

# The Polynomial Growth of an Operator Ideal

Manuel Kauers (RISC)

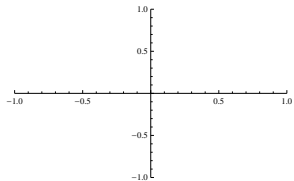
joint work with

Frederic Chyzak and Bruno Salvy (INRIA)

$$\sum_{k=0}^n \frac{2k+1}{k+1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)$$

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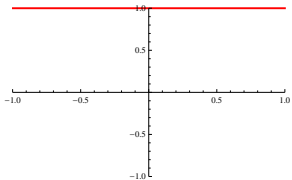
Legendre polynomials:



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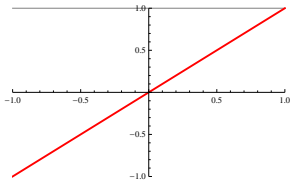
►  $P_0(x) = 1$



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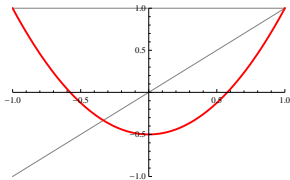
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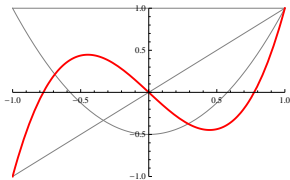
- ▶  $P_0(x) = 1$
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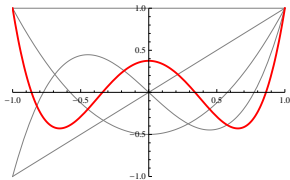
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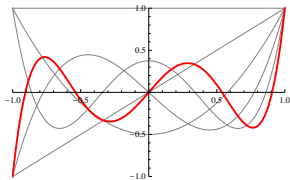
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- ▶  $P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5)$
- ▶ ...



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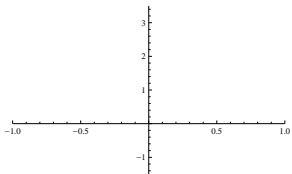
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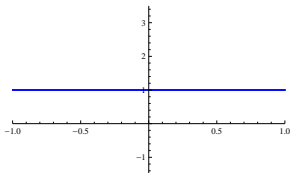
Jacobi polynomials:



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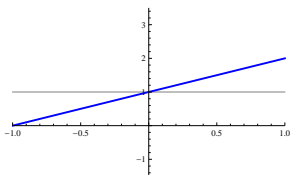
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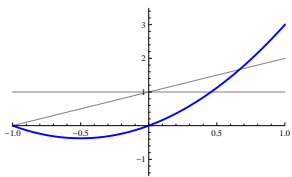
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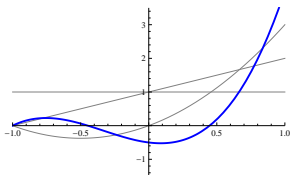
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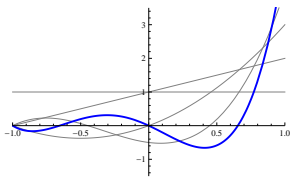
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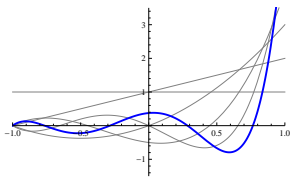
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- ▶  $P_5^{(1,-1)}(x) = \frac{3}{8}(1 + x - 14x^2 - 14x^3 + 21x^4 + 21x^5)$
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How to prove this identity?

$$\sum_{k=0}^n \frac{2k+1}{k+1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)$$

How to prove this identity?  $\longrightarrow$  By induction!

$$\sum_{k=0}^n \frac{2k+1}{k+1} P_k^{(1,-1)}(x) - \frac{1}{1-x} (2 - P_n(x) - P_{n+1}(x)) = 0$$

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How to prove this identity?  $\longrightarrow$  By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.

$$\sum_{k=0}^n \underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence} \\ \text{of order 1}}} P_k^{(1,-1)}(x) - \frac{1}{1-x} (2 - P_n(x) - P_{n+1}(x)) = 0$$

$$\sum_{k=0}^n \underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}} - \frac{1}{1-x} (2 - P_n(x) - P_{n+1}(x)) = 0$$

