

Regular Languages
and Their Generating Functions:
The Inverse Problem

Reguläre Sprachen
und ihre erzeugenden Funktionen:
Das inverse Problem

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Abstract

The technique of determining a generating function for an unambiguous context-free language, is known as the SCHÜTZENBERGER methodology. For regular languages, BARCUCCI ET AL. proposed¹ a technology for inverting this methodology, which allows to give a combinatorial interpretation (by means of a regular expression) of certain positive integer sequences that are defined by a linear recurrence.

In this thesis, we provide an implementation of this inverse methodology in Maple. Therefore, a detailed introduction to the underlying theory, i.e., the theory of formal power series and especially the question of deciding \mathbb{N} -rationality, is given. Further, various aspects and problems concerning the implementation are discussed, and some examples from combinatorics illustrate its applicability.

Kurzzusammenfassung

Mit dem Verfahren von SCHÜTZENBERGER kann zu einer eindeutigen, kontextfreien Sprache eine erzeugende Funktion bestimmt werden. Die Umkehrung dieses Verfahrens im Fall regulärer Sprachen wurde von BARCUCCI ET AL. vorgestellt¹; damit ist es möglich, zu einer durch eine lineare Rekursion gegebenen Reihe positiver ganzer Zahlen eine kombinatorische Interpretation (in Form eines regulären Ausdrucks) anzugeben.

Im Rahmen der vorliegenden Diplomarbeit wurde dieses inverse Verfahren in Maple implementiert. Dazu wird zunächst die zu Grunde liegende Theorie, d.h. die Theorie der formalen Potenzreihen und insbesondere die Frage nach der \mathbb{N} -Rationalität, dargestellt. Anschließend werden verschiedene Aspekte der Implementierung und dabei auftretende Probleme diskutiert. Einige Beispiele aus der Kombinatorik zeigen Anwendungsmöglichkeiten des vorgestellten Maple-Programms auf.

¹in their paper “*A Technology for Reverse-Engineering a Combinatorial Problem from a Rational Generating Function*”, see [BLFR01]

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Chapter 1

Introduction

1.1 English Version

In this thesis, we essentially deal with sequences of positive integers that are defined by a linear recurrence. Such sequences can be identified with the power series expansion of some rational function. The focus of attention is the interrelation between certain power series and regular languages. We call the formal power series

$$S = \sum_{n=0}^{\infty} s_n x^n$$

the generating function of a formal language L , if

$$s_n = \left| \{w \in L : |w| = n\} \right|,$$

i.e., if the n^{th} coefficient of the series S gives the number of words in L having the length n . It is well known how to a given language the generating function of the corresponding power series can be found. The procedure is called the SCHÜTZENBERGER *methodology*, which works on unambiguous context free languages:

Let $G = (V, \Sigma, P, S)$ be an unambiguous context free grammar of the language L_G , where V denotes the set of nonterminals, Σ the set of terminals, P the set of production rules, and S the initial symbol. In order to obtain the generating function for L_G , the morphism Θ is defined:

$$\begin{aligned} \Theta(a) &= x, \quad \forall a \in \Sigma \\ \Theta(\lambda) &= 1 \quad (\lambda \text{ denotes the empty word}) \\ \Theta(A) &= A(x), \quad \forall A \in V \end{aligned}$$

Now Θ is applied to all elements of P , yielding an algebraic system of equations in $A(x), B(x), \dots$:

$$\Theta(A) = \sum_{i=1}^k \Theta(e_i), \text{ where } A \rightarrow e_1|e_2|\dots|e_k \in P.$$

Solving this system for $S(x)$ gives the generating function for L_G . In [CS63] it is proved that if G is an unambiguous regular grammar, then the corresponding generating function is rational.

In this thesis, we tackle the inverse problem by using the algorithms suggested by BARCUCCI ET AL. in [BLFR01]: Given a formal power series which is generated by some rational function, how can we obtain a regular expression for the corresponding regular language (in the case that such a language exists at all), and thus get a combinatorial interpretation of the series?

Chapter 2 is dedicated to the underlying theory: The reader is introduced to the theory of formal power series, with special focus on the rational series, then we treat the famous Theorem of SCHÜTZENBERGER, and give an exact characterization of \mathbb{N} -rational series. We will show that our inverse problem is solely solvable for \mathbb{N} -rational series.

In Chapter 3 we discuss various aspects concerning the implementation of the inverse SCHÜTZENBERGER methodology: Especially deciding if a series is positive and computing a regular expression that defines the corresponding regular language, turn out to be non-trivial tasks. It cost a lot of trouble to handle series which have no dominating root, but nevertheless are \mathbb{N} -rational. In order to illustrate the described algorithms and to demonstrate the functionality of our implementation, some examples are presented in Chapter 4. Chapter 5 contains the manual pages that explain the proper usage of the Maple program, which was developed within the scope of this thesis.

1.2 German Version

Im Mittelpunkt dieser Diplomarbeit stehen Zahlenreihen, welche durch eine lineare Rekursion definiert sind und die aus positiven ganzen Zahlen bestehen. Solche Zahlenreihen können mit der Potenzreihenentwicklung einer rationalen Funktion identifiziert werden. Das Hauptaugenmerk liegt dabei auf der Beziehung zwischen bestimmten Potenzreihen und regulären Sprachen. Eine formale Potenzreihe

$$S = \sum_{n=0}^{\infty} s_n x^n$$

heißt erzeugende Funktion einer formalen Sprache L , wenn

$$s_n = \left| \{w \in L : |w| = n\} \right|$$

gilt, d.h. wenn der n -te Koeffizient der Reihe S die Anzahl von Wörtern der Länge n in L angibt. Mit Hilfe der seit längerem bekannten Methode von SCHÜTZENBERGER lässt sich zu einer gegebenen kontextfreien Sprache die erzeugende Funktion der entsprechenden Potenzreihe ermitteln:

Sei $G = (V, \Sigma, P, S)$ eine eindeutige, kontextfreie Grammatik der Sprache L_G , wobei V die Menge der Nichtterminale, Σ die Menge der Terminale, P die Menge der Produktionsregeln und S das Startsymbol bezeichne. Zur Berechnung der erzeugenden Funktion von L_G wird der Morphismus Θ definiert:

$$\begin{aligned} \Theta(a) &= x, \quad \forall a \in \Sigma \\ \Theta(\lambda) &= 1 \quad (\lambda \text{ bezeichnet das leere Wort}) \\ \Theta(A) &= A(x), \quad \forall A \in V \end{aligned}$$

Θ wird nun auf alle Elemente von P angewendet, was auf ein algebraisches Gleichungssystem in den Variablen $A(x), B(x), \dots$ führt:

$$\Theta(A) = \sum_{i=1}^k \Theta(e_i), \text{ wobei } A \rightarrow e_1 | e_2 | \dots | e_k \in P.$$

Löst man dieses Gleichungssystem nach $S(x)$ auf, erhält man die erzeugende Funktion von L_G . In [CS63] wird gezeigt, dass im Fall einer eindeutigen, regulären Grammatik G die erzeugende Funktion rational ist.

In dieser Diplomarbeit wird das inverse Problem behandelt, wobei die Algorithmen von BARCUCCI ET AL. (s. [BLFR01]) verwendet werden: Wie kann zu einer gegebenen formalen Potenzreihe, die von einer rationalen Funktion erzeugt wird, ein regulärer Ausdruck bestimmt werden, der die zugehörige reguläre Sprache beschreibt (im Fall, dass diese überhaupt existiert), und

somit eine kombinatorische Interpretation gefunden werden?

