Mathematical Methods in Kinematics
Lecture Notes, JKU SS 2018

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1 Computational Problems in Kinematics

Kinematics is the geometric theory of motions of rigid bodies in 3-space or planar figures in the plane. The rigid bodies are combined to mechanical linkages (aka linkages aka mechanisms). A linkage consists of a set of rigid bodies called *links*; two of these bodies may be connected by a joint. A joint between two links restricts the relative position of the two rigid bodies with respect to each other in a certain way. For instance, the relative position might be a obtained by rotation around a fixed axis which is common to both. This gives a one-dimensional set of relative positions, whereas for two links that are not connected by a joint there is a six-dimensional set of relative positions.

A configuration of a linkage is defined as the collection of relative positions of the links with respect to one another. It suffices to specify the relative positions of each pair of links that is connected by a joint; the other relative positions can then be inferred.

Relative positions are mathematically described by elements of the Euclidean group SE_3. This is a group of dimension 6, generated by te three-dimensional group of rotations and by the three-dimensional group of translations (more details follow later). In order to descrive relative positions unambiguously, one needs to specify:

- for each link, a coordinate frame (consisting of an origin and three oriented lines defining the coordinate axes);
- for each joint, an initial relative posiition of the connected links with respect to each other.

We will see this later in more detail; in fact we will use this choices to our advantage, making the equations as simple as possible. For the moment, I would like to draw your attention to the fact that the mathematical description is by no means unique, but it depends on choices which are quite, in principle, quite arbitrary.

2 Differential Manifolds

Let $m,n \in \mathbb{N}$ be natural numbers. Let $U \subset \mathbb{R}^n$ be an open subset. A function $f : U \to \mathbb{R}^m$ is called *differentiable* if it can be locally approximated by a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$. More precisely, for every point $p \in U$, there exists a neighborhood $V \subset U$ and a linear function $L : \mathbb{R}^n \to \mathbb{R}^m$ such that the function from $V \setminus \{p\}$ to $\mathbb{R}^m$

$$y \mapsto \frac{f(x) - f(p) - L(p-x)}{||p-x||}$$

can be continuously extended by zero into p. The linear function L is called the differential of f at p. The matrix representing it is also called the Jacobi-matrix of f at p and we write it as $\left(\frac{\partial f}{\partial x}\right)|_{x=p}$.

The differential of diffential function satisfies the chain rule: if $f : (U \subset \mathbb{R}^n) \to (V \subset \mathbb{R}^m)$ and $g : V \to \mathbb{R}^l$ are differentiable, and $p \in U$, then

$$\left(\frac{\partial g}{\partial y}\right)|_{y=f(p)} \left(\frac{\partial f}{\partial x}\right)|_{x=p}= \left(\frac{\partial (g \circ f)}{\partial x}\right)|_{x=p}$$

If a diffential map is bijective and its inverse is again differentiable, then its Jacobi-matrix is invertible at every point. Such a map is also called a *diffeomorphism*.
We recall the following theorem from Analysis, without proof:

Theorem 2.1 (Inverse Function). *If $f : (U \subset \mathbb{R}^n) \to \mathbb{R}^n$ has an invertible Jacobi-matrix at p, then there is an open neighborhood V of p such that $f|_V : V \to f(V)$ is a diffeomorphism.*

The inverse function has several consequences.

Theorem 2.2 (Implicit Function). *Let $U \subset (\mathbb{R}^n \times \mathbb{R}^n)$, and let $f : U \to \mathbb{R}^m$ be a differentiable function. Assume $(p, q) \in U$ such that $f(p, q) = 0$ and that the submatrix of the Jacobian $\left(\frac{\partial f}{\partial y}\right)|_{(x,y)=(p,q)}$ is invertible. Then there exists a neighborhood V of p and a function $g : V \to \mathbb{R}^m$ such that $g(p) = q$ and $f(x, g(x)) = 0$ for all $x \in V$.*

Proof. We define $h : U \to \mathbb{R}^{n+m}$, $(x, y) \mapsto (x, f(x, y))$. The Jacobi-matrix of h is

$$\left(\frac{\partial h}{\partial (x,y)}\right) = \begin{pmatrix} I_n & 0 \\ \left(\frac{\partial f}{\partial x}\right) & \left(\frac{\partial f}{\partial y}\right) \end{pmatrix}$$

and this is invertible at $(x, y) = (p, q)$. By Theorem 2.2, there is an inverse function, which necessarily has the form $(x, y) \mapsto (x, k(x, y))$, with $k : V \to \mathbb{R}^m$ such that $f(x, k(x, y)) = y$ for all (x, y) in some suitable neighborhood of (p, q). Note that $h(p, q) = (p, 0)$, hence we also have $k(p, 0) = q$.
Exercise: finish the proof by defining g such that the properties stated in the Theorem are fulfilled. □