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recurrence of order 2

$$\sum_{k=0}^n \underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}} - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0$$

recurrence of order 2

recurrence of order 5

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recurrence of order 7

$$\sum_{k=0}^n \frac{2k+1}{k+1} P_k^{(1,-1)}(x) - \frac{1}{1-x} (2 - P_n(x) - P_{n+1}(x)) = 0$$

$$\begin{aligned} \text{lhs}(n+7) &= (\dots \text{messy} \dots) \text{lhs}(n+6) \\ &\quad + (\dots \text{messy} \dots) \text{lhs}(n+5) \\ &\quad + (\dots \text{messy} \dots) \text{lhs}(n+4) \\ &\quad + (\dots \text{messy} \dots) \text{lhs}(n+3) \\ &\quad + (\dots \text{messy} \dots) \text{lhs}(n+2) \\ &\quad + (\dots \text{messy} \dots) \text{lhs}(n+1) \\ &\quad + (\dots \text{messy} \dots) \text{lhs}(n) \end{aligned}$$

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Therefore the identity holds *for all*  $n \in \mathbb{N}$   
if and only if it holds *for*  $n = 0, 1, 2, \dots, 6$ .

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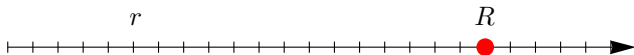
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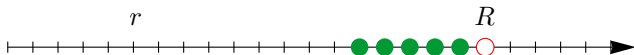
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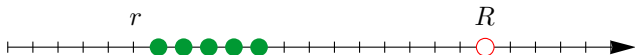
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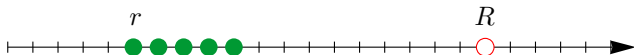
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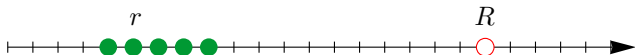
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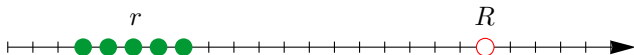
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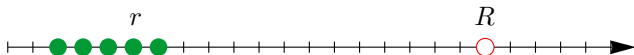
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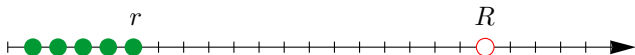
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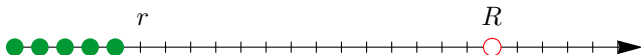
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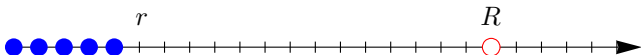
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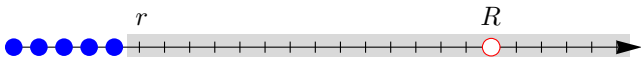
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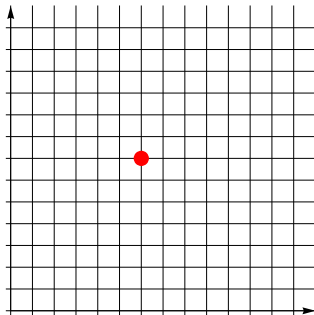
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Also a multivariate recurrence for  $f_{n,k}$  like

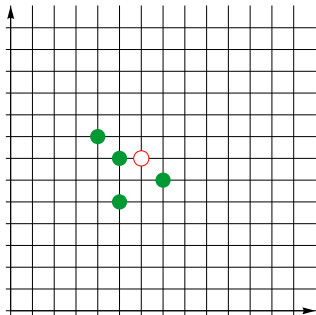
$$p_{2,2}(n, k)f_{n+2, k+2} + p_{0,3}(n, k)f_{n, k+3} + p_{1,2}(n, k)f_{n+1, k+2} \\ + p_{1,0}(n, k)f_{n+1, k} + p_{3,1}(n, k)f_{n+3, k+1} = 0$$

can be used for reducing a term  $f_{n+U, k+V}$  to “smaller” ones.