Kapitel 2 ist der zu Grunde liegenden Theorie gewidmet: Der Leser wird eingeführt in die Theorie der formalen Potenzreihen, wobei die rationalen Reihen im Vordergrund stehen. Anschließend wird der berühmte Satz von SCHÜTZENBERGER behandelt, und eine vollständige Charakterisierung der \mathbb{N} -rationalen Potenzreihen entwickelt. Es wird sich herausstellen, dass unser inverses Problem nur für \mathbb{N} -rationale Reihen lösbar ist.

In Kapitel 3 werden verschiedene Aspekte der Implementierung beleuchtet: Als durchaus nichttrivial erweisen sich dabei insbesondere die Frage, ob eine Reihe positiv ist, sowie die Berechnung eines regulären Ausdrucks für die entsprechende reguläre Sprache. Einige Mühe kostet auch die Behandlung von Reihen, welche keine dominierende Wurzel besitzen, aber dennoch \mathbb{N} -rational sind.

Zur Veranschaulichung der beschriebenen Algorithmen, und um die Funktionsweise der Implementierung zu demonstrieren, werden in Kapitel 4 einige Beispiele vorgestellt.

In Kapitel 5 werden auf Manual Pages die einzelnen Funktionen des Maple-Programms erläutert, welches im Rahmen dieser Arbeit entwickelt wurde.

Chapter 2

Formal Power Series and Regular Languages

In this chapter the basic connections between regular languages and power series are explained. After a few definitions that make us familiar with the notion of a formal power series, we will see that we can connect formal power series and regular languages by means of the characteristic series of a language. Next we have to examine the question which criteria a power series has to satisfy for identifying it with a regular language. This leads to the very important property that a power series must have if we want to build a regular language from it: The \mathbb{N} -rationality. We have to make an effort to get a deep understanding of what the term \mathbb{N} -rational means and how we can find out whether a series is \mathbb{N} -rational or not. At the end of the chapter we will have an exact characterization.

An early view on this theory is given in [Niv69]. Most of the results and proofs are taken from [BR88] and [SS78].

2.1 Formal Power Series

We start with some definitions of terms the reader should be familiar with, then continue to define what a formal power series is, and introduce the notion of a rational series in several noncommuting variables.

First recall that a monoid consists of a set \mathbb{M} , an associative binary operation \circ on \mathbb{M} and of a neutral element 1 such that $1 \circ x = x \circ 1 = x \forall x \in \mathbb{M}$.

In this thesis we will mainly deal with the *free monoid* Σ^* generated by the *alphabet* Σ , i.e., a finite, nonempty set of elements that are named *letters*. Σ^* contains all sequences $x_1 \dots x_n$ of elements $x_i \in \Sigma$, including also the empty sequence denoted by λ (if the empty word should not be included, we

speak about $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$). The elements of Σ^* are called *words*, which can be linked by the operation of concatenation (the monoid operation of Σ^*) defined by

$$(x_1 \dots x_n) \circ (y_1 \dots y_m) = x_1 \dots x_n y_1 \dots y_m.$$

The operation symbol can be skipped and the concatenation of two words w_1 and w_2 is written by $w_1 w_2$. Of course, the empty word λ acts as neutral element of Σ^* . The *length* of a word $w = x_1 \dots x_n, x_i \in \Sigma$ is n ; it is denoted by $|w| = n$. Consequently the empty word λ has length zero.

Since we want to deal with regular languages (counting their words and building up regular expressions), we have to operate on natural numbers. Unfortunately the set of natural numbers \mathbb{N} is not a ring. We therefore have to introduce the notion of a *semiring* which will play an important role in our theory.

Definition 2.1.1 *A semiring is a set \mathbb{K} equipped with two binary operations $+$ (sum) and \cdot (product) and two constant elements 0 and 1 . Furthermore \mathbb{K} has the following properties:*

- $\langle \mathbb{K}, +, 0 \rangle$ is a monoid, where $+$ is commutative,
- $\langle \mathbb{K}, \cdot, 1 \rangle$ is a monoid,
- \cdot is distributive with respect to $+$,
- $0 \cdot x = x \cdot 0 = 0$ for every $x \in \mathbb{K}$.

A semiring is called *commutative* if $x \cdot y = y \cdot x \forall x, y \in \mathbb{K}$.

Thus a semiring is, roughly speaking, a ring without subtraction. As the reader can easily see, the set of natural numbers \mathbb{N} fulfills all criteria of a semiring. Obviously a ring can be defined as a semiring with commutative group $\langle \mathbb{K}, +, 0 \rangle$.

Now it is time to approach the central point of interest and to define what a formal power series is.

Definition 2.1.2 *Given an alphabet Σ and a semiring \mathbb{K} . A formal power series (or formal series) S is a function*

$$S : \Sigma^* \rightarrow \mathbb{K}.$$

The image of a word w under S is called the coefficient of w in S and is denoted by s_w . S itself is written as a formal sum

$$S = \sum_{w \in \Sigma^*} s_w w.$$

The set of all formal power series over Σ^* with coefficients in \mathbb{K} is denoted by $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$.

Given $S \in \mathbb{K}\langle\langle\Sigma^*\rangle\rangle$, the letters of Σ are sometimes referred to as *variables*. Keep in mind that the variables of S in general do not commute. The power series used in this thesis are called formal because it is not of interest to sum up the series and to examine convergence properties. Instead of that we will define various operations for power series.

On the set $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ the following algebraic operations are defined.

Let $S = \sum_{w \in \Sigma^*} s_w w$ and $T = \sum_{w \in \Sigma^*} t_w w$ be two formal series. Then the sum of S and T is

$$S + T = \sum_{w \in \Sigma^*} (s_w + t_w)w.$$

Note that $S + T = T + S$ commute since the operation $+$ in \mathbb{K} does.

Next we define the multiplication of a series $S \in \mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ by a scalar $k \in \mathbb{K}$ (*scalar product*):

$$\begin{aligned} kS &= \sum_{w \in \Sigma^*} (ks_w)w \\ Sk &= \sum_{w \in \Sigma^*} (s_w k)w \end{aligned}$$

The product of S and T (sometimes also referred to as CAUCHY product) is given by

$$ST = \sum_{w \in \Sigma^*} \left(\sum_{uv=w} s_u t_v \right) w.$$

To obtain the n th coefficient of the product, a finite number of algebraic operations with elements in \mathbb{K} must be performed. It is clear that the product in general is not commutative.

Sum and product induce the structure of a semiring on $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$. Its neutral element with respect to the sum is 0, the series which coefficients all equal 0. For any arbitrary element $w \in \Sigma^*$ and for $k \in \mathbb{K}, k \neq 0$, we denote by kw the series whose coefficient for w equals k , the remaining coefficients being equal to 0. Hence kw is referred to as a *monomial*. The neutral element with respect to the product is 1λ . Note that the scalar product kS (Sk) is equivalent to the CAUCHY product $(k\lambda)S$ (resp. $S(k\lambda)$) of S and the monomial $k\lambda$.

The HADAMARD product of two series is defined by

$$S \odot T = \left(\sum_{w \in \Sigma^*} s_w w \right) \odot \left(\sum_{w \in \Sigma^*} t_w w \right) = \sum_{w \in \Sigma^*} s_w t_w w$$

The set of words which have nonzero coefficients is referred to as the *support* $\text{supp}(S)$ of a series. The set of all series with a finite support, i.e., all polynomials, is denoted by $\mathbb{K}\langle\Sigma^*\rangle$, which is a semiring likewise. If \mathbb{K} is a ring, then so are $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ and $\mathbb{K}\langle\Sigma^*\rangle$.

