation (5.1) and moreover that this series  $w$  has a nonzero radius of convergence. Hence  $w$  at  $z = z_0$  is an analytic solution of (5.1).

**Steps of the proof:**

The proof is split into three single steps. In the first step we derive a rule for computing the coefficients  $a_n$ . In the second step we derive an upper bound for the  $a_n$ . The third step presents the convergence analysis for our series  $w$ .

**1. Step:**

We substitute (5.2) as an ansatz into our differential equation (5.1) and obtain

$$\sum_{n=0}^{\infty} a_n n(n-1)(z - z_0)^{n-2} + p(z) \sum_{n=0}^{\infty} a_n n(z - z_0)^{n-1} + q(z) \sum_{n=0}^{\infty} a_n (z - z_0)^n = 0.$$

Plugging in the series expansions for  $p(z)$  and  $q(z)$  we get

$$\begin{aligned} \sum_{n=0}^{\infty} a_n n(n-1)(z - z_0)^{n-2} &+ \left( \sum_{n=0}^{\infty} b_n (z - z_0)^n \right) \sum_{n=0}^{\infty} a_n n(z - z_0)^{n-1} \\ &+ \left( \sum_{n=0}^{\infty} d_n (z - z_0)^n \right) \sum_{n=0}^{\infty} a_n (z - z_0)^n = 0. \end{aligned}$$



Carrying out the multiplications in the second and third sum, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a_n n(n-1)(z-z_0)^{n-2} &+ \sum_{n=0}^{\infty} \sum_{k=0}^n k a_k b_{n-k} (z-z_0)^{n-1} \\ &+ \sum_{n=0}^{\infty} \sum_{k=0}^n a_k d_{n-k} (z-z_0)^n = 0. \end{aligned}$$

To collect the same powers of  $(z-z_0)$ , we shift indices and derive

$$\begin{aligned} \sum_{n=0}^{\infty} a_n n(n-1)(z-z_0)^{n-2} &+ \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} k a_k b_{n-k-1} (z-z_0)^{n-2} \\ &+ \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} a_k d_{n-k-2} (z-z_0)^{n-2} = 0. \end{aligned}$$

The coefficient of each power has to vanish in order to make the left side of above identically zero:

$$\begin{aligned} n=0 : & \quad 0 \cdot a_0 = 0, \\ n=1 : & \quad 0 \cdot a_0 + 0 \cdot a_0 b_0 = 0, \\ n \geq 2 : & \quad n(n-1)a_n + \sum_{k=0}^{n-1} k a_k b_{n-k-1} + \sum_{k=0}^{n-2} a_k d_{n-k-2} = 0. \end{aligned}$$

For  $n=0$  the second and third sum does not enter, and for  $n=1$  only the third sum does not occur. From the first and the second condition we can conclude that  $a_0$  and  $b_0$  are arbitrary constants. From the third condition we derive a rule for computing the coefficient  $a_n$ :

$$a_n = \frac{1}{n(n-1)} \left( - \sum_{k=0}^{n-1} k a_k b_{n-k-1} - \sum_{k=0}^{n-2} a_k d_{n-k-2} \right) \quad \text{for } n \geq 2. \quad (5.3)$$

## 2. Step:

We want to prove an upper bound for the  $a_n$ . To this end we need the following lemma that is folklore in complex analysis.

### Lemma 5 <sup>2</sup>

*The coefficients  $a_n$  in the Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \text{valid in } |z-z_0| < R,$$

<sup>2</sup>[21], E. D. Rainville, p. 54, 1972.

have the property that for any  $r$  in  $0 < r < R$ , there exists a constant  $M$  such that for all  $n \geq 0$ ,

$$|a_n| < \frac{M}{r^n}.$$

The lemma guarantees the existence of positive constants  $r_1, r_2, M_1, M_2$  such that

$$|b_n| < \frac{M_1}{r_1^n} \quad \text{and} \quad |d_n| < \frac{M_2}{r_2^n}.$$

For simplification we introduce the constants  $r$  and  $M$  by suitable choice such that

$$\begin{aligned} r &< \text{Min}(r_1, r_2, 1), \\ M &> \text{Max}(M_1, M_2, 1, |a_1|, r). \end{aligned}$$

$a_0$  has no impact on the convergence of the series, hence without loss of generality we set  $0 < |a_0| < 1$ . With the new variables we can rewrite the restrictions on  $b_n$  and  $d_n$  as follows:

$$|b_n| < \frac{M}{r^n} \quad \text{and} \quad |d_n| < \frac{M}{r^n},$$

because  $M > M_1$  and  $M > M_2$ , respectively  $r < r_1$  and  $r < r_2$ . Additionally we can increase  $M$  (if needed) such that

$$|a_1| < \frac{M}{r}.$$

We will prove an upper bound for  $|a_n|$  by induction.

**Initial condition ( $n = 1$ ):**

By choice, as explained above, we have the initial conditions

$$\begin{aligned} |a_0| &< \left(\frac{M}{r}\right)^0 = 1, \\ |a_1| &< \left(\frac{M}{r}\right)^1. \end{aligned}$$

**Assumption:**

We assume that  $|a_k| < \left(\frac{M}{r}\right)^k$  holds for all  $k$  with  $0 \leq k < n$ .

**To prove:**

We want to prove that the  $n$ -th coefficient has an upper bound of the following form

$$|a_n| < \left(\frac{M}{r}\right)^n.$$

From (5.3) we derive the following inequality

$$|a_n| \leq \frac{1}{n(n-1)} \left( \sum_{k=0}^{n-1} k |a_k| |b_{n-k-1}| + \sum_{k=0}^{n-2} |a_k| |d_{n-k-2}| \right).$$

Plugging in the assumption for  $a_k$  and additionally the restrictions for  $b_n$  and  $d_n$  results in

$$\begin{aligned} |a_n| &< \frac{1}{n(n-1)} \left( \sum_{k=0}^{n-1} k \left(\frac{M}{r}\right)^k \left(\frac{M}{r^{n-k-1}}\right) + \sum_{k=0}^{n-2} \left(\frac{M}{r}\right)^k \left(\frac{M}{r^{n-k-2}}\right) \right) \\ &= \frac{1}{n(n-1)} \left( \sum_{k=0}^{n-1} k \frac{M^{k+1}}{r^{n-1}} + \sum_{k=0}^{n-2} \frac{M^{k+1}}{r^{n-2}} \right). \end{aligned}$$

In view of  $r < 1$  and therefore  $r^{n-1} > r^n$ , respectively  $M > 1$  and therefore  $M^{k+1} < M^n$ , we get the next estimation

$$\begin{aligned} |a_n| &< \frac{1}{n(n-1)} \left( \sum_{k=0}^{n-1} k \frac{M^n}{r^n} + \sum_{k=0}^{n-2} \frac{M^n}{r^n} \right) \\ &= \frac{1}{n(n-1)} \cdot \frac{M^n}{r^n} \left( \sum_{k=0}^{n-1} k + \sum_{k=0}^{n-2} 1 \right). \end{aligned}$$

We know that  $\sum_{k=0}^n k = \frac{n(n+1)}{2}$  and  $\sum_{k=0}^n 1 = n+1$ . Hence our estimation simplifies to

$$|a_n| < \frac{1}{n(n-1)} \cdot \frac{M^n}{r^n} \left( \frac{n(n-1)}{2} + (n-1) \right),$$

respectively,

$$|a_n| < \frac{M^n}{r^n} \cdot \frac{n+2}{2n}.$$

This implies

$$|a_n| < \left(\frac{M}{r}\right)^n,$$

which is valid for all  $n \geq 0$ .

### 3. Step:

Now we have an upper bound for  $|a_n|$ , hence we try to find an upper bound for the  $n$ -th series member  $|a_n(z - z_0)^n|$  of  $w$ .

We specify a circular neighborhood of  $z_0$  by

$$|z - z_0| \leq \frac{r}{2M}.$$

Consequently the following estimation holds for all  $n \geq 0$ :

$$\begin{aligned} |a_n(z - z_0)^n| &\leq \left(\frac{M}{r}\right)^n \cdot \left(\frac{r}{2M}\right)^n \\ &= \frac{M^n r^n}{r^n (2M)^n} \\ &= \frac{1}{2^n}. \end{aligned}$$

Hence we found a convergent series  $\sum_{n=0}^{\infty} \frac{1}{2^n}$  which dominates our series  $w$ , this means

$$\sum_{n=0}^{\infty} |a_n(z - z_0)^n| \leq \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

From this inequality it follows that our series (5.2) is absolutely and uniformly convergent in the region  $|z - z_0| \leq \frac{r}{2M}$ .

Summarizing, this way we found a solution to the differential equation (5.1), namely  $w = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ , which is analytic at the ordinary point  $z_0$  of (5.1).

□

## 5.2 Solutions near a regular singular point

### Theorem 10 <sup>3</sup>

Let  $p(z)$ ,  $q(z)$  be meromorphic functions defined in a neighborhood of  $z_0 \in \mathbb{C}$ .  
If  $z_0$  is a regular singular point of the equation

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0, \quad (5.4)$$

and if the roots  $c_1$  and  $c_2$  of its indicial equation, derived below as equation (5.7), are arranged such that  $c_1 \geq c_2$ , respectively  $\operatorname{Re}(c_1) \geq \operatorname{Re}(c_2)$ . Then there exists a series

$$w_1(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^{n+c_1}, \quad (5.5)$$

<sup>3</sup>[21], E. D. Rainville, p. 86, Theorem 20, 1972.

- satisfying the differential equation (5.4),
- and being convergent within the region  $0 < |z - z_0| < R$ , where  $R$  is the distance from  $z_0$  to the nearest other singular point of (5.4).

**Proof (Theorem 10):**

Given the linear differential equation of order two (5.4) with the regular singular point  $z_0$ . From this we know that  $p(z)$  and  $q(z)$  are not only analytic at  $z_0$  but also possess the following type of expansions:

$$p(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^{n-1} \text{ with } 0 < |z - z_0| < R_1,$$

$$q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^{n-2} \text{ with } 0 < |z - z_0| < R_2,$$

where  $R_1$  is the radius of convergence of the series  $p(z)$ , and  $R_2$  is the radius of convergence of the series  $q(z)$ .  $R_1$  and  $R_2$  are both greater than zero, and  $p_n, q_n \in \mathbb{C}$ . Because of the definition of regular singular points, we get the specific powers above.

**Idea of the proof:**

We will prove that there exists a solution of (5.4), which is convergent in some region  $0 < |z - z_0| < R$ . We look for a solution

$$w = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+c}$$

of (5.4), which is analytic at  $z_0$  except for the factor  $(z - z_0)^c$  with a special  $c$ , which turns out to be a root of the indicial equation of (5.4).  $a_0 \neq 0$  is an arbitrary constant.

**Steps of the proof:**

As before the proof is split into three single steps.