Theorem 2.3 (Linearization). *Let $f : (U \subset \mathbb{R}^n) \to \mathbb{R}^n$ be differentiable. Let $k \in \mathbb{N}$ and assume that for all $x \in U$, the rank of the Jacobi-matrix is k. For every point p, there are open neighborhoods V of p and W of $f(p)$ and diffeomorphisms $s : V' \to V$ and $t : W \to W'$ such that the composed map $t \circ f \circ s : V' \to W'$ is a linear map of rank k.*

Proof. Without loss of generality, we assume that the submatrix consisting of the first k rows and columns is invertible. We decompose $\mathbb{R}^n$ into $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\mathbb{R}^m$ into $\mathbb{R}^k \times \mathbb{R}^{m-k}$, so that f becomes $(x, y) \to (u(x, y), v(x, y))$ for functions $u : U \to \mathbb{R}^k$ and $v : U \to \mathbb{R}^{n-k}$. We write $p = (q, r)$ with $q \in \mathbb{R}^k$ and $r \in \mathbb{R}^{n-k}$. Then $\left(\frac{\partial u}{\partial x}\right)$ is invertible at $p = (q, r)$.
We define $h : U \to R^k \times \mathbb{R}^{n-k}$, $(x, y) \mapsto (u(x, y), y)$. Its Jacobi-matrix is

$$\left(\frac{\partial h}{\partial (x,y)}\right) = \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}\right) \\ 0 & I_{n-k} \end{pmatrix},$$

and this is inverible at p. By Theorem 2.2, there is an inverse. We define s as this inverse on some neighborhood V of p. Because the first k components of f and of h coincide, the first k functions of $f \circ s$ are just the coordinate functions $x_1, \ldots, x_k$. It follows that

$$\left(\frac{\partial (f \circ s)}{\partial (x,y)}\right) = \begin{pmatrix} I_k & 0 \\ A & B \end{pmatrix},$$

for matrices $A \in \mathbb{R}^{(m-k)\times k}$ and $B \in \mathbb{R}^{(m-k)\times(n-k)}$, for all $(x, y) \in V'$. By the chain rule, the rank of the Jacobi-matrix of $f \circ s$ is also k, and it is apparent that the first k columns are linearily independent and generate the column space. But no nontrivial linear combinarion of the first k

columns can have only zero in the first k rows. Therefore the rows $k+1,\dots,m$ are zero. This implies that the value of the function $f\circ s$ does not depend on the coordinates $x_{k+1},\dots,x_n$, in other words $(f\circ s)(x,y)=(x,v(x))$ for some $v:\mathbb{R}^k\to\mathbb{R}^{m-k}$.
Now we set $W:=f(V)$ and $t:W\to\mathbb{R}^m$, $(x,y)\mapsto(x,y-v(x))$. Then $t\circ f\circ s$ maps (x,y) to $(x,0)$. □

We will give two definitions of manifolds, a concrete one and an abstract one. The abstract one is available in `https://en.wikipedia.org/wiki/Manifold` and is used frequently in the literature. But we first give the concrete definition because it provides a better intuition.

Definition 2.1. Let n,k be natural numbers such that $k\le n$. Let $U\subset\mathbb{R}^n$. If a non-empty set $X\subset U$ is the preimage of zero of a differential map $f:U\to\mathbb{R}^{n-k}$ that has maximal Jacobian rank $n-k$, then we say that X is a *manifold* of dimension k.
Let $p\in X$. Then the tangent space of X at p, denoted by T_pX, is defined as the kernel of the Jacobi-matrix. It is a vectorspace of dimension k.

Example 2.4. Let $n=2$, $k=1$, and $U=\mathbb{R}^2$. Then the set $\{(x,y)\mid x^2+y^2-1=0\}$ is a manifold of dimension 1: we can choose $f:(x,y)\mapsto x^2+y^2-1$, its Jacobian is the gradient which has rank everywhere on the circle. On the other hand, the set $\{(x,y)\mid x^2+x^3-y^2=0\}$ is not a manifold: the gradient of the function $(x,y)\mapsto x^2+x^3-y^2$ vanishes at $(0,0)$, so the Jaconi rank is not constant. We can repair this easily by defining U as the complement of the origin: the zero set in U is a manifold.

Exercise 2.2. Prove or disprove that the subset of $\mathbb{R}^3$ defined as $\{(x,y,z)\mid x-yz=xz-y^2=0\}$ is a manifold. If it is, what is its dimension?

If $X\subset\mathbb{R}^n$ and $Y\subset\mathbb{R}^m$ are manifolds, then a differentiable function from X to Y is a function $f:X\to Y$ which is the the restriction of a differentiable function from an open set of $\mathbb{R}^n$ containing X to $\mathbb{R}^m$. If $f:X\to Y$ is a differentiable function, and $p\in X$, then the differential of f at p is a linear map $df|_p:T_pX\to T_{f(p)}Y$. A diffeomorphism of manifolds is a differentiable map with differentiable inverse. As a consequence of the linearization theorem, every manifold of dimension k is locally diffeomorphic to $\mathbb{R}^k$.
For the abstract definition of a manifold, we would like to say something like "a manifold of dimension k is a topological space that is locally diffeomorphic to $\mathbb{R}^k$". Unfortunately, we cannot simply state it in that simple way. The reason is that the definition of a a diffeomorphism requires the definition of a differentiable function, and for abstract topological spaces we have no notion of differentiability available. So we need to set up the definition in a roundabout, as follows.