Definition 2.1.3 A power series (especially a polynomial) $S \in \mathbb{K}\langle\langle\Sigma^*\rangle\rangle$, $S = \sum_{w \in \Sigma^*} s_w w$ is called *quasiregular* (or *proper*) if the coefficient of the neutral element of Σ^* vanishes, i.e., if $s_\lambda = 0$.

To $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ we can add a topological structure such that we can speak about the convergence of a sequence S_1, S_2, \dots of elements of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$: The sequence converges to the limit S , i.e.,

$$\lim_{i \rightarrow \infty} S_i = S \iff \forall n \in \mathbb{N} \exists m \in \mathbb{N} \forall w \in \Sigma^n \forall j > m : (s_j)_w = s_w,$$

where $(s_j)_w$ denotes the coefficient of w in the series S_j .

Now let S be a quasiregular series. We look at the sequence of powers of S : $S^0 = \lambda, S^1, S^2 = S \cdot S, S^3, \dots$, which converges to 0, since the coefficient for any word $w, |w| = n$ equals 0 in all series $S^j, j > n$. Therefore the sum

$$S^* = \lim_{m \rightarrow \infty} \sum_{n=0}^m S^n$$

exists (the sequence S^0, S^1, S^2, \dots is called *summable*) and is named the *star* of S . In the theory of formal languages this expression is termed **KLEENE closure**. Equivalently we define

$$S^+ = \lim_{m \rightarrow \infty} \sum_{n=1}^m S^n.$$

The following equalities hold, which are also well known from the theory of formal languages¹:

$$\begin{aligned} S^* &= \lambda + S^+ \\ S^+ &= SS^* = S^*S \end{aligned}$$

Moreover, if \mathbb{K} is a ring, then $S^* = (1 - S)^{-1}$, since

$$S^*(1 - S) = S^* - S^+ = \lambda.$$

¹see e.g., [HU79]

Definition 2.1.4 *The rational operations in $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ are the sum, the product, and the star. A subsemiring of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ is rationally closed if it is closed for the rational operations. The rational closure of a subset $M \subseteq \mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ is the smallest subset of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ containing M and being rationally closed. A formal series $S \in \mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ is \mathbb{K} -rational if it is an element of the rational closure of $\mathbb{K}\langle\Sigma^*\rangle$. This set of all \mathbb{K} -rational series is denoted by $\mathbb{K}^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$.*

2.2 SCHÜTZENBERGER's Theorem

In this section we fix an alphabet Σ and a semiring \mathbb{K} . We now introduce vectors and matrices over $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$, using the classical operations. Such a vector or matrix is called \mathbb{K} -rational (quasiregular), if all its entries are \mathbb{K} -rational (quasiregular respectively) power series. Hence if A is a quasiregular matrix, then the sequence A, A^2, A^3, \dots is summable and the limit of its sum exists and is denoted by A^* . Note that by writing $A^2 = A \cdot A$ we mean the well known matrix multiplication and not the elementwise multiplication of the matrices entries. By A_w we mean the matrix which results when we replace each entry S by its coefficient s_w . Thus A_w is a matrix over \mathbb{K} and $A = \sum_{w \in \Sigma^*} A_w w$.

The following theorem is (in the context of languages) known as ARDEN's Lemma: Given a linear system $X = B + AX$ and the matrix A is quasiregular, then we have a solution which is unique and which preserves rationality. In the next lemma we show the converse: For every rational series we can find a linear system such that the first component of its solution vector is exactly the given series. With these two results we have already done most of the work for proving the Theorem of SCHÜTZENBERGER, which teaches us that the families of rational and recognizable series coincide.

Theorem 2.2.1 *Let $B \in \mathbb{K}\langle\langle\Sigma^*\rangle\rangle^{n,1}$ and $A \in \mathbb{K}\langle\langle\Sigma^*\rangle\rangle^{n,n}$ a quasiregular matrix. The linear system*

$$X = B + AX \quad (*)$$

*has the unique solution $X = A^*B$. Moreover, if A and B are \mathbb{K} -rational then the solution X is \mathbb{K} -rational.*

Proof: First we show that $X = A^*B$ is a solution:

$$B + AX = B + AA^*B = (I + AA^*)B = A^*B = X.$$

The symbol I represents the identity matrix whose entries $(I)_{i,j}$ are λ for $i = j$ and 0 otherwise.

Next let's consider why the solution is unique. Let X be any solution of (*), and let X_λ denote the vector of the coefficients s_λ which belong to the elements of X (analogously for any word w). Obviously $X_\lambda = B_\lambda$ since A is quasiregular (i.e., $A_\lambda = 0$). Next for $|w| \geq 1$, X_w is completely determined by B_w , A_v and X_v , where $|v| < |w|$. Therefore the solution X must be unique. Now let A and B be \mathbb{K} -rational. We prove the \mathbb{K} -rationality of the solution X by induction on the dimension n .

Consider the case $n = 1$: $S = A^*B$ is \mathbb{K} -rational, since in this case A and B are simply \mathbb{K} -rational series.

Induction step: We now look at a system of dimension n and assume that the theorem holds true always for dimension smaller than n . The last equation of the system is $x_n = b_n + a_{n,1}x_1 + \dots + a_{n,n}x_n$. We multiply this equation by $a_{n,n}^*$ and get

$$\begin{aligned} x_n \lambda + x_n a_{n,n}^+ &= a_{n,n}^* (b_n + a_{n,1}x_1 + \dots + a_{n,n-1}x_{n-1}) + a_{n,n}^+ x_n \\ \implies x_n &= a_{n,n}^* (b_n + a_{n,1}x_1 + \dots + a_{n,n-1}x_{n-1}). \end{aligned}$$

We substitute x_n in the first $n - 1$ equations and get a system of dimension $n - 1$. By induction hypothesis its solution vector $(x_1, \dots, x_{n-1})^T$ is \mathbb{K} -rational, hence x_n must be \mathbb{K} -rational (it is obtained by rational operations, see the above equation). \square

Lemma 2.2.2 *Let S belong to $\mathbb{K}^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$. Then there is a quasiregular linear system*

$$X = B + AX, \quad B = B_\lambda \lambda, \quad A = \sum_{|w|=1} A_w w$$

and S is the first component of its solution vector.

Proof: Let $S \in \mathbb{K}^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$. Let R be the set of all series which are obtained in the manner described. We have to show that R contains all monomials $k \in \mathbb{K}$ and $w \in \Sigma$ and is rationally closed. Let \hat{A} denote the first row of the matrix A (respectively \hat{B} the first element of vector B).

- Monomials: k is the solution of $x = k + 0x$.
Now let $w \in \Sigma$: Since $\begin{pmatrix} 0 \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ \lambda \end{pmatrix} = \begin{pmatrix} w \\ \lambda \end{pmatrix}$, w is the first component of the solution vector of the system $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ (note that $|w| = 1$).