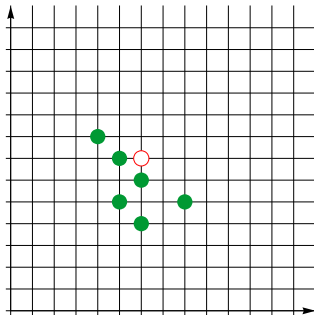
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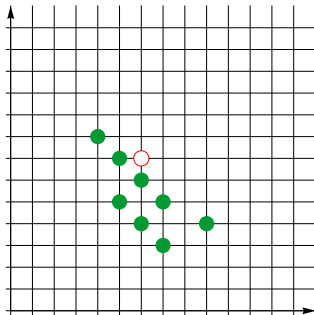
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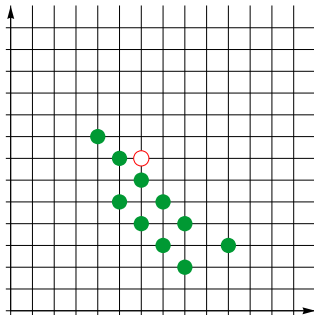
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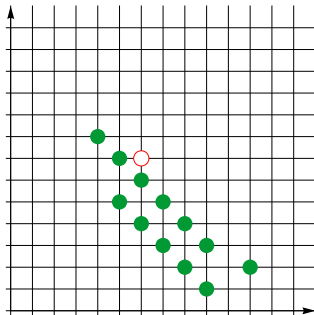
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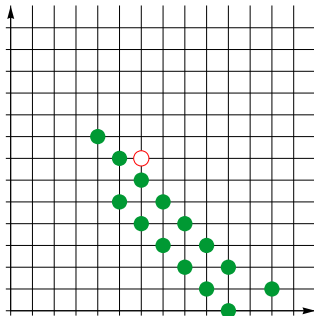
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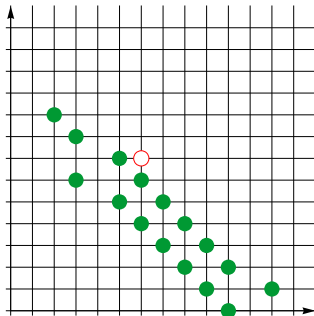
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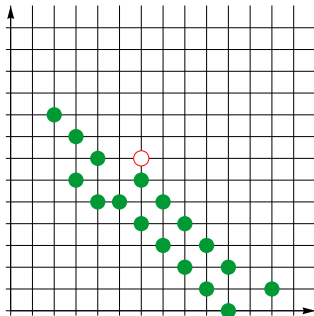
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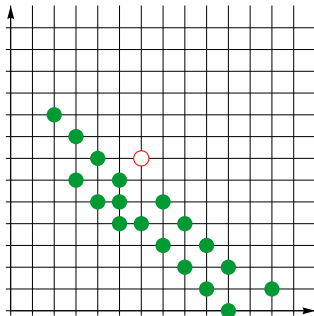
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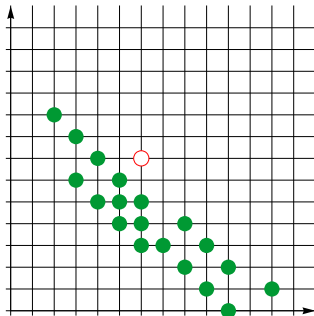
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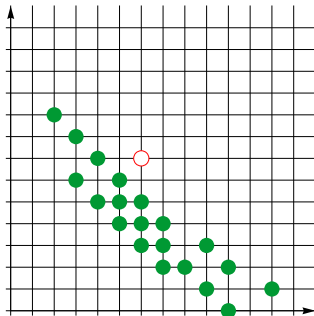
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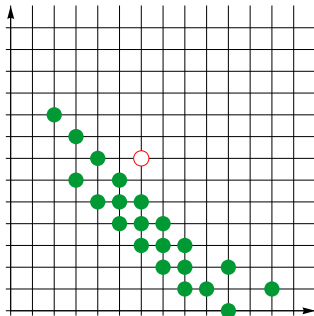
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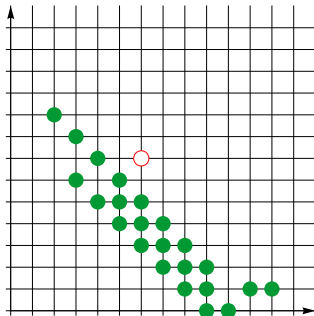
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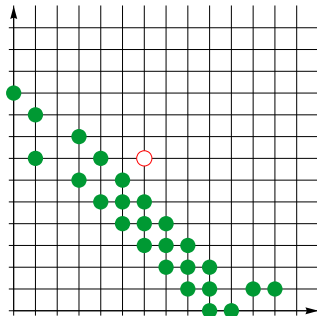