Definition 2.3. Let k be a natural number. A *topological manifold* is a topological space X such that X is Hausdorff, its topology has a countable base, and X is locally homeomorphic to $\mathbb{R}^k$: for every point $p\in X$, there is an open neighborhood U and a map $\phi_U:U\to\mathbb{R}^k$ such that its image $V\subset\mathbb{R}^k$ is open and $\phi:U\to V$ is a homeomorphism. Any such map is called a *chart*. A collection of charts such that the domains cover X is called an *atlas* of X. Finally, a manifold is a topological manifold together with an atlas such that for any two charts $\phi_1:U_1\to V_1$, $\phi_2:U_2\to V_2$, the chart change map $\psi_{12}:=\phi_1\circ\phi_2^{-1}:\phi_2(U_1\cap U_2)\to\phi_1(U_1\cap U_2)$ is differentiable.

Example 2.5. Let $X\subset\mathbb{R}^n$ be a concrete manifold of dimension k. Every subset of $\mathbb{R}^n$ is Hausdorff and its topology has a countable base, namely the intersections with open balls with rational center and rational radius. Let U be an open set containing X and $f:U\to\mathbb{R}^{n-k}$ be differentiable function with maximal Jacobian rank $n-k$ such that $X=U\cap f^{-1}(\{0\})$. Then X is locally diffeomorphic to $\mathbb{R}^k$, and these locally defined diffeomorphism form an atlas making X to an abstract manifold.
In order to construct an atlas explicitly, let $p\in X$ be an arbitrary point. The Jacobian of f has rank $n-k$ at p. Choose a subset of $n-k$ variables whose columns in the Jacobi matrix are linear independent and call them $(x_1,\dots,x_{n-k})=:x$. The remaining variables are denoted by $(y_1,\dots,y_k)=:y$. Recall that X is defined by the equation $f(x,y)=0$. By the implicit function

theorem, there is an open neighborhood of U_p and a function $g_p : V_p \to U_p$, with V_p defined as the projection of U_p to the x-variables, such that $f(x, g(x)) = 0$ for all $x \in V_p$. We define the chart $\phi_p : U_p \to V_p$ as the projection. The inverse is $x \mapsto (x, g_p(x))$. The chart change maps are differential because they are compositions of differentiable maps. (If the chosen subset of variables is the same, than the chart change map is actually the identity.)

Exercise 2.4. Compute an atlas for the unit circle $C := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\}$ by the construction in Example 2.5. How many charts are needed to cover the circle?
Since C is not homeomorphic to an open set in $\mathbb{R}$, there is no atlas with only one chart. Try to construct an atlas consisting of two charts! (Hint: the above construction is not so helpful.)

Example 2.6 (projective space)**.** Let V by an $\mathbb{R}$-vectorspace of finite dimension $n + 1$. The projective space $\mathbb{P}^n$ or $\mathbb{P}(V)$ is defined as the set of all equivalence classes of nonzero vectors by the equivalence $v \sim \lambda v$ for $v \in (V - \{0\}), \lambda \in \mathbb{R}^*$. It is a topological space by the quotient topology: a set of equivalence classes is open if and only if the set of all its vectors is open. In this exercise, we show that it is also a manifold.
Let $H \subset V$ be a subspace of dimension n, and define U_H as the set of all equivalence classes of vectors in $V - H$. Let H_1 be an affine hyperplane parallel to H, but different from it (not passing through the origin). Then every equivalence class in U_H has a unique representative in H_1. This defines a map $U_H \to H_1$. We compose it with some isomorphism $H_1 \to \mathbb{R}^n$ and get the chart ϕ_H. In order to compute the chart change map of $\psi_{HK} = \phi_K \circ \phi_H^{-1}$ for two linear subspaces H, K of dimension n, we choose coordinates $(x_1, \ldots, x_{n+1})$ such that K has equation $x_1 = 0$ and H has equation $x_2 = 0$. A point in $p \in (U_H \cap U_K)$ has projective coordinates $(x_1 : x_2 : \cdots : x_{n+1})$ with $x_1, x_2 \neq 0$. Let me explain this notation: $(x_1 : x_2 : \cdots : x_{n+1})$ is just a funny way to say $[(x_1, \ldots, x_{n+1})]_\sim$. It is commonly used for the the classes corresponding to points in projective space, and the name for it is “projective coordinates”. So, as mentioned above, a point in $U_H \cap U_K$ has projective coordinates $(x_1 : x_2 : \cdots : x_{n+1})$ with $x_1, x_2 \neq 0$. We choose H_1 as the set defined by $x_1 = 1$, and K_1 as the set defined by $x_2 = 1$. Then we obtain

$$\phi_H(p) = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_{n+1}}{x_1}\right),\ \phi_K(p) = \left(\frac{x_1}{x_2}, \frac{x_3}{x_2}, \ldots, \frac{x_{n+1}}{x_2}\right),$$

and therefore ψ_{HK} maps $(y_1, \ldots, y_n)$ to $(y_1^{-1}, y_1^{-1}y_2, \ldots, y_1^{-1}y_n)$ (the map is only defined if $y_1 \neq 0$). In particular, all chart change maps are differentiable and so $\mathbb{P}^n$ is a manifold.