- Sum: For $i = 1, 2$ let $S_i \in R$ be the first component of X_i which is the solution vector of $X_i = B_i + A_i X_i$. Now we set $B := \begin{pmatrix} \widehat{B}_1 + \widehat{B}_2 \\ B_1 \\ B_2 \end{pmatrix}$

and $A := \begin{pmatrix} 0 & \widehat{A}_1 & \widehat{A}_2 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix}$ and show that $\begin{pmatrix} S_1 + S_2 \\ X_1 \\ X_2 \end{pmatrix}$ is the solution of

the corresponding system: $\begin{pmatrix} \widehat{B}_1 + \widehat{B}_2 \\ B_1 \\ B_2 \end{pmatrix} + \begin{pmatrix} 0 & \widehat{A}_1 & \widehat{A}_2 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix} \begin{pmatrix} \widehat{B}_1 + \widehat{B}_2 \\ B_1 \\ B_2 \end{pmatrix}$

$$= \begin{pmatrix} \widehat{B}_1 + \widehat{B}_2 + \widehat{A}_1 X_1 + \widehat{A}_2 X_2 \\ B_1 + A_1 X_1 \\ B_2 + A_2 X_2 \end{pmatrix} = \begin{pmatrix} S_1 + S_2 \\ X_1 \\ X_2 \end{pmatrix}$$

- Product: Let S_i be as before. Now we set $B := \begin{pmatrix} B_1 \widehat{B}_2 \\ B_2 \end{pmatrix}$ and $A := \begin{pmatrix} A_1 & B_1 \widehat{A}_2 \\ 0 & A_2 \end{pmatrix}$ and get: $\begin{pmatrix} B_1 \widehat{B}_2 \\ B_2 \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \widehat{A}_2 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} X_1 S_2 \\ X_2 \end{pmatrix}$
- $$= \begin{pmatrix} B_1 \widehat{B}_2 + A_1 X_1 S_2 + B_1 \widehat{A}_2 X_2 \\ B_2 + A_2 X_2 \end{pmatrix} = \begin{pmatrix} B_1 S_2 + A_1 X_1 S_2 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 S_2 \\ X_2 \end{pmatrix}$$
- Since the first component of X_1 is S_1 , $S_1 S_2$ is the first component of the solution vector $(X_1 S_2, X_2)^T$.

- Star: Let $S \in R$ be quasiregular and be the first component of X which is the solution vector of $X = B + AX$. Since A is quasiregular too (see the definition of A at the beginning of this lemma), it follows that $\widehat{B} = 0$ and hence $S = \widehat{A}X$. We show that $\begin{pmatrix} S^* \\ X S^* \end{pmatrix}$ is the solution vector of the following system:

$$\begin{aligned} \begin{pmatrix} \lambda \\ B \end{pmatrix} + \begin{pmatrix} 0 & \widehat{A} \\ 0 & A + B\widehat{A} \end{pmatrix} \begin{pmatrix} S^* \\ X S^* \end{pmatrix} &= \begin{pmatrix} \lambda + \widehat{A}X S^* \\ B + (A + B\widehat{A})X S^* \end{pmatrix} \\ &= \begin{pmatrix} \lambda + S S^* \\ B + A X S^* + B S S^* \end{pmatrix} \\ &= \begin{pmatrix} \lambda + S^+ \\ B + A X + A X S^+ + B S^+ \end{pmatrix} \\ &= \begin{pmatrix} S^* \\ X + X S^+ \end{pmatrix} = \begin{pmatrix} S^* \\ X S^* \end{pmatrix} \end{aligned}$$

It is easy to verify that each of the occurring vectors B and matrices A satisfy $B = B_\lambda \lambda$ and $A = \sum_{w \in \Sigma} A_w w$. This completes the proof. \square

Definition 2.2.3 A formal series $S \in \mathbb{K}\langle\langle \Sigma^* \rangle\rangle$ is called recognizable if its coefficients can be written as follows:

$$s_w = \alpha \cdot \mu(w) \cdot \beta,$$

where for some $n \geq 1$ we have $\alpha \in \mathbb{K}^{1,n}$ and $\beta \in \mathbb{K}^{n,1}$; $\mu : \Sigma^* \rightarrow \mathbb{K}^{n,n}$ is a multiplicative homomorphism of monoids (note that the set $\mathbb{K}^{n,n}$ of square matrices forms a semiring). (α, μ, β) is called a linear representation of S of dimension n (μ itself is usually called a representation).

We are now prepared to prove the famous Theorem of SCHÜTZENBERGER:

Theorem 2.2.4 A formal series $S \in \mathbb{K}\langle\langle \Sigma^* \rangle\rangle$ is \mathbb{K} -rational if and only if S is recognizable.

Proof: \implies : Let S be a \mathbb{K} -rational series. By Lemma 2.2.2 we know that there is a quasiregular linear system $X = B + AX$ with $B = B_\lambda \lambda$ and $A = \sum_{w \in \Sigma} \mu(w) w$ whose solution vector X has S as its first component. Hereby we set $\mu(w) := A_w$ for $|w| = 1$. Theorem 2.2.1 gives the solution $X = A^* B$. Therefore

$$\begin{aligned} S &= (1, 0, \dots, 0) \cdot A^* B = (1, 0, \dots, 0) \left(\sum_{w \in \Sigma^*} \mu(w) w \right) B \\ &= \sum_{w \in \Sigma^*} ((1, 0, \dots, 0) \cdot \mu(w) \cdot B) w \end{aligned}$$

is recognizable.

\impliedby : Let S be recognizable, hence $S = \sum_{w \in \Sigma^*} (\alpha \mu(w) \beta) w$. Let $A = \sum_{w \in \Sigma} \mu(w) w$. Since every element of A is a sum of monomials, A is \mathbb{K} -rational. Then

$$A^+ = \sum_{k=1}^{\infty} A^k = \sum_{k=1}^{\infty} \left(\sum_{w \in \Sigma} \mu(w) w \right)^k = \sum_{k=1}^{\infty} \sum_{w \in \Sigma^k} \mu(w) w = \sum_{w \in \Sigma^+} \mu(w) w.$$

Let A_i^+ be the i^{th} column of A^+ . Then all A_i^+ are \mathbb{K} -rational because they are solutions of the system $(A_i^+)^+ = A_i + A(A_i^+)^+$. Hence A^+ is \mathbb{K} -rational.

Now we can see that S is \mathbb{K} -rational:

$$\begin{aligned} S &= \sum_{w \in \Sigma^*} (\alpha\mu(w)\beta)w = s_\lambda\lambda + \sum_{w \in \Sigma^+} \alpha\mu(w)w\beta = s_\lambda\lambda + \alpha A^+\beta \\ &= s_\lambda\lambda + \sum_{1 \leq i, j \leq n} \alpha_i\beta_j(A_{i,j})^+. \end{aligned}$$

□

In a later section we will need the following result:

Theorem 2.2.5 *Let $S, T \in \mathbb{K}^{\text{rat}}\langle\langle X^* \rangle\rangle$, with \mathbb{K} a commutative semiring. Then also the HADAMARD product $S \odot T$ is \mathbb{K} -rational.*

Proof: It suffices to prove that the HADAMARD product of two recognizable series is again recognizable (according to Theorem 2.2.4). We first show that we can write a recognizable series $S = \sum_{w \in \Sigma^*} \alpha\mu(w)\beta$ as

$$(*) \quad S = s_\lambda\lambda + \sum_{w \in \Sigma^+} \left(\tilde{\mu}(w) \right)_{1,n} \cdot w \quad \text{with } \mu : \Sigma^* \rightarrow \mathbb{K}^{n,n}$$

We define $\tilde{\mu}(w) := \begin{pmatrix} 0 & \alpha\mu(w) & s_w \\ 0 & \mu(w) & \mu(w)\beta \\ 0 & 0 & 0 \end{pmatrix}$, $w \neq \lambda$.