**1. Step:**

We substitute (5.5) into our differential equation (5.4) and obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(n+c)(n+c-1)(z-z_0)^{n+c-2} &+ \\ p(z) \sum_{n=0}^{\infty} a_n(n+c)(z-z_0)^{n+c-1} &+ \\ q(z) \sum_{n=0}^{\infty} a_n(z-z_0)^{n+c} &= 0. \end{aligned}$$

Plugging in the series expansions for  $p(z)$  and  $q(z)$  we get

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(n+c)(n+c-1)(z-z_0)^{n+c-2} &+ \\ \sum_{n=0}^{\infty} p_n(z-z_0)^{n-1} \sum_{n=0}^{\infty} a_n(n+c)(z-z_0)^{n+c-1} &+ \\ \sum_{n=0}^{\infty} q_n(z-z_0)^{n-2} \sum_{n=0}^{\infty} a_n(z-z_0)^{n+c} &= 0. \end{aligned}$$

Carrying out the multiplications in the second and third sum and thus obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(n+c)(n+c-1)(z-z_0)^{n+c-2} &+ \\ \sum_{n=0}^{\infty} \sum_{k=0}^n (k+c)a_k p_{n-k}(z-z_0)^{n+c-2} &+ \\ \sum_{n=0}^{\infty} \sum_{k=0}^n a_k q_{n-k}(z-z_0)^{n+c-2} &= 0. \end{aligned}$$

To collect the same powers of  $(z-z_0)$ , we rewrite the series as follows:

$$\sum_{n=0}^{\infty} \left[ a_n(n+c)(n+c-1) + \sum_{k=0}^n a_k ((k+c)p_{n-k} + q_{n-k}) \right] (z-z_0)^{n+c-2} = 0.$$

By splitting off the  $n$ -th term, we derive a recurrence relation for computing  $a_n$ :

$$[(n+c)(n+c-1) + (n+c)p_0 + q_0] a_n = - \sum_{k=0}^{n-1} a_k ((k+c)p_{n-k} + q_{n-k}). \quad (5.6)$$

To aim at the indicial equation, we evaluate at  $n=0$  and obtain

$$[c(c-1) + cp_0 + q_0] a_0 = 0.$$

Division by  $a_0 \neq 0$  leads to the quadratic indicial equation

$$c^2 + c(p_0 - 1) + q_0 = 0. \quad (5.7)$$

**Note:** We can assume  $a_0 \neq 0$  without any loss of generality. Because otherwise the problem would just result in a shifted version (i. e., in  $c + N$  instead of  $c$ ) of the indicial equation.

The roots of the indicial equation are called  $c_1$  and  $c_2$ , with  $Re(c_1) \geq Re(c_2)$ . We define the difference of the roots  $s := c_1 - c_2$ . Define

$$F(c) := c(c - 1) + cp_0 + q_0.$$

Because of

$$F(c) = (c - c_1)(c - c_2)$$

we obtain

$$F(n + c_1) = (n + c_1 - c_1)(n + c_1 - c_2) = n(n + s),$$

which is used to derive from (5.6) the equivalent statement

$$n(n + s)a_n = - \sum_{k=0}^{n-1} a_k ((k + c_1)p_{n-k} + q_{n-k}). \quad (5.8)$$

From this recurrence relation we now obtain an upper bound for  $|a_n|$  similar as before in the case of solutions near ordinary points.

## 2. Step:

We want to find an upper bound for  $a_n$ .

Again Lemma 5 guarantees the existence of positive constants  $r_1, r_2, M_1, M_2$  such that

$$|p_n| < \frac{M_1}{r_1^n} \quad \text{and} \quad |q_n| < \frac{M_2}{r_2^n}.$$

For simplification we introduce the new constant

$$r = \text{Min}(r_1, r_2),$$

which causes

$$|p_n| < \frac{M_1}{r^n} \quad \text{and} \quad |q_n| < \frac{M_2}{r^n}.$$

For  $k = 0$  in (5.8) we get the summand  $c_1 p_n + q_n$ , for which we determine an upper bound

$$|c_1 p_n + q_n| \leq |c_1| \cdot |p_n| + |q_n| < |c_1| \cdot \frac{M_1}{r^n} + \frac{M_2}{r^n} = \frac{|c_1| M_1 + M_2}{r^n}.$$

We introduce the new constant

$$M = \text{Max}(M_1, M_2, |c_1| M_1 + M_2, 1),$$

which gives the new estimates

$$|p_n| < \frac{M}{r^n}, \quad |q_n| < \frac{M}{r^n}, \quad |c_1 p_n + q_n| < \frac{M}{r^n}.$$

With this preparatory work we will prove an upper bound for  $|a_n|$  by induction.

**Initial condition ( $n = 1$ ):**

By choosing  $a_0 = 1$  and with the two facts

- $1(1 + s)a_1 = -a_0(c_1 p_1 + q_1)$  (case  $n = 1$ ),
- $|1 + s| \geq \text{Re}(1 + s) = 1 + \text{Re}(s) \geq 1$ ,

we get the following estimation

$$|a_1| = \frac{c_1 p_1 + q_1}{|1 + s|} < \frac{M}{r}.$$

**Assumption:**

We assume that  $|a_k| < \left(\frac{M}{r}\right)^k$  holds for all  $k$  with  $0 \leq k < n$ .

**To prove:**

We want to prove that the  $n$ -th coefficient has an upper bound in the following form

$$|a_n| < \left(\frac{M}{r}\right)^n.$$

From (5.8) we get

$$|a_n| = \frac{1}{n|n + s|} \left| \sum_{k=0}^{n-1} a_k ((k + c_1)p_{n-k} + q_{n-k}) \right|$$



$$\begin{aligned}
|a_n| &= \frac{1}{n|n+s|} \left| \sum_{k=0}^{n-1} k a_k p_{n-k} + \sum_{k=0}^{n-1} a_k (c_1 p_{n-k} + q_{n-k}) \right| \\
&\leq \frac{1}{n|n+s|} \left( \sum_{k=0}^{n-1} k |a_k| \cdot |p_{n-k}| + \sum_{k=0}^{n-1} |a_k| \cdot |c_1 p_{n-k} + q_{n-k}| \right).
\end{aligned}$$

Plugging in the assumption for  $a_k$  and additionally the restrictions for  $p_n$  and  $q_n$  results in

$$\begin{aligned}
|a_n| &< \frac{1}{n|n+s|} \left( \sum_{k=0}^{n-1} k \left(\frac{M}{r}\right)^k \cdot \frac{M}{r^{n-k}} + \sum_{k=0}^{n-1} \left(\frac{M}{r}\right)^k \cdot \frac{M}{r^{n-k}} \right) \\
&= \frac{1}{n|n+s|} \left( \sum_{k=0}^{n-1} k \frac{M^{k+1}}{r^n} + \sum_{k=0}^{n-1} \frac{M^{k+1}}{r^n} \right).
\end{aligned}$$

Because of  $M \geq 1$  and therefore  $M^{k+1} < M^n$ ,  $0 \leq k \leq n-1$ , we get the next estimation:

$$\begin{aligned}
|a_n| &< \frac{1}{n|n+s|} \left( \sum_{k=0}^{n-1} k \frac{M^n}{r^n} + \sum_{k=0}^{n-1} \frac{M^n}{r^n} \right) \\
&= \frac{1}{n|n+s|} \left( \frac{M^n}{r^n} \sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1 \right) \\
&= \frac{1}{n|n+s|} \left(\frac{M}{r}\right)^n \left(\frac{n(n-1)}{2} + n\right).
\end{aligned}$$

Because of  $Re(s) \geq 0$  we get the fact that  $|n+s| \geq n$  and therefore we derive

$$\begin{aligned}
|a_n| &\leq \frac{1}{n^2} \left(\frac{M}{r}\right)^n \frac{n(n-1) + 2n}{2} \\
&= \frac{n+1}{2n} \left(\frac{M}{r}\right)^n.
\end{aligned}$$

Hence we get the estimation

$$|a_n| \leq \left(\frac{M}{r}\right)^n \quad \text{for } n \geq 0.$$

### 3. step:

The third step is analogous to the third step of the case of solutions near ordinary points.

With this approach we found only one solution of (5.4):

$$w_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+c_1}, \quad (5.9)$$

which is convergent in the specified region  $0 < |z - z_0| < R = \frac{r}{2M}$ . But this is sufficient to complete the proof of Theorem 10.  $\square$

The second, linearly independent solution is more complicated to find. The solution method splits into two cases:

$$(A) \ s \notin \mathbb{N} \text{ and } (B) \ s \in \mathbb{N}.$$

In case (A) one can apply the approach described above for the second root  $c_2$  of the indicial equation. This way one obtains another series solution

$$w_2(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^{n+c_2} \quad (5.10)$$

convergent in some region  $0 < |z - z_0| < \tilde{R}$ . Hence for case (A) we summarize:

### Theorem 11

*Let  $c_1$  and  $c_2$  be the roots of the indicial equation (5.7) arranged such that  $\operatorname{Re}(s) \geq 0$  for  $s := c_1 - c_2$ . If  $s \notin \mathbb{N}$ , let  $w_1(z)$  and  $w_2(z)$  be the power series solutions of the form as in (5.9) and (5.10) - obtained by the method described above. Then (5.9) and (5.10) are linearly independent (over  $\mathbb{C}$ ) solutions of (5.4).*

### Proof (Theorem 11):

The two solutions  $w_1$  and  $w_2$  are linearly independent because of the following observation:

Suppose there exist  $\alpha, \beta \in \mathbb{C}$  such that

$$\alpha \cdot w_1(z) = \beta \cdot w_2(z).$$

This means we have

$$\alpha (z - z_0)^{c_1 - c_2} (a_0 + a_1(z - z_0)^1 + \dots) = \beta (b_0 + b_1(z - z_0)^1 + \dots),$$

where  $a_0 \neq 0$  and  $b_0 \neq 0$ . On the left hand side we have a function that is analytic in an open disc around  $z_0$ . The function on the right hand side is not analytic on such a disc since  $c_1 - c_2 \notin \mathbb{Z}$ . Hence we can conclude that  $\alpha = \beta = 0$ .

For the case (B),  $s = c_1 - c_2 \in \mathbb{N}$ , we have to consider the two different cases:

$$(B1) \ c_1 = c_2 =: c \text{ and } (B2) \ c_1 \neq c_2.$$

In case (B1) we only get one power series solution with  $c$ :

$$w_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+c}.$$

In general, the second solution involves a logarithm in the following form:

$$\tilde{w}_2(z) = w_1(z) \ln(z - z_0) + \sum_{n=0}^{\infty} d_n (z - z_0)^{n+c}.$$

Suppose there exist  $\alpha, \beta \in \mathbb{C}$  such that

$$\alpha \cdot w_1(z) = \beta \cdot \tilde{w}_2(z).$$

We divide by  $(z - z_0)^c$  and get

$$\underbrace{\alpha \cdot \sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{analytic}} = \beta \cdot \underbrace{\frac{w_1(z) \ln(z - z_0)}{(z - z_0)^c}}_{\text{not-analytic}} + \beta \cdot \underbrace{\sum_{n=0}^{\infty} d_n (z - z_0)^n}_{\text{analytic}}.$$

We cannot get equality of something not-analytic and something analytic by multiplication of a complex number. Hence we can conclude that  $\alpha = \beta = 0$ .

The case (B2) is similar to case (B1), because in general one can guarantee only the existence of a solution of type (5.9) corresponding to the root  $c_1$ .

□

For the series

$$\tilde{w}_2(z) = w_1(z) \ln(z - z_0) + \sum_{n=0}^{\infty} d_n (z - z_0)^{n+c_2}$$

we omit the details of a further discussion on existence, and restrict ourselves to pointing to the examples given in the Chapters 3 and 4.

**Remark: Computation of the indicial equation**

Assume the following setting:

$z_0$  is a regular singular point of  $w'' + p(z)w' + q(z)w = 0$  with the expansions

$$p(z) = \sum_{n=0}^{\infty} p_n(z - z_0)^{n-1},$$

$$q(z) = \sum_{n=0}^{\infty} q_n(z - z_0)^{n-2}.$$

The summand for  $n = 0$  delivers the indicial equation

$$c^2 + (p_0 - 1)c + q_0 = 0, \quad (5.11)$$

where  $p_0$  and  $q_0$  are the constant terms of  $p(z)$  and  $q(z)$ . In order to compute  $p_0$  and  $q_0$  via taking a limit, we reformulate the expansion for  $p(z)$  and obtain

$$p(z) = \frac{p_0}{z - z_0} + \sum_{n=1}^{\infty} p_n(z - z_0)^{n-1},$$

$$(z - z_0)p(z) = p_0 + \sum_{n=1}^{\infty} p_n(z - z_0)^n,$$

which leads to

$$p_0 = \lim_{z \rightarrow z_0} [(z - z_0)p(z)]. \quad (5.12)$$

We also reformulate the expansion for  $q(z)$  and get

$$q(z) = \frac{q_0}{(z - z_0)^2} + \sum_{n=1}^{\infty} q_n(z - z_0)^{n-2},$$

$$(z - z_0)^2 q(z) = q_0 + \sum_{n=1}^{\infty} q_n(z - z_0)^n,$$

which leads to

$$q_0 = \lim_{z \rightarrow z_0} [(z - z_0)^2 q(z)]. \quad (5.13)$$

The two equations (5.12) and (5.13) allow us to calculate the coefficients  $p_0$  and  $q_0$  in the indicial equation (5.11) at any regular singular point  $z_0$ .