We recapitulate: in order to define manifolds abstractly, in particular to define what differentiability means, we introduced charts and reduced the matter to function from $\mathbb{R}^k$ to $\mathbb{R}^k$ (the chart change maps). One could go one step further and asking: is the topological space X really necessary? Maybe it is enough, in order to specify an atlas defining a manifold, to give only the open sets $V_i \in \mathbb{R}^k$ and the chart change maps, and we do not need the U_i and X at all?
There is indeed a construction of a manifold of dimension k from the chart change maps alone. It starts with *glueing data* consisting of

- a sequence of open sets $(V_i)_{i \in I}$ in $\mathbb{R}^k$, indexed by a finite or countable set index set I;
- for each pair (i, j) of indices, an open subset $V_{ij} \subset V_i$;
- for each pair (i, j) of indices, a homeomorphism $\psi_{ij} : V_{ji} \to V_{ij}$, such that for each triple (i, j, k) of indices, the functions $\psi_{ij} \circ \psi_{jk}$ and ψ_{ik} coincide on the set $V_{ik} \cap V_{jk}$ (in particular, the left hand side is defined at these points).

Unfortunately it is not in general true that glueing data define manifolds.

Example 2.7. We set $k = 1$, $I = \{1, 2\}$, $V_1 = V_2 = \mathbb{R}$, $V_{12} = V_{21} = (-\infty, 0)$, and $\psi_{ij} : V_{ji} \to V_{ij}$ is always the identity on $(-\infty, 0)$. If this gluing data would define a manifold, it better be the union of open subsets $U_1 \cup U_2$, there are isomorphisms $\phi_1 : U_1 \to \mathbb{R}$ and $\phi_1 : U_1 \to \mathbb{R}$, and we have $\phi_1(U_1 \cap U_2) = \phi_2(U_1 \cap U_2) = (-\infty, 0)$. Then the two points $p_1 := \phi_1^{-1}(0)$ and $p_2 := \phi_2^{-1}(0)$

are distinct. Any open neighborhood of p_1 intersects $U_1 \cap U_2$, and the image of this intersection under ϕ_1 (or ϕ_2, does not matter) contains an interval of the form $(-\epsilon, 0)$. The same holds for any open neighborhood of p_2. Hence p_1 and p_2 do not have disjoint neighborhoods, contradicting Hausdorff.

In this lecture we will not use the glueing construction for manifolds.
If X is a manifold of dimension n and Y is a manifold of dimension m, then a continuous map $f : X \to Y$ is said to be differentiable of the composition with the charts and their inverse are differentiable maps from $(U_i \subset \mathbb{R}^n)$ to $\mathbb{R}^m$. The theorems above – inverse function theorem, implicit function theorem, linearisation – can be extended to the manifold case (the rank of the Jacobian does not depend on the choice of the chart). Here is an interesting consequence.

Corollary 2.8. *Let X, Y be manifolds of dimension n, m. Let $f : X \to Y$ be differentiable map. Let k be the maximal rank of the Jacobian. Let $X_0 \in X$ be the set of all points in X where the Jacobian has rank k. Let $y \in Y$ be a point in $f(X_0)$. Then $f^{-1}(y)$ is a manifold of dimension $n - k$.*

Proof. First, the subset $X_0 \subset$ can be defined by inequalities of subdeterminants of the Jacobian, so it is open and therefore it is a manifold. By linearisation, any point in $f^{-1}(y)$ is locally isomorphic to a linear subspace of dimension k in $\mathbb{R}^m$. □

Caution: the image of a differential map need not, in general, be a manifold. It could have self-intersections. Even f is injective, the image need not be a topological manifold, because image points could lie "close to each other". Here is an instructive example.

Example 2.9. Let $X := \mathbb{R}$ and let $Y := S^1 \times S^1$ be the torus. Let $f :\to Y$ be the map $t \mapsto (((\cos(t), \sin(t)), ((\cos(\pi t), \sin(\pi t)))$ (instead of π you can choose any irrational number). Then f is injective, but the image is dense in Y, and so it is not a manifold of dimension 1.

The tangent space of a manifold at a point could be defined by charts, but such a definition would not give good intuition. In the following we give a definition following the idea that "tangent vectors are velocities of curves on a manifold". We represnt tangent vectors at p by curves passing through p. At the same time, we also define co-tangent vectors, which are represented by real-valued functions.