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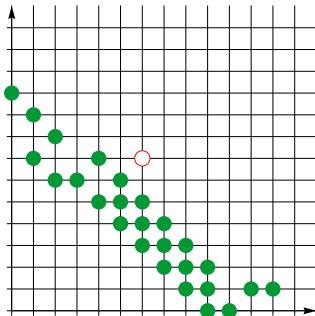
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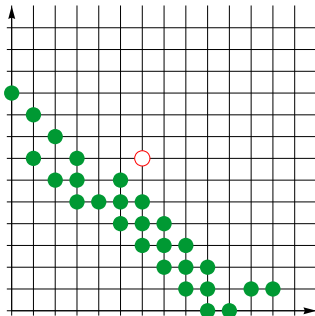
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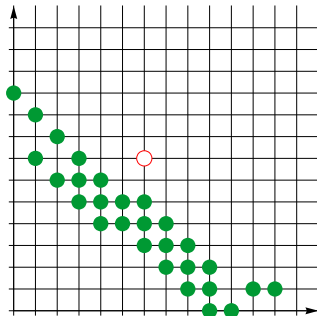
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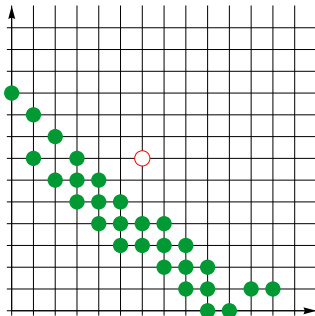
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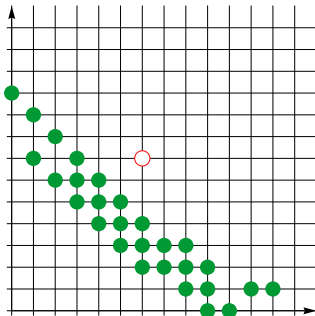
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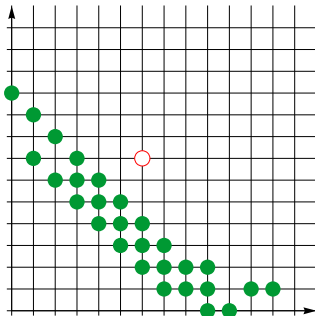
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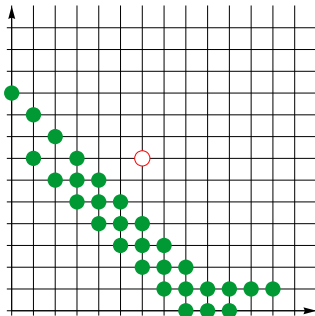
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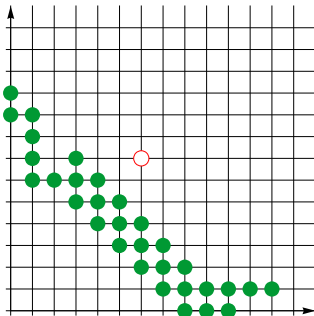
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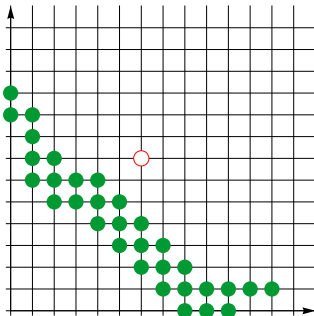
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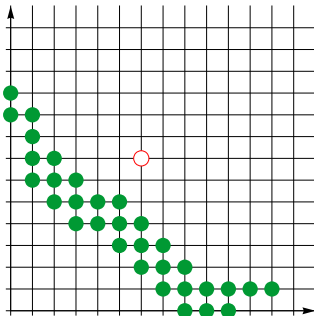
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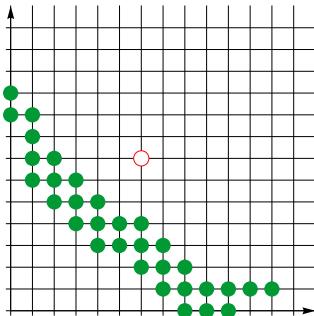
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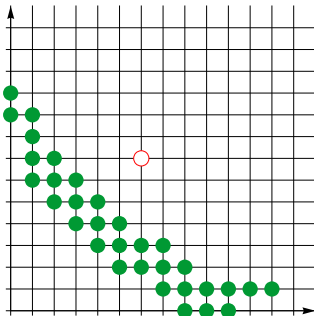
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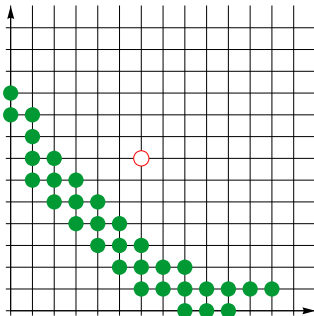