## 5.3 Fuchsian Equations

In the Chapter 3 we introduced a power series method for solving a linear differential equation about any ordinary or regular singular point of the equation. Hence if we study an equation which has no irregular singular points, we are in the position to solve this equation about any point in  $\mathbb{C}$ . The following definition and theorems are taken from [21].

### Definition 17 (Fuchsian Equation)

*A linear equation, for which any singular point is a regular singular point, is called Fuchsian Equation.*

Hence, a Fuchsian Equation has no irregular singular point.

### Theorem 12

*The number of singular points of a Fuchsian Equation of order two is finite.*

For the application of the Frobenius Method for solving also algebraic equations, see Chapter 7, the following theorem is very important. Namely, as a consequence, when dealing with holonomic differential equations, we do not meet irregular singular points.

### Theorem 13 <sup>4</sup>

*The coefficients in a Fuchsian Equation of order two are **rational functions** of the independent variable.*

Theorem 12 and 13 are also true for Fuchsian Equations of order greater than two.

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<sup>4</sup>[21], E. D. Rainville, Theorem 22, page 144, 1972.



## Chapter 6

# The General Algebraic Equation

As an application for power series techniques we envision applications related to algebraic equations. In this chapter we briefly review classical aspects of solving the general univariate algebraic equation of degree  $n$ ,

$$y^n + x_1y^{n_1} + x_2y^{n_2} + \cdots + x_py^{n_p} + x_0 = 0, \quad (6.1)$$

where the coefficients  $x_i$  are complex numbers, and with natural numbers  $n_i$  such that

$$n > n_1 > n_2 > \cdots > n_p \geq 1.$$

In the introduction we pointed out that - because of working over  $\mathbb{C}$ , the constant term  $x_0$  can be normalized to  $-1$ . However, for the purpose of this chapter, we prefer to treat things without this normalization.

Because of the fundamental theorem of algebra (Theorem 1), we know that there exist  $n$  solutions for this problem (counting multiplicities accordingly). The way to solve the problem is dependent on the degree of the equation.

The question arises how to solve such equations dependent on the degree.

For degree 1 up to degree 4 we have formulas or rather algorithms which give us the requested solutions.

## 6.1 Algebraic equations of degree 2

Given an algebraic equation of degree 2,

$$y^2 + ay + b = 0,$$

where  $a, b \in \mathbb{C}$ . There exist at most two solutions of the form

$$y_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}.$$

## 6.2 Algebraic equations of degree 3

If we have an algebraic equation of degree 3 in the form

$$y^3 + ay^2 + by + c = 0,$$

where  $a, b, c \in \mathbb{C}$ , we can get the three solutions by use of the formula of Cardano<sup>1</sup>. Before we use this formula, we make the substitution  $z = y + \frac{a}{3}$ . We do not reduce the degree of the equation, but the term with the power 2 vanishes. The new transformed equation is

$$z^3 + 3pz + 2q = 0$$

with  $2q = \frac{2a^3}{27} - \frac{ab}{3} + c$  and  $3p = \frac{3b-a^2}{3}$ .

The basic step in the procedure of Cardano is a substitution for  $z$ . If we set  $z = u + v$ , then the equation  $z^3 + 3pz + 2q = 0$  becomes

$$u^3 + v^3 + (u + v)(3uv + 3p) + 2q = 0.$$

This equation is fulfilled, if  $u^3 + v^3 = -2q$  and  $uv = -p$  are valid. We can write  $uv = -p$  in the form  $u^3v^3 = -p^3$ . So  $u^3$  and  $v^3$  are not known, but their sum and also their product are given by the equations above. Because of this fact,  $u^3$  and  $v^3$  can be seen as solutions of the quadratic equation

$$(w - u^3) \cdot (w - v^3) = w^2 - (u^3 + v^3)w + u^3v^3 = w^2 + 2qw - p^3 = 0.$$

We get  $w_1 = u^3 = -q + \sqrt{q^2 + p^3}$  and  $w_2 = v^3 = -q - \sqrt{q^2 + p^3}$ . For the solution  $z$  of the equation  $z^3 + 3pz + 2q = 0$  the formula of Cardano arises:

$$z = u + v = \sqrt[3]{-q + \sqrt{q^2 + p^3}} + \sqrt[3]{-q - \sqrt{q^2 + p^3}}.$$

Because the equation  $w_1 = u^3$  yields three solutions, we would get nine different solutions for our equation, but because of the fact  $uv = -p$  the number of solu-

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<sup>1</sup>[18] P. Pilz, 2003.



tions is reduced to three. So with the formula of Cardano we derive the following three solutions for  $z^3 + 3pz + 2q = 0$ :

$$\begin{aligned} z_1 &= u_1 + v_1, \\ z_2 &= u_1\left(-\frac{1}{2} + \frac{i}{2}\right)\sqrt{3} + v_1\left(-\frac{1}{2} - \frac{i}{2}\right)\sqrt{3}, \\ z_3 &= u_1\left(-\frac{1}{2} - \frac{i}{2}\right)\sqrt{3} + v_1\left(-\frac{1}{2} + \frac{i}{2}\right)\sqrt{3}. \end{aligned}$$

$u_1$  and  $v_1$  have to be the cubic roots with the condition  $u_1v_1 = -p$ .

### 6.3 Algebraic equations of degree 4

If we have an algebraic equation of degree 4 in the form

$$x^4 + ax^3 + bx^2 + cx + d = 0, \quad (6.2)$$

where  $a, b, c, d \in \mathbb{C}$ , we can find the four solutions by the use of a trick of *L. Ferrari*<sup>2</sup>. In the first step we utilize the following transformation rule, similar as in the case for equations of degree 3,

$$y = x + \frac{a}{4},$$

to eliminate the term with the third power and obtain the following new equation

$$\left(y - \frac{1}{4}a\right)^4 + a\left(y - \frac{1}{4}a\right)^3 + b\left(y - \frac{1}{4}a\right)^2 + c\left(y - \frac{1}{4}a\right) + d = 0.$$

We collect terms with the same power and rename the coefficients in the following way

$$y^4 + \underbrace{y^2\left(b - \frac{3a^2}{8}\right)}_{=:p} + \underbrace{y\left(c - \frac{ab}{2} + \frac{a^3}{8}\right)}_{=:q} + \underbrace{d - \frac{ac}{4} + \frac{a^2b}{16} - \frac{3a^4}{256}}_{=:r} = 0.$$

<sup>2</sup>[4], J. Bewersdorff, p. 25, 2007.

We transformed our equation in such a way that the degree stays the same, but we removed the term with degree 3 in our equation, and arrived at

$$y^4 + py^2 + qy + r = 0.$$

The next step is to reformulate our equation by manipulating

$$y^4 + py^2 = -qy - r,$$

such that the left side of the equation becomes a perfect square

$$\begin{aligned} y^4 + 2py^2 + p^2 &= -qy - r + py^2 + p^2, \\ (y^2 + p)^2 &= py^2 - qy - r + p^2. \end{aligned}$$

Now we add a new variable  $z$  on the left side and correct the right hand side by the missing members

$$\begin{aligned} (y^2 + p + z)^2 &= py^2 - qy - r + p^2 + 2pz + 2y^2z + z^2 \\ &= \underbrace{y^2(p + 2z) - qy + (p^2 - r + 2pz + z^2)}_{=:RHS}. \end{aligned} \quad (6.3)$$

In the next step we aim at writing  $RHS$  also as a perfect square. To this end, choose  $s$  and  $t$  such that

$$s^2 = p + 2z, \quad (6.4)$$

$$2st = -q, \quad (6.5)$$

$$t^2 = p^2 - r + 2pz + z^2. \quad (6.6)$$

Then

$$\begin{aligned} RHS &= s^2y^2 + 2sty + t^2 \\ &= (sy + t)^2. \end{aligned}$$

Having in mind that  $(2st)^2 - 4s^2t^2 = 0$ , we combine the 3 equations (6.4), (6.5) and (6.6) into one, and obtain:

$$(-q)^2 - 4(p + 2z)(p^2 - r + 2pz + z^2) = 0.$$

This equation can be rewritten as a polynomial equation of degree 3 in  $z$ :

$$z^3 + \frac{5}{2}pz^2 + z(2p^2 - r) + \frac{1}{2}(p^3 - rp - \frac{q^2}{4}) = 0, \quad (6.7)$$

which we solve for  $z$ . For the next step, for simplicity we group and rename the coefficients in (6.3) as follows:

$$(y^2 + \underbrace{p + z}_{\alpha})^2 = y^2 \underbrace{(p + 2z)}_{\beta} - qy + \underbrace{(p^2 - r + 2pz + z^2)}_{\gamma}. \quad (6.8)$$

We form the perfect square on the right hand side of equation (6.8), and obtain

$$(y^2 + \alpha)^2 = (y\sqrt{\beta} - \sqrt{\gamma})^2,$$

respectively the quadratic equations in  $y$ :

$$y^2 - y\sqrt{\beta} + \sqrt{\gamma} + \alpha = 0,$$

and

$$y^2 + y\sqrt{\beta} - \sqrt{\gamma} + \alpha = 0.$$

These can be solved by

$$y_{1,2} = +\frac{\sqrt{\beta}}{2} \pm \sqrt{\frac{\beta}{4} - \sqrt{\gamma} - \alpha},$$

and

$$y_{3,4} = -\frac{\sqrt{\beta}}{2} \pm \sqrt{\frac{\beta}{4} + \sqrt{\gamma} - \alpha}.$$

One can easily verify that for any solution  $z$  of (6.7):  $y$  solves (6.8) if and only if  $x = y - \frac{a}{4}$  solves (6.2). Hence with the backward transformation  $x = y - \frac{a}{4}$  we get four (taking multiplicities in account) solutions for  $x$  independent of the choice of the three possible  $z$ . In other words, we only need one solution of (6.7) to obtain a complete set of solutions.

As an illustration we calculate the four solutions of

$$x^4 - 2x^3 - x^2 + 2x = 0, \quad (6.9)$$

where Mathematica provides the following result:

`In:= Solve[x4 - 2x3 - x2 + 2x == 0, x]`

Out=  $\{\{x \rightarrow -1\}, \{x \rightarrow 0\}, \{x \rightarrow 1\}, \{x \rightarrow 2\}\}$ .

We define the coefficients of (6.9) as follows:

$$a := -2, \quad b := -1, \quad c := 2, \quad d := 0.$$

To eliminate the term with the third power, we perform the transformation

$$y = x + \frac{a}{4} = x - \frac{1}{2}.$$

We derive the new equation

$$y^4 + py^2 + qy + r = 0, \tag{6.10}$$

with the coefficients

$$\begin{aligned} p &= b - \frac{3a^2}{8} = -\frac{5}{2}, \\ q &= c - \frac{ab}{2} + \frac{a^3}{8} = 0, \\ r &= d - \frac{ac}{4} + \frac{a^2b}{16} - \frac{3a^4}{256} = \frac{9}{16}. \end{aligned}$$

With the above remodeling we obtain the equation (6.7) of degree 3

$$z^3 - 6.25z^2 + 11.9375z - 7.109375 = 0,$$

which we can solve by use of the formula of Cardano, which is described in Section 6.2.

In this example we solve this equation with Mathematica and obtain

In:= `Solve[z3 - 6.25z2 + 11.9375z - 7.109375 == 0, z]`

Out=  $\{\{z \rightarrow 1.25\}, \{x \rightarrow 1.75\}, \{x \rightarrow 3.25\}\}$ .