Then we get:

$$\begin{aligned} \tilde{\mu}(u) \cdot \tilde{\mu}(v) &= \begin{pmatrix} 0 & \alpha\mu(u) & s_u \\ 0 & \mu(u) & \mu(u)\beta \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \alpha\mu(v) & s_v \\ 0 & \mu(v) & \mu(v)\beta \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha\mu(u)\mu(v) & \alpha\mu(u)\mu(v)\beta \\ 0 & \mu(u)\mu(v) & \mu(u)\mu(v)\beta \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha\mu(uv) & s_{uv} \\ 0 & \mu(uv) & \mu(uv)\beta \\ 0 & 0 & 0 \end{pmatrix} = \tilde{\mu}(uv) \end{aligned}$$

This proves formula (*).

Now back to the HADAMARD product: Write $S = s_\lambda\lambda + \sum_{w \in \Sigma^+} \left(\mu_1(w) \right)_{1,n_1} w$

and $T = t_\lambda\lambda + \sum_{w \in \Sigma^+} \left(\mu_2(w) \right)_{1,n_2} w$ in the above form. Then:

$$S \odot T = \sum_{w \in \Sigma^*} s_w t_w w = s_\lambda t_\lambda \lambda + \sum_{w \in \Sigma^+} \left(\mu_1(w) \right)_{1,n_1} \left(\mu_2(w) \right)_{1,n_2} w$$

$$= s_\lambda t_\lambda \lambda + \sum_{w \in \Sigma^+} \left(\mu(w) \right)_{1, n_1 n_2} w$$

where $\mu(w)$ is the **KRONECKER product** of $\mu_1(w)$ and $\mu_2(w)$. The **KRONECKER product** of two matrices $A \in \mathbb{K}^{n,n}$ and $B \in \mathbb{K}^{m,m}$ is defined by:

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,n}B \end{pmatrix} \in \mathbb{K}^{nm, nm}$$

where the $a_{i,j}$'s are the entries of A . Since \mathbb{K} is commutative, μ is a representation, i.e.,

$$\left(\mu_1(u) \otimes \mu_2(u) \right) \left(\mu_1(v) \otimes \mu_2(v) \right) = \mu_1(uv) \otimes \mu_2(uv)$$

holds². □

2.3 Regular Languages

In this section we present the basic interrelations between regular languages and rational series. It turns out that a language is regular if and only if its characteristic series is the support of an \mathbb{N} -rational series. This result is strongly connected to **KLEENE's Theorem**, since later on we will show how for any arbitrary \mathbb{N} -rational series a regular expression can be computed.

Every subset $L \subseteq \Sigma^*$ is referred to as a formal language. Thus the support of a series belonging to $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle$ is a language over the alphabet Σ . Otherwise for any language L the series $S \in \mathbb{B}\langle\langle \Sigma^* \rangle\rangle$, whose coefficients equal either 0 or 1, is termed the *characteristic series* $\text{char}(L)$ of L , if

$$s_w = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{if } w \notin L \end{cases} \quad \forall w \in \Sigma^*$$

Recall that a language is called regular (or of type 3 in the **CHOMSKY hierarchy**) if it is generated by a grammar $G = (V, \Sigma, S, P)$, where V is the set of variables (or nonterminals), Σ is the alphabet of terminals, S is the initial letter, and P is the set of productions (or rewriting rules) that are of the form $A \rightarrow xB$ or $A \rightarrow x$, where $A, B \in V$ and $x \in \Sigma$.

A well known essential from language theory is that a language is regular if and only if it is accepted by some deterministic finite automaton

²see [KS86, p. 64] or [Smi60, p. 222]

$(Q, \Sigma, \delta, q_1, F)$, where Q is the set of states, $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, q_1 the initial state, and F the set of accepting states. We will use this characterization to show the basic interconnections between regular languages and rational series.

Theorem 2.3.1 *Let L be a regular language and \mathbb{K} a semiring. Then the characteristic series of L is \mathbb{K} -rational.*

Proof: Since L is regular, it is accepted by a deterministic finite automaton $(\{q_1, \dots, q_m\}, \Sigma, \delta, q_1, F)$. Let $\mu(x)$ be the transition matrix with respect to the letter x :

$$\left(\mu(x)\right)_{ij} := \begin{cases} 1 & \text{if } \delta(q_i, x) = q_j, \\ 0 & \text{otherwise} \end{cases}$$

Furthermore we set $\alpha := (1, 0, \dots, 0)$ and $\beta := (\beta_1, \dots, \beta_m)^T$ with

$$\beta_i = \begin{cases} 1 & \text{if } q_i \in F, \\ 0 & \text{otherwise} \end{cases}$$

Now let $w = w_1 \dots w_k \in L$:

$$\delta(q_1, w) = \delta(\delta(\dots \delta(q_1, w_1), \dots, w_{k-1}), w_k) = q_j \in F.$$

Equivalently we can write

$$\alpha \cdot \mu(w) = \alpha \cdot \mu(w_1) \cdot \dots \cdot \mu(w_k) = e_j^T,$$

where e_j denotes the j^{th} unit vector. Thus $\sum_{w \in \Sigma^*} (\alpha \mu(w) \beta) w$ is the characteristic series of L , since $e_j^T \beta$ equals 1 if $q_j \in F$ and 0 otherwise. By Theorem 2.2.4 this series is \mathbb{K} -rational. \square

Theorem 2.3.2 *The support of any series $S \in \mathbb{N}^{\text{rat}} \langle\langle \Sigma^* \rangle\rangle$ is a regular language.*

Proof: First we show that from the \mathbb{N} -rationality of S it follows that $\text{char}(S)$ is \mathbb{B} -rational. Let $h : \mathbb{N} \rightarrow \mathbb{B}$ be the semiring homomorphism that maps \mathbb{N} onto $\mathbb{B} = \{0, 1\}$. Let $X = B + AX$ be the linear system from Lemma 2.2.2 and let S be the first component of its solution vector. Then $\text{char}(S) = h(S)$ is the first component of the solution vector of the linear system $X = h(B) + h(A)X$. Thus the \mathbb{B} -rationality of $\text{char}(S)$ is implied by Theorem 2.2.1. It is clear that $\text{supp}(\text{char}(S)) = \text{supp}(S)$.

By Theorem 2.2.4 $\text{char}(S)$ is recognizable and can be written as

$$\text{char}(S) = \sum_{w \in \Sigma^*} s_w^c w = \sum_{w \in \Sigma^*} (\alpha \mu(w) \beta) w,$$

where $\alpha \in \mathbb{B}^{1,m}$, $\mu(w) \in \mathbb{B}^{m,m}$, and $\beta \in \mathbb{B}^{m,1}$. Define the deterministic finite automaton $(\{q_1, \dots, q_{2^m}\}, \Sigma, \delta, q_1, F)$, where $q_1 = \alpha, q_2, \dots, q_{2^m}$ are the vectors of $\mathbb{B}^{1,m}$. Analogously to the preceding theorem the transition function δ corresponds to the representation μ

$$\delta(q_i, x) = q_i \cdot \mu(x)$$

and the set of accepting states is

$$F = \{q_i | q_i \beta = 1\}.$$

Then

$$s_w^c = 1 \iff \delta(q_1, w) \in F,$$

and this proves that $\text{supp}(S)$ is a regular language. \square

Note that in language theory, the operation corresponding to the HADAMARD product is the intersection of languages.