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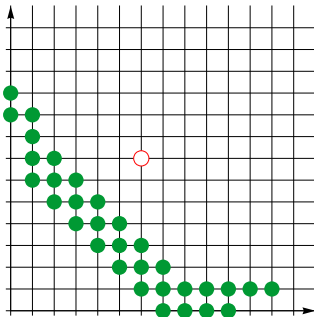
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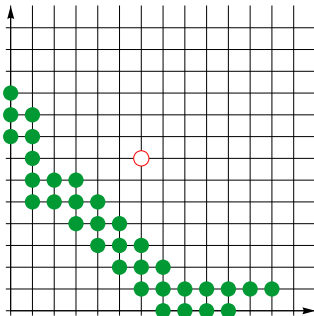
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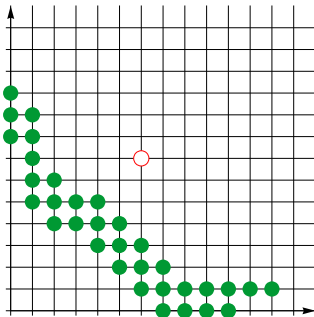
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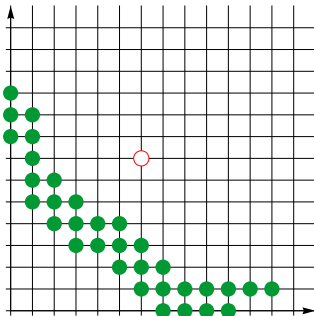
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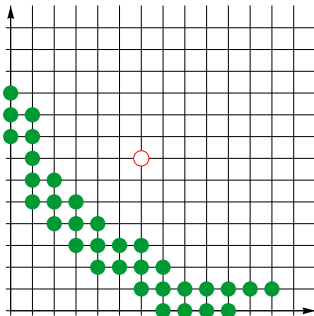
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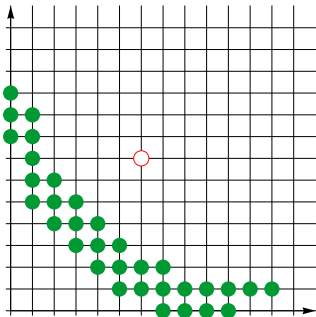
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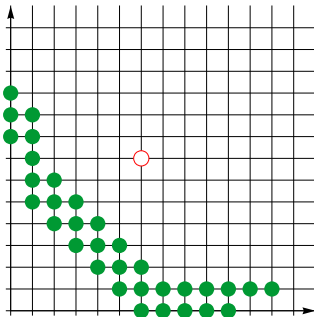
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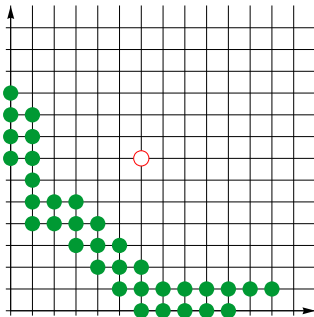
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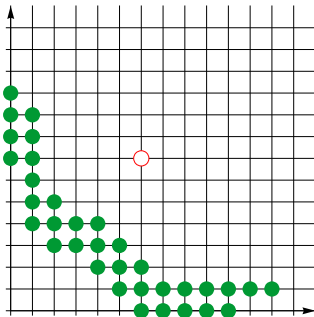
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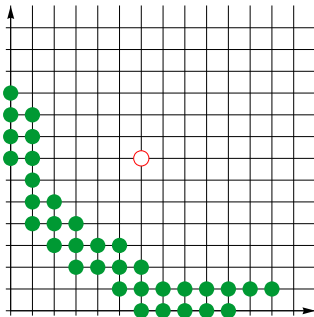
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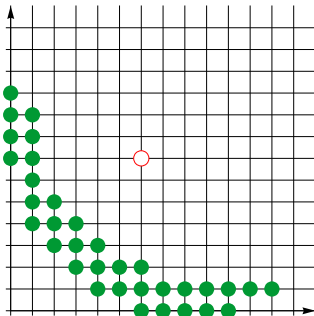
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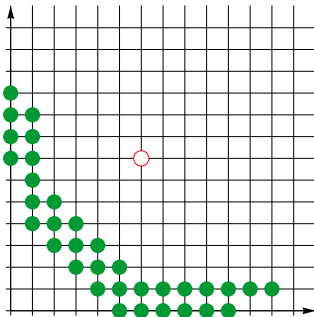
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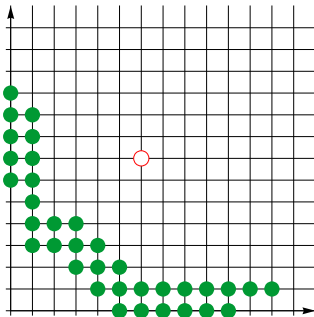