Therefore we derived three different solutions for  $z$ , i. e.,

$$z_1 = 1.25, \quad z_2 = 1.75, \quad z_3 = 3.25.$$

To get the four solutions of (6.10):

$$y_{1,2} = +\frac{\sqrt{\beta}}{2} \pm \sqrt{\frac{\beta}{4} - \sqrt{\gamma} - \alpha},$$

and

$$y_{3,4} = -\frac{\sqrt{\beta}}{2} \pm \sqrt{\frac{\beta}{4} + \sqrt{\gamma} - \alpha},$$

we have to calculate  $\alpha$ ,  $\beta$  and  $\gamma$ :

$$\alpha = p + z_1 = -1.25,$$

$$\beta = p + 2z_1 = 0,$$

$$\gamma = p^2 - r + 2pz_1 + z_1^2 = 1.$$

Step by step we compute the four solutions  $y_1, y_2, y_3$  and  $y_4$  of (6.10) and the corresponding solutions to our original equation (6.9)  $x_1, x_2, x_3$  and  $x_4$ :

$$\begin{aligned} y_1 &= +\frac{\sqrt{\beta}}{2} + \sqrt{\frac{\beta}{4} - \sqrt{\gamma} - \alpha} = 0.5, \\ x_1 &= y_1 - \frac{a}{4} = 0.5 + 0.5 = 1, \\ y_2 &= +\frac{\sqrt{\beta}}{2} - \sqrt{\frac{\beta}{4} - \sqrt{\gamma} - \alpha} = -0.5, \\ x_2 &= y_2 - \frac{a}{4} = -0.5 + 0.5 = 0, \\ y_3 &= -\frac{\sqrt{\beta}}{2} + \sqrt{\frac{\beta}{4} + \sqrt{\gamma} - \alpha} = 1.5, \\ x_3 &= y_3 - \frac{a}{4} = 1.5 + 0.5 = 2, \\ y_4 &= -\frac{\sqrt{\beta}}{2} - \sqrt{\frac{\beta}{4} + \sqrt{\gamma} - \alpha} = -1.5, \\ x_4 &= y_4 - \frac{a}{4} = -1.5 + 0.5 = -1. \end{aligned}$$

Hence we derived the right solutions to (6.9):

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 2, \quad x_4 = -1,$$

with the choice  $z_1$ . With the options  $z_2$  and  $z_3$  we derive up to the order the same solutions  $y_1, y_2, y_3$  and  $y_4$  of (6.10) and therefore the same solutions  $x_1, x_2, x_3$  and  $x_4$  of (6.9).

Depending on the application, numerical methods might be applied also for solving equations of degree 3 and 4. Short descriptions of numerical methods applicable also for higher degrees are given below.

## 6.4 Algebraic equations of degree 5 or higher

The general algebraic equations of degree 5 or of higher degree than 5 cannot be solved by a general formula involving only arithmetical operations and roots. We have to apply numerical methods, e. g. the Newton Method or the Householder Method.

The following methods, for which we give only informal descriptions, are iteration methods, which proceed via approximated values  $x_k$  towards an approximated solution of the equation  $f(x) = 0$ . In other words, we produce a sequence of approximate solutions  $x_k$ , which converge to an exact solution  $x^*$ , i.e.  $f(x^*) = 0$ , feasibly fast.

### 6.4.1 Newton Method

For solving the equation  $f(x) = 0$ , the Newton Method has the following iteration rule:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots,$$

where some suitable  $x_0$  has to be given as a starting point. So for calculating the new approximation  $x_{n+1}$ , only the values of  $f(x)$  and of the derivative of  $f(x)$  at  $x = x_n$  are necessary. The Newton Method converges locally, so the approximate values  $x_i$  have to be in a neighborhood of the exact solution, denoted by  $x^*$ .

If the Newton Method converges, then the number of accurate positions doubles in each iteration step. In this case one speaks of quadratic convergence.

### 6.4.2 Householder Method

The Householder Method is an extension of the Newton Method. The iteration rule contains the  $d$ -th and the  $(d-1)$ -th derivative of  $f_H(x) := 1/f(x)$  at  $x = x_n$ .

$$x_{n+1} = x_n + d \frac{f_H^{(d-1)}(x_n)}{f_H^{(d)}(x_n)}, \quad n \geq 0.$$

The convergence rate of the method depends only on the choice of  $d$ , more precisely, the rate of convergence can be proven to be  $d + 1$ .

## 6.5 A concrete example: numerical experiments

Now we consider a concrete example, solving the algebraic equation

$$y^5 + x \cdot y^2 - 1 = 0. \quad (6.11)$$

In fact, we want to solve (6.11) by means of an algorithm, which transforms the algebraic equation into a differential equation, which is described in the next chapter, and afterwards solve the differential equation by the Frobenius Method. However first we want to analyse the problem with help of our general knowledge in algebra.

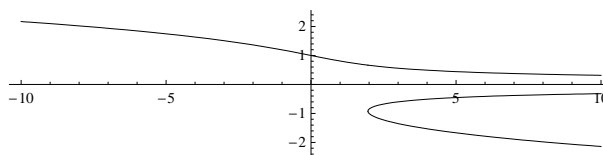
We use the Mathematica command *ImplicitPlot*, which is included in the Mathematica package *Graphics'ImplicitPlot'*, to get an idea how the graph looks like.

### Example:

```
In:= << Graphics'ImplicitPlot'
```

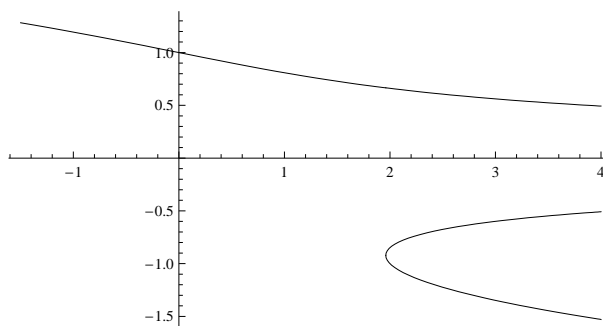
```
In:= ImplicitPlot[y5 + xy2 - 1 == 0, {x, -10, 10}]
```

```
Out=
```



```
In:= ImplicitPlot[y5 + xy2 - 1 == 0, {x, -1.5, 4}]
```

```
Out=
```



We do not want to enter the theory of algebraic curves. However, we note that already from the plots we can see that we only get one real solution in the interval  $[-\infty, \approx 1.9]$ . The singularity in the neighbourhood of 1.9 is confirmed by the numerical tests. With the Mathematica function *NSolve* we can solve the equation for specific  $x$ -values.

```
In:= NSolve[y5 + xy2 - 1 == 0/.x → 1.96, y]
```

```
Out= {{y → -0.922101 - 0.00436396i}, {y → -0.922101 + 0.00436396i},
      {y → 0.589157 - 1.19124i}, {y → 0.589157 + 1.19124i},
      {y → 0.665889}}
```

```
In:= NSolve[y5 + xy2 - 1 == 0/.x → 1.97, y]
```

```
Out= {{y → -0.960384}, {y → -0.884863},
      {y → 0.590295 - 1.19247i}, {y → 0.590295 + 1.19247i},
      {y → 0.664657}}
```

In the interval [*singularity*  $\approx 1.96$ ,  $+\infty$ ] three real solutions pop up. The singularity has to lie between 1.96 and 1.97.

Now we want to compare these observations with the numerics of Mellin series. The Mellin series for a solution of our algebraic equation (6.11) is

$$y = \sum_{k=0}^{\infty} \binom{(2k+1)/5}{k} \frac{(-x)^k}{2k+1}.$$

The general form of the Mellin series for a principal solution of a general algebraic equation can be found in the Chapter 8.

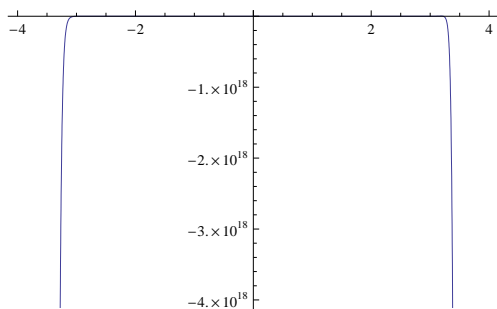
We try to get a closer look at the solution with help of the following plots from Mathematica:

**Example:**

```
In:= Y[x_, N_] := Sum[Binomial[(2k + 1)/5, k](-x)k/(2k + 1), {k, 0, N}]
```

```
In:= Plot[Y[x, 100], {x, -4, 4}]
```

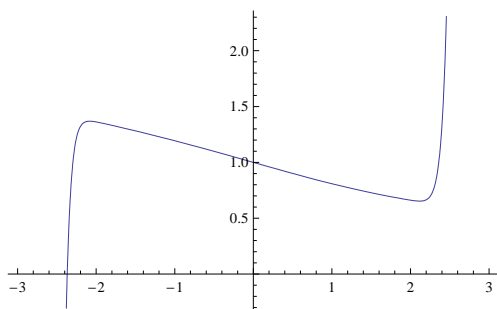
```
Out=
```



```
In:= Plot[Y[x, 40], {x, -3, 3}]
```

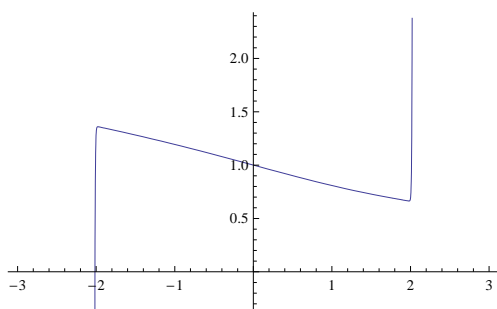
```
Out=
```





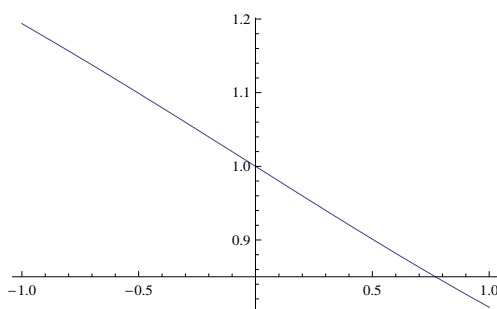
```
In:= Plot[Y[x, 400], {x, -3, 3}]
```

```
Out=
```



```
In:= Plot[Y[x, 100], {x, -1, 1}]
```

```
Out=
```



These plots give us a first impression about convergence. For a precise statement we need to calculate the radius of convergence.

First of all we find out that the ratio test cannot be applied.

```
In:= c[k_] := Abs[Binomial[(2k + 1)/5, k]/(2k + 1)]
```

```
In:= N [ c[k+1]/c[k] /.k -> 30000 ]
```

```
Out= 0.825431
```

```
In:= N [ c[k+1]/c[k] /.k -> 30001 ]
```

```
Out= 0.
```

```
In:= N [ c[k+1]/c[k] /.k -> 30002 ]
```

```
Out= ComplexInfinity
```

```
In:= N [ c[k+1]/c[k] /.k -> 30003 ]
```

Out= 0.315287

In:= N  $\left[ \frac{c[k+1]}{c[k]} / .k \rightarrow 30004 \right]$

Out= 0.510144

In:= N  $\left[ \frac{c[k+1]}{c[k]} / .k \rightarrow 30005 \right]$

Out= 0.825431

Consequently, we apply Cauchy's test ("Wurzelkriterium").

In:= N  $\left[ (c[k])^{(1/k)} / .k \rightarrow 30000 \right]$

Out= 0.509877

In:= N  $\left[ (c[k])^{(1/k)} / .k \rightarrow 30001 \right]$

Out= 0.509886

In:= N  $\left[ (c[k])^{(1/k)} / .k \rightarrow 30002 \right]$

Out= 0.