Definition 2.5. Let X be a manifold of dimension n. Let $p \in X$ be a point. Let $\mathrm{Hom}_{0,p}(\mathbb{R}, X)$ be the set of all differentiable maps $\mathbb{R} \to X$ sending 0 to p. Let $\mathrm{Hom}_{p,0}(X, \mathbb{R})$ be the set of all differentiable maps $X \to \mathbb{R}$ sending p to 0. For $\alpha, \beta \in \mathrm{Hom}_{0,p}(\mathbb{R}, X)$, we write $\alpha \sim \beta$ if for all $f \in \mathrm{Hom}_{p,0}(X, \mathbb{R})$ the equation $(f \circ \alpha)'|_0 = (f \circ \beta)'|0$ holds. For $f, g \in \mathrm{Hom}_{p,0}(X, \mathbb{R})$, we write $f \sim g$ if for all $\alpha \in \mathrm{Hom}_{0,p}(\mathbb{R}, X)$ the equation $(f \circ \alpha)'|_0 = (g \circ \alpha)'|_0$ holds. Then the quotient sets $\mathrm{Hom}_{0,p}(\mathbb{R}, X)/\sim$ and $\mathrm{Hom}_{p,0}(X, \mathbb{R})/\sim$ are vector spaces over $\mathbb{R}$ with dimension n and they are dual to each other; the duality is defined by composing representatives and evaluating the derivative at 0. The first space is called the tangent space T_pX, and the second cotangent space T_p^*X of X at p.
Let $f : X \to Y$ be a differentiable map. Let $p \in X$. Composition with f gives the functions $f^* : \mathrm{Hom}_{0,p}(\mathbb{R}, X) \to \mathrm{Hom}_{0,f(p)}(\mathbb{R}, Y)$ and $\mathrm{Hom}_{f(p),0}(Y, \mathbb{R}) \to \mathrm{Hom}_{p,0}(X, \mathbb{R})$. The maps are well-defined modulo $\sim$ and give linear maps $df|_p : T_p(X) \to T_{f(p)}Y$ and its dual $f^* : T^*_{f(p)}Y \to T^*_pX$.

The definition above requires a little proving, but the required statement are straightforward when one composes with a chart containg p.

3 Groups

Let G be a group, and let $H \subset G$ be a subgroup (which is a subset closed under multiplication and under taking the inverse). Then we say that H is normal if it is invariant under conjugation. Here, conjugation by an element $g \in G$ is the map $\mathrm{conj}_a : G \to G$, $x \mapsto g^{-1}xg$. Conjugation is a group automorphism of G, i.e. a group homomorphism from G to itself: for all $x, y, g \in G$, we have $\mathrm{conj}_g(xy) = \mathrm{conj}_g(x)\mathrm{conj}_g(y)$. And: the conjugation map $g \mapsto \mathrm{conj}_g$ is a group antihomomorphism from the group G to the group $\mathrm{Aut}(G)$ of automorphisms of G: for all $g, h \in G$, we have $\mathrm{conj}_{gh} = \mathrm{conj}_h \circ \mathrm{conj}_g$. It is also possible to define conjugation in such a way that $g \mapsto \mathrm{conj}_g$ is a homomorphism (one just needs to put the inverse to the other side). We stick to the convention that defines it as an antihomomorphism because it has other advantages (conjugation is a right action).

The trivial subgrous $\{e\}$ and G are always normal. If G is abelian, then conjugation by any element is always the identity. Consequently, all subgroups are normal.

The normal subgroups of the permutation group S_n, where n is a natural number bigger than 1, are completely known. If $n \neq 4$, then the only normal subgroups are the trivial subgroups and the alternating group A_n, which is the subgroup of permutations with positive sign. For $n = 4$, there is another normal subgroup V_4 with 4 elements, going under the name "Klein group".

Exercise 3.1. For $n \leq 5$, the proof of the above classification is a "fun exercise". First, every permutation g can be written in cycle notation: you draw the elements as nodes of a graph and make a directed edge from i to j if g send i to j. The sum of the cycle lengths is equal to n (we consider fixed points as 1-cycles), so they form a partition of n. We observe that two permutations are conjugate if and only if the sizes of their cycles are equal. If H is a normal subgroup of S_n, and $h \in H$, then any permutation with same cycle lengths is also contained in H. Hence, in order to identify a normal subgroup, it is enough to know the partitions that arise as cycle length partition.
For $n = 2$, S_2 is abelian, and there are only two subgroups; the alternating group coincides with the trivial subgroup $\{e\}$. The statement is trivially true.
For $n = 3$, we have partitions 3 (3-cycles), $1 + 2$ (transpositions), and $1 + 1 + 1$ (the identitiy element). If H is a nontrivial normal subgroup, then it cannot contain the transpositions because they generate S_3. Any subgroup contains the identity, so the only nontrivial subgroup is the one formed by 3-cyles and the identity. This is A_3.
For $n = 4$, we have partitions 4 (4-cycles), $1+3$ (3-cycles), $2+2$ (double transpositions), $1+1+2$ (transpositions), and $1 + 1 + 1 + 1$ (the identity). The transpostions generate S_4 (this holds for every n). Show that the 4-cycles generate the transposition. Hint: cut out little pieces of paper and write numbers $1, 2, 3, 4$ on them, then you can compose the permutations easily by hand. Show that the 3-cycles generate the double transpositions. This is enough: any nontrivial normal subgroup cannot contain the transpostions neither nor the 4-cycles, but it has to contain the double transpositions. There are two choices, and both are normal subgroups: if H contains the 3-cycles, then it is A_4, else its V_4.