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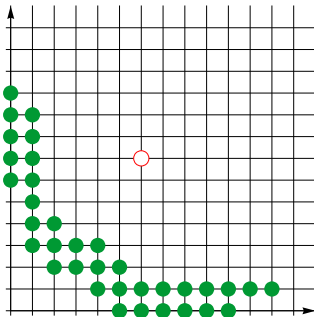
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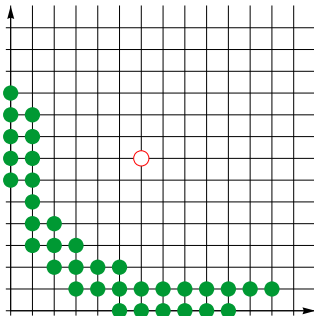
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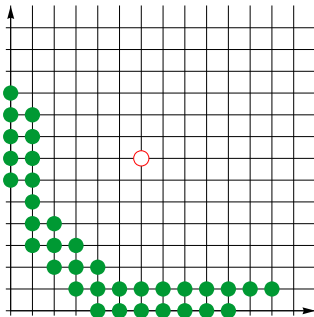
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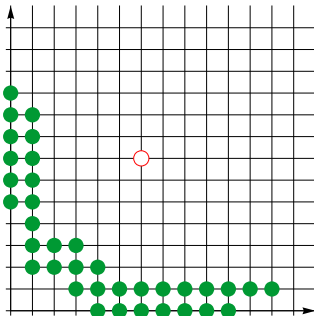
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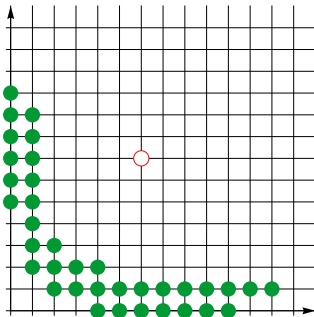
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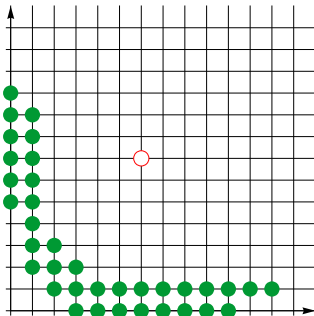
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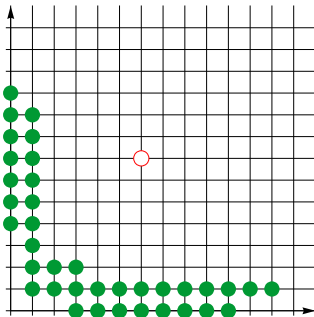
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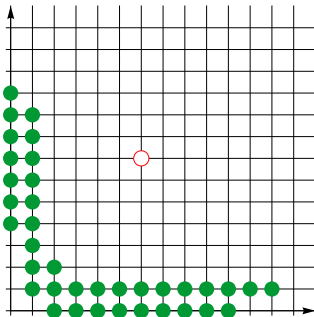
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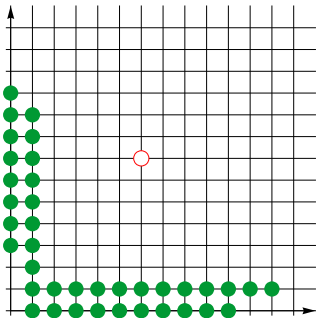
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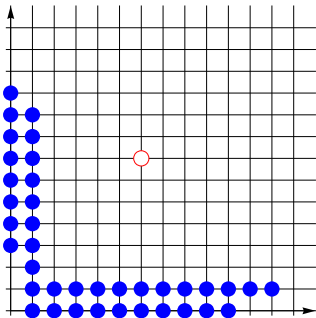
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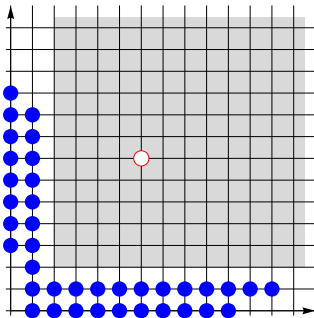
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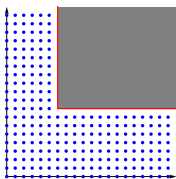
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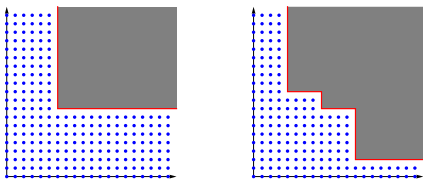