In:= N  $\left[ (c[k])^{(1/k)} / .k \rightarrow 30003 \right]$

Out= 0.509886

In:= N  $\left[ (c[k])^{(1/k)} / .k \rightarrow 30004 \right]$

Out= 0.509877

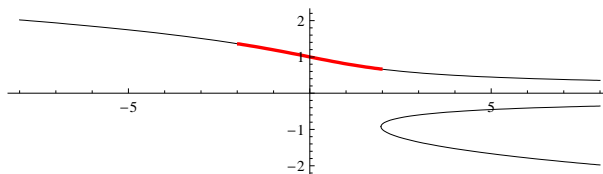
In:= N  $\left[ (c[k])^{(1/k)} / .k \rightarrow 30005 \right]$

Out= 0.509877

For the inverse of the largest value we obtain  $R = 1.96122$  as an experimentally determined estimate for the radius of convergence. We note that this numerical value is quite good in comparison to that given by (8.12), namely 1.96013.

```
In:=Show[ImplicitPlot[y5 + xy2 - 1 == 0, {x, -8, 8}],
  ImplicitPlot[y5 + xy2 - 1 == 0, {x, -1.96, 1.96}, PlotStyle -> {Red, Thick}]]
```

Out=



Next we deal with the imaginary solutions. The question arises how to obtain on the interval  $[-R, R]$  the four imaginary solutions by the Mellin approach. The

computational experiments above suggest that the Mellin series solution does not converge on a significantly bigger interval than  $[-R, R]$ . Therefore we restrict our attention to  $[-R, R]$ . For example, for  $x = \frac{1}{2} \in [-R, R]$  the Mathematica command *NSolve* computes

```
In:=NSolve[y5 + xy2 - 1 == 0/.x → 0.5, y]
Out={{y → -0.839678 - 0.491654i}, {y → -0.839678 + 0.491654i},
      {y → 0.389139 - 1.01031i}, {y → 0.389139 + 1.01031i},
      {y → 0.901077}}
```

The real solution for our choice  $x = \frac{1}{2}$  can be (alternatively to *NSolve*) derived by the following calculation.

```
In:=sol1 = Sum[Binomial[ $\frac{2k+1}{5}$ , k]  $\frac{(-0.5)^k}{2k+1}$ , {k, 0, 100}]
Out=0.901077
```

Finally, the imaginary solutions (at  $x = \frac{1}{2}$ ) can be simply achieved by modifying the Mellin series solution (at  $x = \frac{1}{2}$ ) with the 5th roots of unity, denoted by  $ru[i]$ . The general formula for this multiplication can be found in the Chapter 8.

```
In:=ru[i_] := (Cos[2π * i/5] + i * Sin[2π * i/5])//N
In:=sol2 = (Sum[Binomial[ $\frac{2k+1}{5}$ , k]  $\frac{-0.5*(ru[1])^2)^k}{2k+1}$ , {k, 0, 100}]) * ru[1]
Out=0.389139 + 1.01031i
In:=sol3 = (Sum[Binomial[ $\frac{2k+1}{5}$ , k]  $\frac{-0.5*(ru[2])^2)^k}{2k+1}$ , {k, 0, 100}]) * ru[2]
Out=-0.839678 + 0.491654i
In:=sol4 = (Sum[Binomial[ $\frac{2k+1}{5}$ , k]  $\frac{-0.5*(ru[3])^2)^k}{2k+1}$ , {k, 0, 100}]) * ru[3]
Out=-0.839678 - 0.491654i
In:=sol5 = (Sum[Binomial[ $\frac{2k+1}{5}$ , k]  $\frac{-0.5*(ru[4])^2)^k}{2k+1}$ , {k, 0, 100}]) * ru[4]
Out=0.389139 - 1.01031i
```

Hence, the numerical solutions coincide with the solutions obtained by the Mellin series, even the imaginary ones.

In the following chapter we transform a general algebraic equation into a differential equation. Hence we can apply the Frobenius Method also to solve algebraic equations.



## Chapter 7

# Transforming Algebraic Equations into Differential Equations

### 7.1 Strategy

The goal of this chapter is to describe the transformation of a general algebraic equation into a differential equation. With this approach we, for instance, apply the transformation routine on an algebraic equation, like  $y^5 + xy^2 - 1 = 0$ , and afterwards solve the gained differential equation by the Frobenius Method, which is described in Chapter 3.

We want to transform an algebraic equation defining a function  $y = y(x)$  into a linear differential equation with rational function coefficients, satisfied by  $y(x)$ . To this end we use an algorithm described by D.V. Chudnovsky and G.V. Chudnovsky in [7] without a reference. As noted in Mallinger [12], it has been stated by Comptet [8] in 1964. However, according to Flajolet its story is much older and traces back to Abel.<sup>1</sup> Despite based on a simple idea, the steps are non-trivial and therefore we present a more detailed account of it.

In the following,  $(K, D)$  is a differential field extension of the differential field  $(k(x), \frac{d}{dx})$  of rational functions in  $k(x)$  with derivative  $\frac{d}{dx}$  being the standard derivative.

Elements of  $K$  are denoted by  $f(x)$  and the derivative  $D$  on  $K$  is viewed as a derivative with respect to  $x$ . We indicate this explicitly by writing alternatively  $df/dx$  or  $f'$  for  $Df(x)$ .

---

<sup>1</sup>P.Paule: private communication

**Definition 18**

An element  $f = f(x) \in K$  is called algebraic over  $k(x)$ , if there exists a polynomial  $P(x, y) \in k[x, y]$  such that

$$P(x, f) = 0.$$

## 7.2 Algorithm description

**Context:**  $(K, D)$  is a differential field extension of the differential field  $(k(x), \frac{d}{dx})$  of rational functions in  $k(x)$  with derivative  $\frac{d}{dx}$  being the standard derivative.

**Input:**

$P(x, y) \in k[x, y]$  of degree  $m$  in  $y$ , defining an algebraic element  $f = f(x) \in K$  in the sense of Definition 18.

**Output:**

A linear differential equation with polynomial coefficients  $b_i \in k[x]$  satisfied by  $f = f(x)$ :

$$\sum_{i=0}^{m-1} b_i(x) \frac{d^i f}{dx^i} = 0. \quad (7.1)$$

**Note:** Chudnovsky and Chudnovsky claim that this algorithm yields a linear differential equation corresponding to  $P(x, f) = 0$  of the smallest possible order.

We implemented this algorithm in Mathematica 5.0. We do not need additional packages.

**Notation:** `Polynom:=P(x, y)`

The routine `AEtoDE[Polynom, y, x]` transforms the first input variable `Polynom` into a differential equation.

**Remark:** Note that this algorithm is a constructive version of the following statement:

**Theorem 14**<sup>2</sup>

If  $f(x) \in K[[x]]$  is algebraic, then  $f(x)$  is holonomic.

---

<sup>2</sup>[12], C. Mallinger, p.16, 1996.

**Note:** Because of Definition 11  $f(x)$  fulfills the following equation:

$$p_m(x)f^{(m)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

with  $p_0(x), p_1(x), \dots, p_m(x) \in K[x]$ , not all zero.

### 7.3 Sketch of the idea of the algorithm

For a given algebraic function  $f = f(x)$  from  $K$  we show below that its derivative  $f' = df/dx$  is again an element of  $K$  and expressible as a linear combination of powers of  $f$  with coefficients from  $k(x)$ , i.e.,

$$f' \in \langle f^0, f^1, \dots, f^{m-1} \rangle_{k(x)},$$

where  $m$  is the degree of  $f$ , more precisely:  $m = \deg_y P(x, y)$ . This results in the following representation for  $f'$ :

$$f' = r_0 f^0 + r_1 f^1 + \cdots + r_{m-1} f^{m-1},$$

where  $r_0, r_1, \dots, r_{m-1} \in k(x)$ . This gives us the following representation for  $f''$ :

$$f'' = r'_0 f^0 + r_0 \underbrace{(f^0)'}_{=0} + r'_1 f^1 + r_1 f' + \cdots + r'_{m-1} f^{m-1} + r_{m-1} (m-1) f^{m-2} f'.$$

Consequently, for  $f'' = d^2 f/dx^2$ :

$$f'' \in \langle f^0, f^1, \dots, f^{m-1} \rangle_{k(x)}.$$

Proceeding in the same way we derive the same condition for the  $N$ -th derivative:

$$f^{(N)} \in \langle f^0, f^1, \dots, f^{m-1} \rangle_{k(x)}.$$

We found out that  $\langle f^0, f', f'', f^{(3)}, \dots \rangle_{k(x)}$  is a subspace of  $\langle f^0, f^1, \dots, f^{m-1} \rangle_{k(x)}$ .

Hence

$$\begin{aligned} \langle f^0, f', f'', f^{(3)}, \dots \rangle_{k(x)} &\leq \langle f^0, f^1, \dots, f^{m-1} \rangle_{k(x)} \quad \text{and} \\ \dim \langle f^0, f', f'', f^{(3)}, \dots \rangle_{k(x)} &\leq \dim \langle f^0, f^1, \dots, f^{m-1} \rangle_{k(x)} \leq m. \end{aligned}$$

Now then we can express all derivatives  $f^{(0)}, f', f'', \dots, f^{(m)}$  as powers of  $f$ . From this step we derive a system of equations for the coefficients  $c_i(x)$  in

$$c_0(x)f^{(0)} + c_1(x)f'(x) + \dots + c_m(x)f^{(m)} = 0.$$

We have  $m + 1$  unknowns

$$(c_0(x), c_1(x), \dots, c_m(x)) \in k(x)^{m+1}$$

and we know that  $\dim \langle f^0, f^1, \dots, f^{m-1} \rangle_{k(x)} = m$ . Hence this shows existence of a nontrivial differential equation (7.1).

This theoretical idea can be made constructive as shown below.

## 7.4 Description of the subroutine:

### ExtendedGCDPoly

The routine *AEToDE* needs the subroutine *ExtendedGCDPoly*[ $a, b, c$ ].

#### Input:

Two polynomials  $a$  and  $b \in k[x, y]$  and  $c$  being either  $x$  or  $y$ .

#### Output:

If  $c = x$  it computes the Extended GCD of  $a$  and  $b$  in  $F[x]$  where  $F = k(y)$ .

If  $c = y$  it computes the Extended GCD of  $a$  and  $b$  in  $G[y]$  where  $G = k(x)$ .

In both cases the output is a triple  $\{g, \{r, s\}\}$  such that  $r \cdot a + s \cdot b = g$ .

#### Example:

We calculate the ExtendedGCDPoly of the polynomial  $y^5 + xy^2 - 1$  and its derivative  $5y^4 + 2xy$  with respect to the variable  $y$ .

```
In:= ExtendedGCDPoly[y^5 + x * y^2 - 1, 5 * y^4 + 2 * x * y, y]
```

```
Out= {3125 - 108x^5, {-3125 + 108x^5 + 450x^3y - 1875xy^2 + 270x^4y^3,
      225x^2 + (625 - 54x^5)y - 90x^3y^2 + 375xy^3 - 54x^4y^4}}
```

#### Verification:

```
In:= FullSimplify[(-3125 + 108x^5 + 450x^3y - 1875xy^2 + 270x^4y^3)*
      (y^5 + x * y^2 - 1) + (225x^2 + (625 - 54x^5)y - 90x^3y^2 + 375xy^3 - 54x^4y^4)*
```



$$(5 * y^4 + 2 * x * y)]$$

$$\text{Out} = 3125 - 108x^5$$

### Definition 19

Given  $a, b \in k[x, y]$  we define

$$\gcd_x(a, b) := g,$$

where  $g$  is determined by  $\text{ExtendedGCDPoly}[a, b, y] = \{g, \{r, s\}\}$ . Analogous we define

$$\gcd_y(a, b) := g,$$

where  $g$  is determined by  $\text{ExtendedGCDPoly}[a, b, x] = \{g, \{r, s\}\}$ .