If H is a normal subgroup of G, then the cosets $\{gH\}_{g \in G}$ form a group, called the quotient group G/H. Conversely, the kernel of any group homomorphism is a normal subgroup.

Assume now that G has two normal subgroups H, K such that $H \cap K = \{e\}$ and $HK = G$ - here the set product is the set of all products of an element in the first set with an element in the second set. Then we say that G is the direct product of H and H. Any element in $g \in G$ can uniquely by written as $g = hk$ with $h \in H, k \in K$. Exercise: prove that any $h \in H$ and any $k \in K$ commute: $hk = kh$. As a consequence, G is isomorphic to the cartesian product of the groups H_1 and H_2, with multiplication defined componentwise.

Now, let us assume that G has a normal subgroup H and another subgroup K which is not normal. Also, still assume $H \cap K = \{e\}$ and $HK = G$. Then we can still say that every element $g \in G$ can uniquely be written as $g = hk$ with $h \in H, k \in K$. But now elements in H and in K do not commute. Since H is normal, at least we have that the conjugation maps leave H invariant, so that $\mathrm{conj} : G \to \mathrm{Hom}(H, H)$ is a homomorphism. In the case when H is abelian, this homomorphism has H in its kernel, so it factors through G/H which in turn is equivalent to K. Let us write this conjugation homomorphism as exponent: $\mathrm{conj}_k(h) = h^k$. In this situation, we say that G is the semidirect product of H and K with respect to the action. The group G can be reconstructed from H, K, and the action: as a set, G is the carteesion product, and the group multiplication is

$$(h_1, k_1) \cdot (h_2, k_2) = h_1 k_1 h_2 k_2 = h_1 h_2 h_2^{-1} k_1 h_2 k_2 = h_1 h_2 k_1^{h_2} k_2 = (h_1 h_2, k_1^{h_2} k_2)$$

The Euclidean group SE_3 is an example which will appear frequently in the lecture. Here, $H \sim \mathbb{R}^3$ is the group of translation and $K \sim \mathrm{SO}$ is the subgroup fixing a single point (the "origin"; but one can choose any point as the origin). The action of K on H is the same as common vector-matrix multiplication when we identify elements in H with vectors and elements in K with orthogonal matrices.
A Lie group is a manifold such that the group multiplication is differentiable. It follows from the implicit function theorem that the function assigning to every element its inverse is also differentiable. Examples are $(\mathbb{R}^k, +)$, $(\mathbb{R}^*, \cdot)$, or the general linear group $\mathrm{GL}(k, \mathbb{R})$ of all invertible linear maps $\mathbb{R}^k \to \mathbb{R}^k$, which is isomorphic to the group of nonsingular matrices. This last group has many subgroups which are also differentiable manifolds: the orthogonal group $\mathrm{O}(k, \mathbb{R})$, the special linear group $\mathrm{SL}(k, \mathbb{R})$ consisting of all matrices with determinant 1, the special orthogonal group $\mathrm{SO}(k, \mathbb{R})$. Every Lie group has an atlas with smooth charts, even stronger, an atlas with analytic charts, hence we do not distinguish between smooth and just differentiable Lie groups.
The 0-dimensional Lie groups are groups with discrete topology: any point is a neighborhood of itself. Because of the countable basis axiom, such a Lie group is necessarily finite or countable. Conversely, every finite or countable group, with the discrete topology is a Lie group. But you can imagine that these Lie groups are not in the center of interest in Lie group theory, even thouh they arise frequently. The reason is that these objects are more conveniently studied as groups, the topology is not helpful at all.
The set of connected componets of a Lie group is countable – this is even true for every manifold. The connected component of the unit element e is a normal subgroup. Examples are the set $\mathrm{GL}_+(k, \mathbb{R})$ and the special orthogonal group as a subgroup of the orthogonal group.
Any closed subgroup is also a submanifold, therefore a Lie group. If H is a normal closed subgroup of G, then the quotient group is also a Lie group of dimension $\dim(G) - \dim(H)$. Direct products and semidirect products are defined as in the group case, only with closed subgroups.
Let G e a Lie group of dimension n. The tangent space at the identity T_eG is called its Lie algebra $\mathfrak{g}$. Before we define the operations of a Lie algebra, look at a few examples: $\mathrm{GL}(k, \mathbb{R})$ $(n = k^2)$, $\mathrm{SL}(k, \mathbb{R})$ $(n = k^2 - 1)$, $\mathrm{SO}(k, \mathbb{R})$ (what is n)?
For any element $g \in G$, $\mathrm{conj}_g : G \to G$ is a differentiable map mapping e. So its differential is a linear map $d\mathrm{conj}_g : \mathfrak{g} \to \mathfrak{g}$. Fix an element $x \in \mathfrak{g}$ and consider the funcion $f := g \mapsto d\mathrm{conj}_g(x)$, which goes from G to $\mathfrak{g}$ and maps e to 0. Then df is a linear map from $\mathfrak{g}$ to itself (since $\mathfrak{g}$ is a vectorspace, its tangent space is itself). Varying x, we get a bilinear function on $\mathfrak{g}$ with values in $\mathfrak{g}$: the Lie bracket, written as $(x, y) \mapsto [x, y]$. It fulfills the axioms of a Lie algebra: it is bilinear, anticommutativem and satisfies the Jacobi identity