2.4 Rational Series in One Variable

From now on we examine rational series over an alphabet that consists only of one single letter: $\Sigma = \{x\}$. Instead of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$ we write $\mathbb{K}\langle\langle x^*\rangle\rangle$. A series $S \in \mathbb{K}\langle\langle x^*\rangle\rangle$ is written as $S = \sum_{n=0}^{\infty} s_n x^n$. Of course, x^* is a commutative monoid. In this section we give three different characterizations of \mathbb{K} -rational series that are valid in the case that \mathbb{K} is a commutative ring: The existence of a rational generating function, the existence of a recurrence relation for the series' coefficients and the possibility to compute the coefficients by means of the exponential polynomial. Theorem 2.4.7 states that changing finitely many coefficients of a rational series preserves rationality.

Definition 2.4.1 *The rational function $f(x) = \frac{p(x)}{q(x)}$ with $p, q \in \mathbb{K}\langle x^*\rangle$ is*

called a generating function of the formal power series $S = \sum_{n=0}^{\infty} s_n x^n$, if $p = qS$.

Theorem 2.4.2 *Let \mathbb{K} be a commutative ring and $S \in \mathbb{K}\langle\langle x^*\rangle\rangle$. Then*

$$S \in \mathbb{K}^{\text{rat}}\langle\langle x^*\rangle\rangle \iff S \text{ has a generating function } f(x) = \frac{p(x)}{1 - q(x)},$$

where $p, q \in \mathbb{K}\langle x^\rangle$ are polynomials and q is quasiregular, in other words the denominator of f is not divisible by x and its constant term is 1.*

Proof: \implies : We show that the set of power series which have a generating function of the mentioned form, is closed under the rational operations sum, product, and star. Therefore all rational power series must have such a generating function. Let S and T be two power series (not necessarily infinite) and $f(x) = \frac{p_f(x)}{1 - q_f(x)}$, $g(x) = \frac{p_g(x)}{1 - q_g(x)}$ their generating functions. Then the following relations hold:

$$\begin{aligned} \bullet \quad f + g &= \frac{p_f(1 - q_g) + p_g(1 - q_f)}{(1 - q_f)(1 - q_g)} = \frac{p_f + p_g - p_f q_g - p_g q_f}{1 - (q_f + q_g - q_f q_g)} \\ \bullet \quad f \cdot g &= \frac{p_f p_g}{1 - (q_f + q_g - q_f q_g)} \\ \bullet \quad f^* &= \left(1 - \frac{p_f}{1 - q_f}\right)^{-1} = \frac{1 - q_f}{1 - (q_f + p_f)} \end{aligned}$$

In every case we get a rational function of the desired form with coefficients in \mathbb{K} .

\Leftarrow : We write S as $S = \frac{p(x)}{1 - q(x)} = p \cdot q^*$. Hence S is rational, because it is obtained by rational operations (product and star) from the polynomials $p(x)$ and $q(x)$. \square

A rational function $p(x)/q(x)$ is called *normalized*, if p and q have no common factor in $\mathbb{K}\langle x^* \rangle$ and if $q(0) = 1$. In this case, $q(x)$ is termed the *minimal denominator* of the series. In the following we consider generating functions to be given always in normalized form.

Theorem 2.4.3 *Let \mathbb{K} be a commutative ring and $S = \sum_{n=0}^{\infty} s_n x^n$ a power series with coefficients $s_n \in \mathbb{K}$. Then we have for large n :*

$$S \text{ is } \mathbb{K}\text{-rational} \iff s_n = q_1 s_{n-1} + \dots + q_k s_{n-k}, \quad q_i \in \mathbb{K}.$$

Proof: \implies : Let S be \mathbb{K} -rational. Hence it has a generating function

$$S = \frac{p_0 + p_1 x + \dots + p_l x^l}{1 - (q_1 x + q_2 x^2 + \dots + q_k x^k)} \quad (\text{Theorem 2.4.2}).$$

Multiplying the denominator to the left side gives

$$(s_0 + s_1 x + s_2 x^2 + \dots)(1 - q_1 x - q_2 x^2 - \dots - q_k x^k) = p_0 + p_1 x + \dots + p_l x^l.$$

Comparing the coefficients

$$\underbrace{s_0}_{=p_0} + \underbrace{(s_1 - q_1 s_0)}_{=p_1} x + \underbrace{(s_2 - s_1 q_1 - s_0 q_2)}_{=p_2} x^2 + \dots + \underbrace{(s_l - \dots - s_0 q_l)}_{=p_l} x^l$$

teaches us that $s_n - s_{n-1}q_1 - \dots - s_{n-k}q_k = 0 \forall n \geq l + 1$.

\Leftarrow : By reading the first part of the proof backwards it can be seen that the other direction holds also. \square

Definition 2.4.4 Let S be a rational power series and $f(x) = p(x)/q(x)$ its normalized generating function. Then the roots of the denominator $q(x) = 1 - q_1x - q_2x^2 - \dots - q_kx^k$ are called poles of S . The roots of the reciprocal polynomial $\bar{q}(x) = x^k - q_1x^{k-1} - \dots - q_{k-1}x - q_k$ are called roots of S .

Theorem 2.4.5 Let $S = \sum_{n=0}^{\infty} s_n x^n \in \mathbb{K}\langle\langle x^* \rangle\rangle$, \mathbb{K} again a commutative ring, be an infinite power series (i.e., not a polynomial) and let $\lambda_0, \dots, \lambda_r$ its distinct roots with the multiplicities m_0, \dots, m_r . Then we have for large n :

$$S \text{ is } \mathbb{K}\text{-rational} \iff s_n = \sum_{i=0}^r P_i(n) \lambda_i^n.$$

The P_i 's are complex nonzero polynomials with $\deg P_i = m_i - 1$ and with coefficients that are algebraic over \mathbb{K} . This exponential representation is unique.

Proof: Let $f(x)$ be the generating function of S . We expand $f(x)$ in partial fractions, and remember that the roots of the denominator are $\lambda_0^{-1}, \dots, \lambda_r^{-1}$:

$$f(x) = \sum_{i=0}^r \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(x - \lambda_i^{-1})^j} = \sum_{i=0}^r \sum_{j=1}^{m_i} \frac{\alpha_{i,j} (-\lambda_i)^j}{(1 - \lambda_i x)^j} \text{ with } \alpha_{i,m_i} \neq 0$$

We get:

$$\begin{aligned} (1 - \lambda_i x)^{-j} &= \sum_{n=0}^{\infty} \binom{-j}{n} (-\lambda_i x)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(-j)(-j-1) \cdots (-j-n+1)}{n!} \lambda_i^n x^n \\ &= \sum_{n=0}^{\infty} \frac{j(j+1) \cdots (j+n-1)}{n!} \lambda_i^n x^n \\ &= \sum_{n=0}^{\infty} \frac{(n+1) \cdots (n+j-1)}{(j-1)!} \lambda_i^n x^n \end{aligned}$$

Therefore we have for large n :

$$s_n = \sum_{i=0}^r \underbrace{\left(\sum_{j=1}^{m_i} \alpha_{i,j} (-\lambda_i)^j \frac{(n+1) \cdots (n+j-1)}{(j-1)!} \right)}_{=: P_i(n)} \lambda_i^n$$

It can be seen that n^{m_i-1} is the highest power in $P_i(n)$ (remember that $\alpha_{i,m_i} \neq 0$). The coefficients of the P_i 's are algebraic over \mathbb{K} , because the λ_i 's and the $\alpha_{i,j}$'s are.

The uniqueness follows from the uniqueness of the partial fraction expansion.