In the new version, Mathematica 6.0, the subroutine *ExtendedGCDPoly* is already implemented, called *PolynomialExtendedGCD*.

## 7.5 The steps of *AEtoDE*

For those with a special interest in the algorithm, we explain the procedure step-by-step. We start with the input as specified above.

For simplicity we write  $P$  for  $P(x, y)$ ,  $P_y$  for  $dP/dy$  and  $P_x$  for  $dP/dx$ , where  $d/dy$  and  $d/dx$  are the standard partial derivatives with respect to  $y$ , respectively  $x$ .

In the first step we have to calculate the derivative of our polynomial  $P$  with respect to  $y$ . Without loss of generality we assume that  $P(x, y)$  is square-free with respect to  $y$ . This means,  $r(x) := \gcd_y(P, P_y)$  is independent of  $y$ , i. e.,  $r(x) \in k[x]$ . By means of the routine *ExtendedGCDPoly* we compute  $(A(x, y), B(x, y))$ , such that

$$A(x, y) \cdot P(x, y) + B(x, y) \cdot P_y(x, y) = r(x). \quad (7.2)$$

$A(x, y)$  and  $B(x, y)$  are polynomials in  $x$  and  $y$  of degrees  $m - 2$  and  $m - 1$  in  $y$ . Recall that  $f = f(x)$  is a solution of our algebraic equation  $P(x, f) = 0$ . On the one hand with the trivial relation

$$A(x, f(x)) \cdot P(x, f(x)) = 0$$

our equation (7.2) simplifies to

$$B(x, f(x)) \cdot P_y(x, f(x)) = r(x),$$

where  $r(x) := \gcd(P, P_y)$ . On the other hand by differentiating  $P(x, f) = 0$  with respect to  $x$  we obtain

$$P_x(x, f(x)) + \frac{df}{dx} \cdot P_y(x, f(x)) = 0 \quad \text{or} \quad \frac{df}{dx} = -\frac{P_x(x, f(x))}{P_y(x, f(x))}.$$

Taking the two facts from above into account we obtain

$$\frac{df}{dx} = \frac{-B(x, f(x)) \cdot P_x(x, f(x))}{r(x)}. \quad (7.3)$$

Up to now the method is expressed in [8], from now on the steps can be found in [12].

With the Mathematica command *PolynomialRemainder* and by help of our algebraic equation  $P(x, f) = 0$  we can eliminate from the numerator of (7.3) all powers of  $f(x)$  higher than  $m - 1$  and get the following result

$$\frac{df}{dx} = \frac{R_1(x, f)}{r(x)}, \quad (7.4)$$

where  $R_1 = R_1(x, f)$  is a polynomial in  $f$  of degree  $m - 1$ . The coefficients of  $R_1$  are rational functions in  $x$ . By differentiating (7.4) we get

$$\frac{d^2 f}{dx^2} = \frac{1}{r(x)^2} \left( R_1 \frac{dR_1}{dy} + r(x) \frac{dR_1}{dx} - R_1 \frac{dr(x)}{dx} \right).$$

Again we use the command *PolynomialRemainder* and  $P(x, f) = 0$  to eliminate all powers of  $f(x)$  higher than  $m - 1$  and obtain

$$\frac{d^2 f}{dx^2} = \frac{R_2(x, f)}{r(x)^2},$$

where  $R_2 = R_2(x, f)$  is a polynomial in  $f$  of degree  $m - 2$ . The coefficients of  $R_2$  are rational functions in  $x$ . We have to repeat this procedure  $m - 2$  times more. Consequently, we obtain a system of  $m - 1$  equations

$$\frac{d^i f}{dx^i} = \frac{R_i}{r(x)^i} = r_{i,0} f^0 + \cdots + r_{i,m-1} f^{m-1}, \quad i = 1, \dots, m - 1.$$

By solving this system of equations in the unknowns  $r_{i,j} \in k(x)$  we derive a linear relation

$$\sum_{i=0}^{m-1} c_i(x) \frac{d^i f}{dx^i} r(x)^i = 0.$$

By multiplying this relation by the common denominator of the  $m$  rational functions  $c_0, c_1, \dots, c_{m-1}$  in  $x$ , we obtain a linear differential equation on  $y$  with polynomial coefficients in  $x$

$$\sum_{i=0}^{m-1} b_i(x) \frac{d^i f}{dx^i} = 0, \quad (7.5)$$

with an order at most  $m - 1$ . At least one of the coefficients  $b_0(x), \dots, b_{m-1}(x)$  should be nonzero to end with a differential equation. Our derived differential equation (7.5) corresponds to the output as specified in (7.1). Hence the solution of the differential equation (7.5) solves the algebraic equation  $P(x, f) = 0$ .

By use of our routine *AEtoDE* we now transform our algebraic equation

$$y(x)^5 + x * y(x)^2 - 1 = 0$$

into a differential equation.

**Example:**

In:= AEtoDE[y<sup>5</sup> + x \* y<sup>2</sup> - 1, y, x] == 0

Out= (-3125 + 108x<sup>5</sup>)<sup>3</sup>(-72x<sup>3</sup>y[x] + 1080x<sup>4</sup>y'[x] - 6250y''[x] + 2706x<sup>5</sup>y''[x] + 6250xy<sup>(3)</sup>[x] - 3125x<sup>2</sup>y<sup>(4)</sup>[x] + 108x<sup>7</sup>y<sup>(4)</sup>[x]) == 0

We simplify the coefficients with the Mathematica command *FullSimplify*.

In:= FullSimplify[AEtoDE[y<sup>5</sup> + x \* y<sup>2</sup> - 1, y, x] == 0]

Out= (-3125 + 108x<sup>5</sup>)(-72x<sup>3</sup>y[x] + 1080x<sup>4</sup>y'[x] + (-6250 + 2706x<sup>5</sup>)y''[x] + x(2(3125 + 567x<sup>5</sup>)y<sup>(3)</sup>[x] + x(-3125 + 108x<sup>5</sup>)y<sup>(4)</sup>[x])) == 0

To compare our result with the Mellin series solution, we plug the Mellin series or rather the corresponding derivatives of the Mellin series into our output of the two procedures. Correctness is already shown with the first ten series members.

**Example:**

```

In:= sol[n_] := Sum[Binomial[(2k + 1)/5, k] * (-x)^k / (2k + 1), k, 0, n]
In:= dgl = AEtoDE[y^5 + x * y^2 - 1, y, x];
In:= dgl/.y[x]- > sol[10]/.Derivative[i_][y][x]- > D[sol[10], x, i]//Expand
Out= 2871618750000x^9 - 1320944625000x^11 - 530630100000x^12+
      337549212000x^13 - 297729432000x^14 + 136955538720x^16 + ...

```

We observe that the first series members vanish. Hence the Mellin series solves the differential equation.

## 7.6 The steps of AEtoDE: a concrete example

We have a closer look at the particular steps of the method by computing a differential equation satisfying the algebraic function

$$P(x, y(x)) = y^5 + xy^2 - 1.$$

Derivation with respect to  $y$  gives

$$P_y(x, y) = 5y^4 + 2xy.$$

By applying the subroutine *ExtendedGCDPoly*[ $P, P_y, y$ ] we obtain

$$\underbrace{(-3125 + 108x^5 + 450x^3y - 1875xy^2 + 270x^4y^3)}_{A(x,y)}(y^5 + xy^2 - 1) + \underbrace{(225x^2 + 625y - 54x^5y - 90x^3y^2 + 375xy^3 - 54x^4y^4)}_{B(x,y)}(5y^4 + 2xy) = 3125 - 108x^5,$$

and with the command *PolynomialRemainder*[- $B*P_x, P, y$ ] we eliminate powers of  $y$  higher than 4 and we derive

$$\frac{dy}{dx} = \frac{\overbrace{-375x + 54x^4y + 150x^2y^2 - 625y^3 + 90x^3y^4}^{=:R_1}}{\underbrace{3125 - 108x^5}_{=:r(x)}}. \quad (7.6)$$

By differentiating (7.6) further we compute  $\frac{d^2y}{dx^2}, \dots, \frac{d^4y}{dx^4}$  and the corresponding  $R_2, \dots, R_4$

$$\frac{d^2y}{dx^2} = \frac{1}{(3125 - 108x^5)^2} (3(2916x^8y + 546875xy^2 - 187500x^4y^3 + 125000x^2y^4 + 14580x^7y^4 + 390625(-1 + y^5) + 6750x^5(-9 + 4y^5) + 2700x^6y^2(9 + 4y^5) - 18750x^3y(-10 + 7y^5))),$$

$$R_2 = -101250x^5 + (168750x^3 + 8748x^8)y + (468750x + 24300x^6)y^2 - 168750x^4y^3 + (375000x^2 + 11340x^7)y^4,$$

$$\frac{d^3y}{dx^3} = \frac{1}{(3125 - 108x^5)^3} ((168750x^3 + 8748x^8 + 2(468750x + 24300x^6)y - 506250x^4y^2 + 4(375000x^2 + 11340x^7)y^3)(-375x + 54x^4y + 150x^2y^2 - 625y^3 + 90x^3y^4) + (3125 - 108x^5)(-506250x^4 + (506250x^2 + 69984x^7)y + (468750 + 145800x^5)y^2 - 675000x^3y^3 + (750000x + 79380x^6)y^4) + 1080x^4(-101250x^5 + (168750x^3 + 8748x^8)y + (468750x + 24300x^6)y^2 - 168750x^4y^3 + (375000x^2 + 11340x^7)y^4)),$$

$$R_3 = -1019531250x^4 - 46777500x^9 + (292968750x^2 + 263250000x^7 + 2361960x^{12})y + (1464843750 + 686250000x^5 + 7338600x^{10})y^2 + (-1699218750x^3 - 77962500x^8)y^3 + (1757812500x + 426937500x^6 + 2828520x^{11})y^4,$$

$$\frac{d^4y}{dx^4} = \frac{1}{(3125 - 108x^5)^4} ((292968750x^2 + 263250000x^7 + 2361960x^{12} + 2(1464843750 + 686250000x^5 + 7338600x^{10})y + 3(-1699218750x^3 - 77962500x^8)y^2 + 4(1757812500x + 426937500x^6 + 2828520x^{11})y^3) \cdot (-375x + 54x^4y + 150x^2y^2 - 625y^3 + 90x^3y^4) + (3125 - 108x^5) \cdot (-4078125000x^3 - 420997500x^8 + (585937500x + 1842750000x^6 + 28343520x^{11})y + (3431250000x^4 + 73386000x^9)y^2 + (-5097656250x^2 - 623700000x^7)y^3 + (1757812500 + 2561625000x^5 + 31113720x^{10})y^4) + 1620x^4(-1019531250x^4 - 46777500x^9 + (292968750x^2 + 263250000x^7 + 2361960x^{12})y + (1464843750 + 686250000x^5 + 7338600x^{10})y^2 + (-1699218750x^3 - 77962500x^8)y^3 + (1757812500x + 426937500x^6 + 2828520x^{11})y^4)),$$

$$\begin{aligned}
R_4 = & -8349609375000x^3 - 2099671875000x^8 - 28179495000x^{13} + \\
& (-3662109375000x + 4144921875000x^6 + 296611875000x^{11} + \\
& 892820880x^{16})y + (11337890625000x^4 + 799621875000x^9 + \\
& 3109914000x^{14})y^2 + (-13916015625000x^2 - 3499453125000x^7 - \\
& 46965825000x^{12})y^3 + (3662109375000 + 8813671875000x^5 + \\
& 430170750000x^{10} + 1027190160x^{15})y^4,
\end{aligned}$$

and receive a system of equations. The solution-vector is derived by the command *NullSpace* from Mathematica and the entries are exactly the coefficients of the differential equation, which we are looking for. We multiply by the common denominator of the entries and in conclusion we obtain the differential equation corresponding to the algebraic equation  $y^5 + xy^2 - 1 = 0$

$$\begin{aligned}
& (-3125 + 108x^5)^3(-72x^3y + 1080x^4y' - 6250y'' + 2706x^5y'' + 6250xy^{(3)} + \\
& 1134x^6y^{(3)} - 3125x^2y^{(4)} + 108x^7y^{(4)}) = 0.
\end{aligned}$$

In the end we cancel out common factors and join the same powers to get

$$\begin{aligned}
& -72x^3y + 1080x^4y' + (-6250 + 2706x^5)y'' + \\
& x(2(3125 + 567x^5)y^{(3)} + x(-3125 + 108x^5)y^{(4)}) = 0.
\end{aligned}$$

## 7.7 Example: AEtoDE in connection to Frobenius Method

By use of the routine *AEtoDE* we transform the algebraic equation

$$y^4 + xy^2 - x = 0$$

into a differential equation and solve this derived differential equation with the Frobenius Method.