$$\forall x, y, z : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

The differential of a Lie group homomorphism is a Lie algebra homomorphism. The tangent space of a closed Lie subgroup is a Lie subalgebra. If the subgroup is normal, then the Lie algebra is a Lie ideal, i.e. it is not just closed under the Lie bracket but closed under taking Lie bracket with arbitrary elements from the bigger Lie algebra.
The Lie algebra counterpart of a semidirect product is the sum. Assume that the Lie algebra $\mathfrak{g}$ has two Lie subalgebras $\mathfrak{h}$ and $\mathfrak{k}$, and $\mathfrak{h}$ is a Lie ideal. Assume that $\mathfrak{g}$ is the direct sum of its subspaces $\mathfrak{h}$ and $\mathfrak{k}$. For any element $k \in \mathfrak{k}$, taking Lie bracket with k is a linear map $\mathfrak{h} \to \mathfrak{h}$ which we write as exponentiation by k. The Lie bracket on $\mathfrak{g}$ can be expressed in terms of this action and the Lie brackets on the subalgebras:

$$\forall h_1, h_2 \in \mathfrak{h}, k_1, k_2 \in \mathfrak{k} : \; [h_1 + h_2, k_1 + k_2] = [h_1, h_2] + h_1^{k_2} - h_2^{k_1} + [k_1, k_2].$$

In our main example of $\mathfrak{se}(3, \mathbb{R})$, we have the Lie ideal $\mathbb{R}^3$ of translations (with trivial Lie bracket since translations commute), and the Lie subalgebra $\mathfrak{so}(3, \mathbb{R})$ of skew symmetric matrices.
We can use the algebraic structure in order to determine the connected closed Lie subgroups of $\mathrm{SE}_3(3, \mathbb{R})$, up to conjugation. Any such Lie subgroup is uniquely determined by its Lie algebra. As a basis for $\mathfrak{se}(3, \mathbb{R})$, we fix $(e_x, e_y, e_z, \mathbf{i}, \mathbf{j}, \mathbf{k})$. The first three vectors generated the trivial Lie

algebra of translations. The other three basis vectors stand for the skew symmetric matrices

$$\mathbf{i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The action of the Lie subalgebra $\mathfrak{so}(3,\mathbb{R})$ generated by $\mathbf{i},\mathbf{j},\mathbf{k}$ is then also described by vector-matrix multiplication.

Let $\mathfrak{g}$ be a Lie subalgebra. Let $\mathfrak{h} = \mathfrak{g} \cap \mathbb{R}^3$, and let $\mathfrak{k} \subset \mathfrak{so}(3,\mathbb{R})$ be the image of the projection to $\langle \mathbf{i},\mathbf{j},\mathbf{k}$. We make a case by case analysis according to the dimensions of $\mathfrak{h}$ and $\mathfrak{k}$. Note that $\mathfrak{k}$ must be a Lie subalgebra of $\mathfrak{so}(3,\mathbb{R})$, which cannot be two-dimensional because the cross product of two linear independent vectors is not on the linear span of the two vectors.

$\dim(\mathfrak{k}) = 3$ **and** $\dim(\mathfrak{h}) = 3$ In this case $\mathfrak{h} = \mathfrak{se}(3,\mathbb{R})$ and we have the full Lie algebra (and also the full Lie group). For the remaining cases, it is important to point out that the subspace $\mathfrak{h}$ needs to be invariant under the action of $\mathfrak{k}$. The action of $\mathfrak{so}(3,\mathbb{R})$ is transitive on nonzero vectors, hence if $\dim(\mathfrak{k}) = 3$ then $\dim(\mathfrak{h})$ can only be 3 or 0.

$\dim(\mathfrak{k}) = 3$ **and** $\dim(\mathfrak{h}) = 0$ In this case $\mathfrak{g}$ is an isomorphic copy of $\mathfrak{so}(3,\mathbb{R})$ inside $\mathfrak{se}(3,\mathbb{R})$, containing no translations. The Lie group is the group of rotations leaving some point fixed. The orbits of this group are spheres.

$\dim(\mathfrak{k}) = 1$ **and** $\dim(\mathfrak{h}) = 3$ The corresponding group has all translations but only rotations around a fixed axis (say vertical). This group is called "waiter's group", the possible motions of a tablet with drinks on it that must not be tilted.