□

Corollary 2.4.6 *Let \mathbb{K} be a subring of \mathbb{R} , $\mathbb{K}_+ \subseteq \mathbb{K}$ the (commutative) semiring of all nonnegative numbers in \mathbb{K} , and $S = \sum_{n=0}^{\infty} s_n x^n \in \mathbb{K}_+^{\text{rat}} \langle\langle x^* \rangle\rangle \setminus \mathbb{K}_+ \langle x^* \rangle$. Then we have for large n :*

$$s_n = \sum_{i=0}^r P_i(n) \lambda_i^n.$$

The λ_i 's are the roots of S with the multiplicities m_i , and the P_i 's are complex nonzero polynomials with $\deg P_i = m_i - 1$.

Theorem 2.4.7 *Given a commutative semiring \mathbb{K} .*

$$S \in \mathbb{K} \langle\langle x^* \rangle\rangle \text{ is } \mathbb{K}\text{-rational} \iff \exists h \geq -1 : S^{[h]} := \sum_{n>h} s_n x^n \text{ is } \mathbb{K}\text{-rational.}$$

Proof: \implies : S is \mathbb{K} -rational since it is the HADAMARD product $S \odot (x^{h+1} x^*)$ (see Theorem 2.2.5).

\impliedby : Let $S^{[h]}$ be \mathbb{K} -rational. Then $S = S^{[h]} + P$, where P is a polynomial and $\deg P \leq h$. Thus S is \mathbb{K} -rational. □

Example 2.4.8 *Given the series*

$$1 + 6x^2 - 9x^3 + 39x^4 - 102x^5 + 324x^6 - 951x^7 + \dots = S \in \mathbb{Z} \langle\langle x^* \rangle\rangle.$$

The coefficients can be computed by the recurrence equation

$$s_n = -s_{n-1} + 5s_{n-2} - 3s_{n-3}.$$

All s_n 's with $n \geq 5$ fulfill this equation. The generating function of S is

$$f(x) = \frac{x^2 + x + 1}{1 + x - 5x^2 + 3x^3}$$

where we can find the coefficients of our recurrence equation in the denominator. Since the denominator factors to $(1 + 3x)(1 - x)^2$ the roots of S are $\lambda_0 = -3$ and $\lambda_1 = 1$. The exponential polynomial therefore is:

$$s_n = \sum_{i=0}^1 P_i(n)\lambda_i^n = \frac{7}{16}(-3)^n + \left(\frac{9}{16} + \frac{3}{4}n\right)1^n,$$

$$\text{e.g., } s_6 = \frac{7}{16} \cdot 729 + \left(\frac{9}{16} + \frac{18}{4}\right) = 324.$$

In this example, this representation is valid for all s_n .

We have seen that for \mathbb{K} being a commutative ring, it is very easy to verify if a given series is \mathbb{K} -rational or not. Indeed in most cases it will be obvious by the form how the series is “given”.

2.5 Positive Series

The last section showed that for a formal power series the question of \mathbb{K} -rationality is not difficult, if \mathbb{K} is a commutative ring. In the following let \mathbb{K} be a subring of \mathbb{R} . In this section we want to examine the case of \mathbb{K}_+ which is only a semiring (e.g. $\mathbb{K}_+ = \mathbb{N}$ or $\mathbb{K}_+ = \mathbb{R}_+$). It will be seen that this case is much more difficult and it will take some efforts to work out a criterion for deciding rationality. In general it is not sufficient to show that a series in $\mathbb{K}^{\text{rat}}\langle\langle x^* \rangle\rangle$ has positive coefficients. It may happen that such a series nevertheless is not \mathbb{K}_+ -rational.

Example 2.5.1 Consider the series A094423 from [Slo]:

$$x + 4x^2 + x^3 + 144x^4 + 361x^5 + 484x^6 + 19321x^7 + 28224x^8 + 128881x^9 + \dots$$

which is generated by the function

$$\frac{x + 5x^2}{1 + x - 5x^2 - 125x^3}.$$

Although all coefficients of this series are positive integers the series is not \mathbb{N} -rational. Later (in Section 4.4) we will see why.

In general a semiring \mathbb{K} is called a *Fatou extension* to a subsemiring \mathbb{K}' of \mathbb{K} iff all \mathbb{K} -rational series with coefficients in \mathbb{K}' are \mathbb{K}' -rational too, i.e., if $\mathbb{K}^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle \cap \mathbb{K}'\langle\langle\Sigma^*\rangle\rangle = \mathbb{K}'^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$. We have seen that \mathbb{Z} is not a FATOU extension of \mathbb{N} . An example for a FATOU extension of \mathbb{N} is the semiring \mathbb{Q}_+ of nonnegative rationals.

Most proofs here are carried out for $\mathbb{K} = \mathbb{R}_+$, but at the end of the section it will be obvious that all results are valid for $\mathbb{K} = \mathbb{N}$, too.

Theorem 2.5.2 gives us a remarkable property of (infinite) \mathbb{R}_+ -rational series: The poles with minimal modulus are of the form $\varrho\vartheta$ where $\varrho > 0$ and ϑ is a root of unity. This fact makes it possible to obtain a dominating root by decomposing the series such that all the roots $\varrho\vartheta$ become real (Theorem 2.5.12). We then introduce the notion of a decomposition of a series (Theorem 2.5.8). For the proof of Theorem 2.5.11, which gives an exact characterization of \mathbb{R}_+ -rationality, we will need some preparing lemmata (2.5.4, 2.5.5, 2.5.9, and 2.5.10).

Theorem 2.5.2 *Let $S \in \mathbb{R}_+^{\text{rat}}\langle\langle x^* \rangle\rangle \setminus \mathbb{R}_+\langle x^* \rangle$ have the generating function $f(x)$ and the roots $\lambda_0, \dots, \lambda_r$ and let $\varrho := \min_{0 \leq i \leq r} |\lambda_i^{-1}|$. Then the following statement holds:*

ϱ is a pole of S (let m_ϱ be its multiplicity) and all other poles of modulus ϱ have the form $\varrho\vartheta$ and a multiplicity $\leq m_\varrho$. ϑ denotes a complex root of unity, i.e., $\exists p \in \mathbb{N} : \vartheta^p = 1$. (*)

Proof: We first show that the multiplicity of any pole $\varrho\vartheta$ cannot be greater than the multiplicity of ϑ . Consider the inequality

$$|f(x)| = \left| \sum_{n=0}^{\infty} s_n(x)^n \right| \leq \sum_{n=0}^{\infty} s_n|x|^n = f(|x|).$$

If $\varrho\vartheta$ is a pole then $\lim_{t \rightarrow 1^-} |f(t\varrho\vartheta)| = \infty$. The inequality shows that then also ϱ itself must be a pole. Next let $x = \varrho\vartheta$:

$$|(tx - x)^{m_\varrho} f(tx)| = |(tx - x)^{m_\varrho} f(tx)| \leq (|x| - t|x|)^{m_\varrho} f(t|x|) \quad (t < 1)$$

Since m_ϱ is the multiplicity of the pole ϱ , the right side remains bounded when $t \rightarrow 1^-$. The same holds for the left side and therefore the multiplicity of x is at most m_ϱ .