### Example:

```

In:= << GeneratingFunctions.m
In:= dgl = AEtoDE[y^4 + x * y^2 - x == 0, y, x]
Out= 3y[x] + 2x^2y'[x] + 4(4x^2 + x^3)y''[x] == 0

```

We derived a holonomic differential equation of second order with  $x = 0$  a regular singular point.

We compute the indicial equation and its roots like in the Chapter 4.

```
In:= rec = DE2RE[dgl, y[x]a[n]]
Out= 2n(-1 + 2n)a[n] + (1 + 4n)(3 + 4n)a[1 + n] == 0
In:= rec = rec/.n- > n - 1
Out= 2(-1 + 2(-1 + n))(-1 + n)a[-1 + n]+
      (1 + 4(-1 + n))(3 + 4(-1 + n))a[n] == 0
In:= rec := 2(-1 + 2(-1 + n + c))(-1 + n + c)a[-1 + n]+
      (1 + 4(-1 + n + c))(3 + 4(-1 + n + c))a[n] == 0
```

We seek for the indicial equation. For  $n = 0$  the first part

$$2(-1 + 2(-1 + n + c))(-1 + n + c)a[-1 + n]$$

vanishes and we get:

$$(1 + 4(-1 + c))(3 + 4(-1 + c))a[0] = 0.$$

We divide by  $a[0] \neq 0$  and by simplifications we derive the indicial equation

$$3 + 16(c - 1)c = 0.$$

We proceed with Mathematica to compute the roots of our indicial equation and the difference of the roots.

```
In:= indequ := 3 + 16(c - 1)c
In:= roots = c/.Solve[indequ == 0, c]
Out= {1/4, 3/4}
In:= c1 := roots[[2]], c2 := roots[[1]]
In:= s = c1 - c2
Out= 1/2
```

Since the difference is nonintegral, we will get two linearly independent solutions with each choice for  $c$ :  $c = c_1 = \frac{3}{4}$  and  $c = c_2 = \frac{1}{4}$ . In the following we compute a

closed form for the coefficient  $a(n)$  for each choice of  $c$  and plug it into the general form for the solution

$$y = x^c + \sum_{n=1}^{\infty} a(n) \cdot x^{n+c},$$

where  $c$  stands for the two choices  $c_1$  and  $c_2$  from above. We get the following two linearly independent solutions.

In:= c := 0.75

In:= C1 = a[n]/.FullSimplify[RSolve[rec == 0, a[0] == 1, a[n], n]]

$$\text{Out} = \left\{ \frac{\left(-\frac{1}{4}\right)^n \Gamma\left[\frac{1}{2} + 2n\right]}{\sqrt{\pi} \Gamma[2 + 2n]} \right\}$$

In:= sol1 = x^c + \sum\_{n=1}^{\infty} C1 \cdot x^{n+c}

$$\text{Out} = x^{0.75} + \sum_{n=1}^{\infty} \left\{ \frac{\left(-\frac{1}{4}\right)^n \Gamma\left[\frac{1}{2} + 2n\right] x^{0.75+n}}{\sqrt{\pi} \Gamma[2 + 2n]} \right\}$$

In:= c := 0.25

In:= C2 = a[n]/.FullSimplify[RSolve[rec == 0, a[0] == 1, a[n], n]]

$$\text{Out} = \left\{ -\frac{(-1)^n \text{Pochhammer}\left[\frac{3}{4}, -1+n\right] \text{Pochhammer}\left[\frac{5}{4}, -1+n\right]}{16 \Gamma[1+2n]} \right\}$$

In:= sol2 = x^c + \sum\_{n=1}^{\infty} C2 \cdot x^{n+c}

$$\text{Out} = x^{0.25} + \sum_{n=1}^{\infty} \left\{ -\frac{(-1)^n \text{Pochhammer}\left[\frac{3}{4}, -1+n\right] \text{Pochhammer}\left[\frac{5}{4}, -1+n\right] x^{0.25+n}}{16 \Gamma[1+2n]} \right\}$$

We tried to get a simplified representation with the Mathematica command *FullSimplify*, but we did not succeed.

**Note:** *Pochhammer*[ $a, n$ ] gives the Pochhammer symbol  $(a)_n$ , which can be found in Definition 3 in Chapter 1.

The general solution can be written as

$$y = A \cdot \text{sol1} + B \cdot \text{sol2},$$

with arbitrary constants  $A$  and  $B$ .



## Chapter 8

# Mellin Series

In this chapter we want to solve algebraic equations again with the help of power series, more precise, we derive the solutions of an algebraic equation via Mellin series.

Mellin [13] solved our equation (see (6.1) in Chapter 6)

$$y^n + x_1 y^{n_1} + x_2 y^{n_2} + \cdots + x_p y^{n_p} - 1 = 0 \quad (8.1)$$

in terms of multivariate power series in the coefficients  $x_1, \dots, x_p \in \mathbb{C}$ , not all 0. This means, the form of the solution is

$$y(x_1, \dots, x_p) = \sum_{k_1, \dots, k_p \geq 0} c(k_1, \dots, k_p) x_1^{k_1} \cdots x_p^{k_p},$$

where  $c(k_1, \dots, k_p)$  is given in explicit form. All other solutions can be obtained from  $y$  by multiplication with the  $n$ th roots of unity; see below.

In 1915 H. Mellin utilized power series to obtain solutions of an algebraic equation. Due to complicated integrals occurring in his approach, Mellin's method has not been studied in detail for a long time. 1988 J. D. Louck found a new derivation of Mellin series without using integrals. In his problem transformation an identity had arisen, for which he had not found a proof in the literature. At the Oberwolfach conference on combinatorics in July 1988 he stated the respective formula as a conjecture:

Let  $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{N}^p$ ,  $\mathbf{b} = (b_1, \dots, b_p) \in \mathbb{N}^p$ ,  $\mathbf{z} = (z_1, \dots, z_p) \in \mathbb{C}^p$ .

We define  $[x]_0 := 1$  and if  $n \geq 1$ ,

$$[x]_n := x(x-1) \cdots (x-n+1).$$

In addition, the following rules for multivariate variables are valid:

$$\begin{aligned} |\mathbf{a}| &:= a_1 + \cdots + a_p, \\ \mathbf{a} \cdot \mathbf{z} &:= a_1 z_1 + \cdots + a_p z_p, \\ \binom{\mathbf{a}}{\mathbf{b}} &:= \binom{a_1}{b_1} \binom{a_2}{b_2} \cdots \binom{a_p}{b_p}. \end{aligned}$$

We define  $r_{\mathbf{0}}(u, \mathbf{z}) := 1$  if  $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^p$ , and if  $\mathbf{a} \neq \mathbf{0}$ ,

$$r_{\mathbf{a}}(u, \mathbf{z}) := u[u - 1 + \mathbf{a} \cdot \mathbf{z}]_{|\mathbf{a}|-1}.$$

### Louck's conjecture:

For all  $u, v \in \mathbb{C}$ :

$$r_{\mathbf{a}}(u + v, \mathbf{z}) = \sum_{\mathbf{b} \in \mathbb{N}^p} \binom{\mathbf{a}}{\mathbf{b}} r_{\mathbf{b}}(u, \mathbf{z}) r_{\mathbf{a}-\mathbf{b}}(v, \mathbf{z}).$$

V. Strehl [23] gave a combinatorial proof of Louck's conjecture. P. Paule proved the identity by use of multivariate Lagrange inversion [14]. Moreover, he discovered in [16] that one can obtain the Mellin Series without the integrals via a univariate setting of Lagrange inversion.

Below we follow Paule's approach from [16], this means, inspired by Mellin and Louck we derive power series solutions to the general algebraic equation. We get the principal solution of (8.1), i. e., the Mellin series solution, where the constant term of the series is equal to 1.

The series

$$g(z) = g_1 z^1 + g_2 z^2 + \cdots$$

is an analytic function defined in some open set  $U \subseteq \mathbb{C}$  containing the origin. If  $g_1 \neq 0$  then  $g : U \rightarrow V$  is a conformal map for some open set  $V \subseteq \mathbb{C}$ . We can deduce that there exists a conformal map  $h : V \rightarrow U$ ,

$$h(z) = h_1 z^1 + h_2 z^2 + \cdots,$$

where  $h_1 \neq 0$ , such that

$$g(h(z)) = h(g(z)) = z.$$

In other words, for the setting  $w := g(z)$  we obtain

$$z = h_1 w^1 + h_2 w^2 + \cdots. \tag{8.2}$$

Lagrange inversion provides a formula not only for computing the coefficients  $h_k$  in (8.2), but also for the coefficients of any analytic function  $y(z)$  expanded in terms of  $w = g(z)$ ; namely:

$$y(z) = y(0) + \sum_{k=1}^{\infty} c_k \frac{w^k}{k!}, \quad (8.3)$$

with

$$c_k = LD^{k-1}y'(z)\varphi^k(z), \quad (8.4)$$

where

$$w = g(z) = \frac{z}{\varphi(z)}$$

for some power series  $\varphi(z)$  with  $\varphi(0) \neq 0$ .

Our goal is to solve the general algebraic equation (8.1) and represent the principle solution in the form of a multivariate power series in the coefficients  $x_i$ . All other solutions will be obtained from the principle solution by multiplication with the  $n$ th roots of unity.

We define

$$w := g(z) := \frac{1 - y(z)^n}{x_1y(z)^{n_1} + x_2y(z)^{n_2} + \cdots + x_py(z)^{n_p}}. \quad (8.5)$$

If we choose  $y(z)$  such that (8.5) suits the setting of Lagrange inversion, we can find a series expansion of  $y(z)$  like in (8.3), where the coefficients  $c_k$  are dependent on the  $x_i$  and  $n$ , and such that

$$1 - y(z)^n = (x_1y(z)^{n_1} + x_2y(z)^{n_2} + \cdots + x_py(z)^{n_p})w. \quad (8.6)$$

This can be achieved by setting

$$y(z) := (1 - z)^{1/n} = \sum_{k=0}^{\infty} \binom{1/n}{k} (-1)^k z^k \quad (8.7)$$

and

$$\varphi(z) := x_1y(z)^{n_1} + x_2y(z)^{n_2} + \cdots + x_py(z)^{n_p}. \quad (8.8)$$

Hence  $g(z)$  is an analytic function of the form

$$g(z) = \frac{z}{\varphi(z)} = g_1z^1 + g_2z^2 + \cdots,$$

where  $g_1 \neq 0$ . In addition,

$$y(z) = 1 + \sum_{k=1}^{\infty} c_k \frac{w^k}{k!},$$

with

$$\begin{aligned} c_k &= LD^{k-1} y'(z) \varphi^k(z) \\ &= -\frac{1}{n} LD^{k-1} (1-z)^{\frac{1}{n}-1} \left( x_1 (1-z)^{\frac{n_1}{n}} + \cdots + x_p (1-z)^{\frac{n_p}{n}} \right)^k \\ &= -\frac{1}{n} LD^{k-1} \sum_{\substack{j_1+\cdots+j_p=k \\ j_1, \dots, j_p \geq 0}} \frac{k!}{j_1! \cdots j_p!} x_1^{j_1} \cdots x_p^{j_p} (1-z)^{\frac{1}{n}-1 + \frac{n_1}{n} j_1 + \cdots + \frac{n_p}{n} j_p} \\ &= (-1)^k \frac{1}{n} \sum_{\substack{j_1+\cdots+j_p=k \\ j_1, \dots, j_p \geq 0}} \frac{k!}{j_1! \cdots j_p!} x_1^{j_1} \cdots x_p^{j_p} \left[ \frac{1}{n} - 1 + \frac{n_1}{n} j_1 + \cdots + \frac{n_p}{n} j_p \right]_{k-1}. \end{aligned}$$

For the above expansion in terms of a multivariate sum, we used the multinomial theorem; see Theorem 3 in Chapter 1.