$\dim(\mathfrak{k}) = 1$ **and** $\dim(\mathfrak{h}) = 2$ Without loss of generality, we may assume $\mathfrak{k} = \langle \mathbf{k} \rangle$. The only 2-dimensional subspace of $\mathbb{R}^3$ which is invariant under $\mathbf{k}$ is $\langle e_x, e_y \rangle$, so we know that $\mathfrak{h} = \langle e_x, e_y \rangle$. The Lie algebra could be $\mathfrak{g} = \langle e_x, e_y, \mathbf{k} \rangle$; this is the Lie algebra of the subgroup $\mathrm{SE}_3(2,\mathbb{R})$ fixing any horizontal plane. Its orbits are, of course, precisely these planes. However, it could also be that the Lie algebra is $\mathfrak{g} = \langle e_x, e_y, \mathbf{k} + ce_z \rangle$ for some $c \neq 0$. In this case, we also get a closed subgroup, which is transitive on points in $\mathbb{R}^3$.

$\dim(\mathfrak{k}) = 1$ **and** $\dim(\mathfrak{h}) = 1$ Again, we may assume $\mathfrak{k} = \langle \mathbf{k} \rangle$, and we can then prove $\mathfrak{h} = \langle e_z \rangle$. The Lie algebra is conjugate (by some translation) to the Lie algebra $\mathfrak{g} = \langle, \mathbf{k}, e_z \rangle$, and the Lie group is genetrated by rotations around the z-axis and vertical translations. Its orbits are cylinders around the rotation axis.

$\dim(\mathfrak{k}) = 1$ **and** $\dim(\mathfrak{h}) = 0$ The Lie algebra is one-dimensional, genrated by an element such as $\mathbf{k} + ce_z$, $e \in \mathbb{R}$. If $c = 0$, then we have a subgroup of rotations around a fixed axis. Its orbits are circles around the rotation axis, so this os the symmetry group of surfaces of revolution. If $c \neq 0$, then the orbits are helices, and the group is the symmetry group of a helical surfaces which is the union of these helices.

$\dim(\mathfrak{k}) = 0$ **and** $\dim(\mathfrak{h}) = 3$ This is the subalgebra/subgroup of translations.

$\dim(\mathfrak{k}) = 0$ **and** $\dim(\mathfrak{h}) = 2$ The subgroup of horizontal translations. Its orbits are the horizontal planes, just as in the case $\dim(\mathfrak{k}) = 1$ and $\dim(\mathfrak{k}) = 2$ for $c = 0$.

$\dim(\mathfrak{k}) = 0$ **and** $\dim(\mathfrak{h}) = 1$ The subgroup of tranlations in one direction (say vertical). Its orbits are vertical lines, and the surfaces composed of these lines are prisms and cylinders with general base shape.

$\dim(\mathfrak{k}) = 0$ **and** $\dim(\mathfrak{h}) = 0$ The trivial subgroup (trivial Lie subalgebra).

For kinematics. the subgroups which leave a surface invariant are of special interest. In order to physically manufacture a joint connecting two links, one ensures that the two corresponding rigid bodies touch each other along a contact surface. The admissibe motions map the contact surface into itself. The subgroup can be recovered as the symmetry group of the contact surface. This leads to the following types of joints:

spherical joints spherical contact surfaces. In the above list, this is $\dim(\mathfrak{k}) = 3$ and $\dim(\mathfrak{h}) = 0$.

planar joints planar contact surfaces. In the above list, this is $\dim(\mathfrak{k}) = 1$ and $\dim(\mathfrak{h}) = 2$, with $c = 0$.

cylinderical joints circular cylinders. In the above list, this is $\dim(\mathfrak{k}) = 1$ and $\dim(\mathfrak{h}) = 1$.

revolute joints the joint surfaces are surfaces of revolution. In the above list, this is $\dim(\mathfrak{k}) = 1$ and $\dim(\mathfrak{h}) = 0$, with $c = 0$.

helical joints the joint surfaces helical surfaces. In the above list, this is $\dim(\mathfrak{k}) = 1$ and $\dim(\mathfrak{h}) = 0$, with $c \neq 0$.

prismatic joints the joint surfaces are cylinders of irregular base or prisms. In the above list, this is $\dim(\mathfrak{k}) = 1$ and $\dim(\mathfrak{h}) = 0$, with $c \neq 0$.

4 The Forward Kinematic Map

Consider a linkage consisting of 7 links, connected in a sequence by 6 revolute joints: the open 6R chain. Tranditionally, the zeroth link is called the base, and the sixth link is the end effector. Each joint allows the change of a single parameter, the angle ϕ_i, $i = 1, \ldots, 6$. The precise deifintion of the angle requires the choice of an "initial relative position". The actual relative position can be achieved from the relative position by a rotation around a fixed axis by an arbitary angle (see also Section **??**). An angle is an element of $\mathbb{R}/(2\pi\mathbb{N})$.
The relative position of the end effector with respect to the base is an element that can be computed as a function depending on the six angles, so that we get a function $f : (\mathbb{R}/(2\pi\mathbb{N}))^6 \to \mathrm{SE}_3(3, \mathbb{R})$. This is the forward kinematic map.
In order to compute f, we introduce dual quaternions as a convenient way of writing coordinates for $\mathrm{SE}_3(3, \mathbb{R})$. We start by introducing quaternions as coordinates for $\mathrm{SO}(3, \mathbb{R})$.