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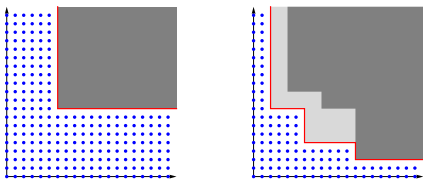
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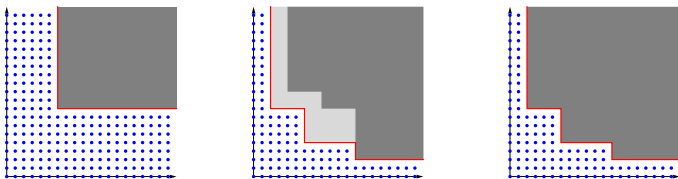
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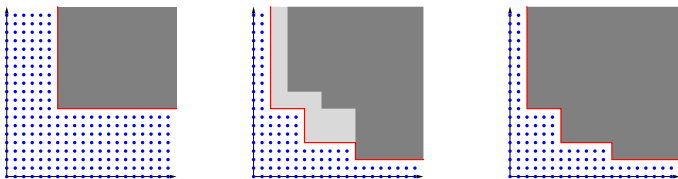
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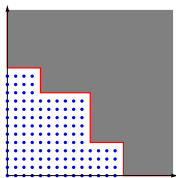
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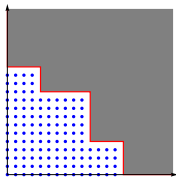
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  - ▶ From now on, all systems are assumed to be Gröbner bases.