Consequently,

$$\begin{aligned} y(z) &= 1 + \frac{1}{n} \sum_{k=1}^{\infty} (-w)^k \sum_{\substack{j_1+\cdots+j_p=k \\ j_1, \dots, j_p \geq 0}} \frac{x_1^{j_1} \cdots x_p^{j_p}}{j_1! \cdots j_p!} \left[ \frac{1}{n} - 1 + \frac{n_1}{n} j_1 + \cdots + \frac{n_p}{n} j_p \right]_{k-1} \\ &= 1 + \frac{1}{n} \sum_{\substack{j_1+\cdots+j_p \geq 1 \\ j_1, \dots, j_p \geq 0}} (-w)^{j_1+\cdots+j_p} \frac{x_1^{j_1} \cdots x_p^{j_p}}{j_1! \cdots j_p!} \left[ \frac{1}{n} - 1 + \frac{n_1}{n} j_1 + \cdots + \frac{n_p}{n} j_p \right]_{j_1+\cdots+j_p-1}. \end{aligned}$$

Referring to Mellin's original work, Louck [10] states that the series

$$\sum_{\substack{j_1+\cdots+j_p \geq 1 \\ j_1, \dots, j_p \geq 0}} (-1)^{j_1+\cdots+j_p} \frac{x_1^{j_1} \cdots x_p^{j_p}}{j_1! \cdots j_p!} \left[ \frac{1}{n} - 1 + \frac{n_1}{n} j_1 + \cdots + \frac{n_p}{n} j_p \right]_{j_1+\cdots+j_p-1}$$

is absolutely convergent for all  $(x_1, \dots, x_p) \in U$ , where  $U$  is the generalized complex disc

$$U := \{(x_1, \dots, x_p) \in \mathbb{C}^p : |x_j| < \min(r_1, \dots, r_p)\},$$

where

$$r_j := \frac{1}{j} \left(\frac{n_j}{n}\right)^{-\frac{n_j}{n}} \left(1 - \frac{n_j}{n}\right)^{\frac{n_j}{n}-1}. \quad (8.9)$$

We apply Abel's Theorem (see Theorem 4 in Chapter 2) with  $w_0 = 1$  to the expansion  $y(z) = 1 + \sum_{k=1}^{\infty} c_k \frac{w^k}{k!}$  and derive the following theorem:

**Theorem 15**<sup>1</sup>

Let  $n_1, \dots, n_p$  and  $n$  be positive integers. The multivariate power series

$$y = y(x_1, \dots, x_p) :=$$

$$1 + \frac{1}{n} \sum_{\substack{j_1+\dots+j_p \geq 1 \\ j_1, \dots, j_p \geq 0}} (-1)^{j_1+\dots+j_p} \frac{x_1^{j_1} \cdots x_p^{j_p}}{j_1! \cdots j_p!} \left[ \frac{1}{n} - 1 + \frac{n_1}{n} j_1 + \cdots + \frac{n_p}{n} j_p \right]_{j_1+\dots+j_p-1}$$

satisfies the algebraic equation (8.1) as a relation in  $\mathbb{C}[[x_1, \dots, x_p]]$ . Furthermore, for  $(x_1, \dots, x_p) \in U$  the series  $y = y(x_1, \dots, x_p)$  is absolutely convergent in  $U$  and is a solution of the algebraic equation (8.1). The remaining  $n-1$  roots  $y_1, \dots, y_{n-1}$  are pairwise different and are obtained from  $y = y(x_1, \dots, x_p)$  by multiplication of the roots of unity  $\rho_m := e^{2\pi im/n}$  as follows:

$$y_m = y_m(x_1, \dots, x_p) := \rho_m y(\rho_m^{n_1} x_1, \dots, \rho_m^{n_p} x_p), \quad 1 \leq m \leq n-1.$$

The proof, that the  $y_m$ ,  $1 \leq m \leq n-1$ , are indeed the remaining roots of the algebraic equation (8.1), can be found in [17].

This Theorem guarantees the existence of  $n$  different roots of the equation (8.1) for two different algebraic settings, firstly if (8.1) is viewed as an algebraic equation in the ring  $\mathbb{C}[[x_1, \dots, x_p]]$  of formal power series in the  $x_j$  and secondly if (8.1) is viewed as the usual algebraic equation with complex coefficients  $(x_1, \dots, x_p)$  taken from  $U$ .

Finally we note that in the analog way we can also obtain the multivariate series expansion for the  $c$ -th power  $y^c$  of the principle solution  $y$  of (8.1).

We define

$$f(z) := y(z)^c,$$

---

<sup>1</sup>[17], P. Paule, 2009.

where  $y(z)$  and  $\varphi(z)$  as in (8.7) and (8.8). Then we obtain

$$f(z) = y(z)^c = 1 + \sum_{k=1}^{\infty} C_k \frac{w^k}{k!},$$

with

$$\begin{aligned} C_k &= LD^{k-1} f'(z) \varphi^k(z) \\ &= -\frac{c}{n} LD^{k-1} (1-z)^{\frac{c}{n}-1} \left( x_1 (1-z)^{\frac{n_1}{n}} + \cdots + x_p (1-z)^{\frac{n_p}{n}} \right)^k \\ &= -\frac{c}{n} LD^{k-1} \sum_{\substack{j_1+\cdots+j_p=k \\ j_1, \dots, j_p \geq 0}} \frac{k!}{j_1! \cdots j_p!} x_1^{j_1} \cdots x_p^{j_p} (1-z)^{\frac{c}{n}-1+\frac{n_1}{n}j_1+\cdots+\frac{n_p}{n}j_p} \\ &= (-1)^k \frac{c}{n} \sum_{\substack{j_1+\cdots+j_p=k \\ j_1, \dots, j_p \geq 0}} \frac{k!}{j_1! \cdots j_p!} x_1^{j_1} \cdots x_p^{j_p} \left[ \frac{c}{n} - 1 + \frac{n_1}{n}j_1 + \cdots + \frac{n_p}{n}j_p \right]_{k-1}. \end{aligned}$$

We apply Abel's Theorem (see Theorem 4 in Chapter 2) again with  $w_0 = 1$  and derive:

$$\begin{aligned} y(z)^c &= \tag{8.10} \\ 1 + \frac{c}{n} \sum_{\substack{j_1+\cdots+j_p \geq 1 \\ j_1, \dots, j_p \geq 0}} (-1)^{j_1+\cdots+j_p} \frac{x_1^{j_1} \cdots x_p^{j_p}}{j_1! \cdots j_p!} \left[ \frac{c}{n} - 1 + \frac{n_1}{n}j_1 + \cdots + \frac{n_p}{n}j_p \right]_{j_1+\cdots+j_p-1}. \end{aligned}$$

Again this series is absolutely convergent for all  $(x_1, \dots, x_p) \in U$ .

We will now have a closer look at a simple example to show how powerful the technique of Mellin's series is. We consider

$$y^p + xy^q - 1 = 0 \quad \text{with } p > q \geq 1.$$

The Mellin series for the principal solution is

$$y(x) = \sum_{k \geq 0} \binom{(qk+1)/p}{k} \frac{(-x)^k}{qk+1}.$$

This formula also pops up in Ramanujan's notes; see e. g. Ramanujan's Quaterly Report 1.6 in [2].

From Theorem 15 we obtain

$$y(x) = 1 + \frac{1}{p} \sum_{k>0} (-1)^k \left[ \frac{1}{p} - 1 + \frac{q}{p}k \right]_{k-1} \frac{x^k}{k!}$$

The expression  $\left[ \frac{1}{p} - 1 + \frac{q}{p}k \right]_{k-1}$  has to be interpreted as falling factorials (see Definition 4 in Chapter 1), this means in fact that

$$\left[ \frac{1}{p} - 1 + \frac{q}{p}k \right]_{k-1} = \frac{\left(\frac{q}{p}k + \frac{1}{p} - 1\right)!}{\left(\frac{q}{p}k + \frac{1}{p} - 1 - k + 1\right)!} = \frac{\left(\frac{qk+1-p}{p}\right)!}{\left(\frac{qk+1-kp}{p}\right)!}$$

Next,

$$y(x) = 1 + \frac{1}{p} \sum_{k>0} \frac{\left(\frac{qk+1-p}{p}\right)!}{\left(\frac{qk+1-kp}{p}\right)!} \frac{(-x)^k}{k!},$$

because of

$$\frac{\left(\frac{qk+1-p}{p}\right)!}{\left(\frac{qk+1-kp}{p}\right)!p} = \frac{\left(\frac{qk+1}{p} - 1\right)!}{\left(\frac{qk+1}{p} - k\right)!p} = \frac{\left(\frac{qk+1}{p}\right)!}{\left(\frac{qk+1}{p} - k\right)!(qk+1)},$$

is equivalent to

$$y(x) = 1 + \sum_{k>0} \frac{\left(\frac{qk+1}{p}\right)!}{\left(\frac{qk+1}{p} - k\right)!(qk+1)} \frac{(-x)^k}{k!}.$$

Finally, because of

$$\binom{\frac{qk+1}{p}}{k} = \frac{\left(\frac{qk+1}{p}\right)!}{k! \left(\frac{qk+1}{p} - k\right)!},$$

we obtain

$$y(x) = \sum_{k \geq 0} \binom{(qk+1)/p}{k} \frac{(-x)^k}{qk+1}, \quad (8.11)$$

the Mellin series for our simple example  $y^p + xy^q - 1 = 0$ .

**Note:** Here  $z! = \Gamma(z+1)$ , i. e., the gamma function introduced in Definition 2 in Chapter 1.

A problem with the Mellin series arises when we consider convergence of the series. The coefficients have to lie in a circle with a small radius, depending on the given powers of the equation. For example, according to (8.9), the series (8.11) converges only for all  $x \in U$ , with

$$U = \{x \in \mathbb{C} : |x| < r_1\},$$

where

$$r_1 = \left(\frac{q}{p}\right)^{-\frac{q}{p}} \left(1 - \frac{q}{p}\right)^{\frac{q}{p}-1}. \quad (8.12)$$

Finally consider the algebraic equation<sup>2</sup>

$$xqy^p + y^q - 1 = 0, \quad (8.13)$$

where  $x \in \mathbb{R}$ ,  $x > 0$ , and  $p, q \in \mathbb{N}^*$  with  $0 < q < p$ , Ramanujan found the following expansion for  $y^n$ ,  $n \in \mathbb{N}^*$ , in nonnegative powers of  $x$ :

$$y^n(x) = \frac{n}{q} \sum_{k=0}^{\infty} \frac{\Gamma((n+pk)/q) \cdot (-qx)^k}{\Gamma((n+pk)/q - k + 1) \cdot k!}. \quad (8.14)$$

This is nothing but a variant of the generalized Mellin Series (8.10).

According to (8.9) the series (8.14) converges for all

$$x < (p)^{-\frac{p}{q}} (p - q)^{\frac{p-q}{q}}.$$

Ramanujan derived (8.14) differently; see [B.C.Berndt, chapter 3, Entry 14].

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<sup>2</sup>[3], B.C.Berndt, page 306, 1985.







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