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$f(x, y)$  is *D-finite* if it satisfies a system of multivariate differential equations with polynomial coefficients of this form.

*Main feature:* If  $f_{n,k}$  and  $g_{n,k}$  are D-finite then so are

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Defining systems for all these can be computed from defining systems of  $f$  and  $g$ .

The results generalize to functions

$$f_{n_1, n_2, \dots, n_s}(x_1, x_2, \dots, x_r)$$

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We can exploit that in general  $\infty \neq \infty$ .

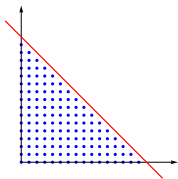
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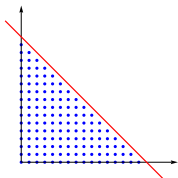
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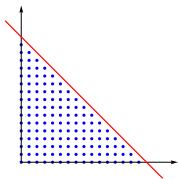
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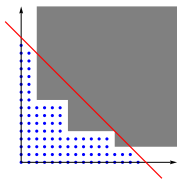
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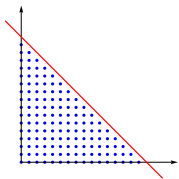


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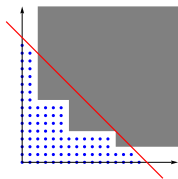


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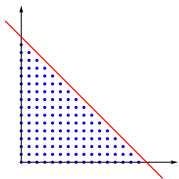
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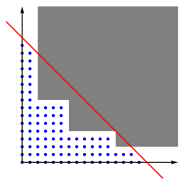
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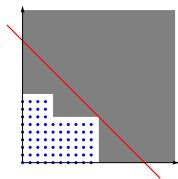
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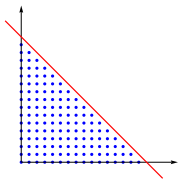


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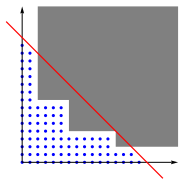


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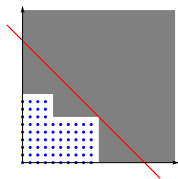
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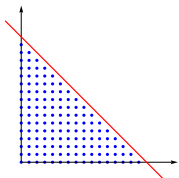
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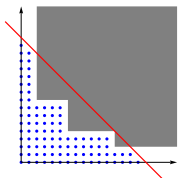
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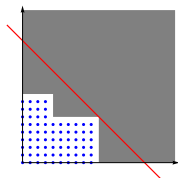
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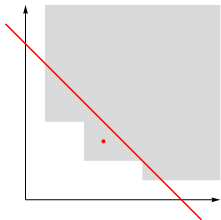
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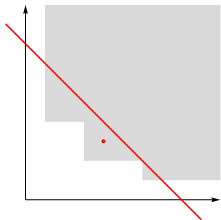
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Answer: It's a number we call the *polynomial growth* of  $A(f)$ .

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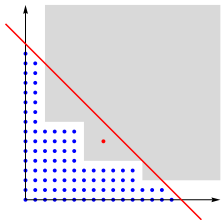


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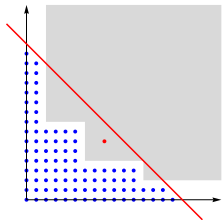
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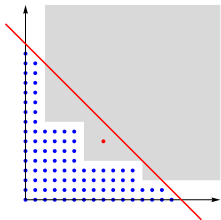
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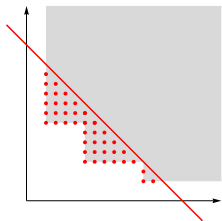
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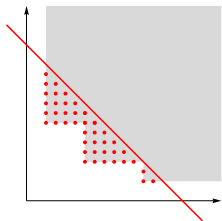


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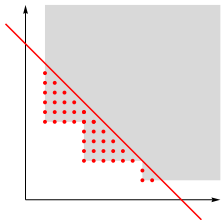


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😊 If  $f_{n,k}$  is hypergeometric then

$$\text{pol } A(f) = 1 \iff f_{n,k} \text{ is proper}$$

😊 If  $f_{n,k}$  is D-finite then

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