Let $R \subseteq \mathbb{R}_+\langle\langle x^* \rangle\rangle$ be the set of power series, that are either polynomials or have the form described in (*) and have nonnegative coefficients. We have to show that R is rational closed. We first look at the cases sum and product:

Let S and T be power series in R and $f(x) = \frac{p_f(x)}{q_f(x)}$, $g(x) = \frac{p_g(x)}{q_g(x)}$ their generating functions. Then their sum and product are

- $S + T: f(x) + g(x) = \frac{p_f(x)q_g(x) + p_g(x)q_f(x)}{q_f(x)q_g(x)}$
- $S \cdot T: f(x)g(x) = \frac{p_f(x)p_g(x)}{q_f(x)q_g(x)}$

We see that the poles of $S + T$ ($S \cdot T$) are among the poles of S and T . Furthermore the pole with minimal modulus ϱ_{S+T} (resp. $\varrho_{S \cdot T}$) is the minimum of ϱ_S and ϱ_T . Some poles may be canceled down. But can the pole ϱ_{S+T} (resp. $\varrho_{S \cdot T}$) itself be eliminated and then have a multiplicity smaller than any of the poles ϱ_{S+T}^ϑ (resp. $\varrho_{S \cdot T}^\vartheta$), $\vartheta \neq 1$? No, because this would contradict the first part of the proof. Hence $S + T$ and $S \cdot T$ satisfy (*).

To show that R is rationally closed with respect to the star is little more complicated: We consider the quasiregular generating function $f(x) = \sum_{n=1}^{\infty} s_n x^n$.

Since $f^*(x) = \frac{1}{1-f(x)}$, we have to examine the roots of $1 - f(x)$, i.e., the values of x for which $f(x) = 1$. The quasiregularity of $f(x)$ implies $f(0) = 0$. Next we know that $f(\varrho) = \infty$. Since the coefficients s_n are positive $f(x)$ grows monotonically from 0 to ∞ if x goes from 0 to ϱ . There must be a real number $0 \leq \mu \leq \varrho$ such that $f(\mu) = 1$. Thus μ is a pole of $f^*(x)$.

We show now that f^* has no poles with smaller modulus than μ and that all poles with modulus μ have the form μ^ϑ . Let z be a pole of f^* and $|z| \leq \mu$. Then:

$$\begin{aligned} 1 &= f(z) = \sum_{n=1}^{\infty} s_n z^n = \operatorname{Re} \left(\sum_{n=1}^{\infty} s_n z^n \right) \\ &= \sum_{n=1}^{\infty} s_n \operatorname{Re}(z^n) \leq \sum_{n=1}^{\infty} s_n |z|^n \leq \sum_{n=1}^{\infty} s_n \mu^n = f(\mu) = 1 \end{aligned}$$

Consequently everywhere equality must hold. From

$$\sum_{n=1}^{\infty} s_n \operatorname{Re}(z^n) - \sum_{n=1}^{\infty} s_n \mu^n = \sum_{n=1}^{\infty} s_n (\operatorname{Re}(z^n) - \mu^n) = 0$$

follows $\operatorname{Re}(z^n) = \mu^n \forall n \geq 1$, since all $s_n \geq 0$. For some n with $s_n \neq 0$ we have $\operatorname{Re}(z^n) = \mu^n$. But here is $|z| \leq \mu$, therefore $z^n = \mu^n$. We see now that $z = \mu^\vartheta$ and ϑ is a n th complex root of unity. \square

Definition 2.5.3 Let $\lambda_0, \dots, \lambda_r$ be the roots of a rational power series S . λ_0 is called dominating root of S if $\lambda_0 \in \mathbb{R}_+$ and $\lambda_0 > |\lambda_i|, 1 \leq i \leq r$ holds. Moreover λ_0 is called strictly dominating root if $\lambda_0 \in \mathbb{R}$, $\lambda_0 > 1$ and $|\lambda_0 \lambda_i| < 1, 1 \leq i \leq r$.

Lemma 2.5.4 Let $S = \sum_{n=0}^{\infty} s_n x^n$ be a rational infinite series (thus S is not a polynomial) and let S have the dominating root λ_0 . Then there exists a real number $c > 0$, such that the series $\sum_{n=0}^{\infty} \frac{s_n}{c^n} x^n$ has a strictly dominating root.

Proof: Let c' be a real number with $\lambda_0 > c' > |\lambda_i|, 1 \leq i \leq r$ and $\lambda'_i := \frac{\lambda_i}{c'}, 0 \leq i \leq r$. Thus $\lambda'_0 > 1 > |\lambda'_i|$ holds. Moreover let c'' be a real number which satisfies $1 < c'' < \lambda'_0 < c''^2$ and $\lambda''_i := \frac{\lambda'_i}{c''} = \frac{\lambda_i}{c'c''}, 0 \leq i \leq r$. Now we have:

$$|\lambda''_0 \lambda''_i| = \frac{|\lambda'_0 \lambda'_i|}{c''^2} < \frac{\lambda'_0}{c''^2} < 1 \quad \text{und} \quad \lambda''_0 > 1.$$

It follows that λ''_0 is strictly dominating (compared to the $\lambda''_i, 1 \leq i \leq r$). According to Theorem 2.4.5 the coefficients of S have the following representation (for large n):

$$s_n = \sum_{i=0}^r P_i(n) \lambda_i^n.$$

We substitute herein the $\lambda''_0, \dots, \lambda''_r$ and obtain the desired power series $\sum_{n=0}^{\infty} \frac{s_n}{c^n} x^n$ with $c := c'c''$. □

Lemma 2.5.5 Let $S = \sum_{n=0}^{\infty} s_n x^n \in \mathbb{R}_+ \langle\langle x^* \rangle\rangle$ be a rational series with non-negative coefficients having a dominating root $\lambda_0 > 1$. Then $s_{n+1} > cs_n$ holds for large n , where c is an arbitrary real number smaller than λ_0 .

Proof: According to Theorem 2.4.5 for large n we can write

$$s_n = P_0(n) \lambda_0^n + \sum_{i=1}^r P_i(n) \lambda_i^n,$$

where the P_i 's are nonzero polynomials. Let d be the degree of P_0 and $p_{0,d}$ its leading coefficient. Then:

$$s_n \sim p_{0,d} n^d \lambda_0^n, \quad \text{i.e.,} \quad \lim_{n \rightarrow \infty} \frac{s_n}{p_{0,d} n^d \lambda_0^n} = 1.$$

and

$$\frac{s_{n+1}}{cs_n} \sim \frac{p_{0,d}(n+1)^d \lambda_0^{n+1}}{cp_{0,d} n^d \lambda_0^n} \sim \frac{\lambda_0}{c} > 1.$$

Since $s_n > 0$ we have $s_{n+1} > cs_n$ for large enough n . □

Definition 2.5.6 Given a formal series $S = \sum_{n=0}^{\infty} s_n x^n$. For any $p \in \mathbb{N}$ the list of series S_0, \dots, S_{p-1} is called a decomposition of S if

$$S_i = \sum_{n=0}^{\infty} s_{i+np} x^n.$$

Thus to build up the subseries S_i one has to take every p^{th} coefficient, beginning at index i . On the other hand S is termed the merge of S_0, \dots, S_{p-1} :

$$S(x) = \sum_{i=0}^{p-1} x^i S_i(x^p).$$

Example 2.5.7 Let $p = 3$ and $S = \sum_{n=0}^{\infty} s_n x^n$. Then we get a decomposition:

$$\begin{aligned} S_0 &= s_0 + s_3 x + s_6 x^2 + \dots = \sum_{n=0}^{\infty} s_{3n} x^n \\ S_1 &= s_1 + s_4 x + s_7 x^2 + \dots = \sum_{n=0}^{\infty} s_{3n+1} x^n \\ S_2 &= s_2 + s_5 x + s_8 x^2 + \dots = \sum_{n=0}^{\infty} s_{3n+2} x^n \end{aligned}$$

How the generating functions of the S_i 's can be computed is described in Chapter 3 (see Theorem 3.3.3).

Theorem 2.5.8 Let \mathbb{K} be a semiring. $S \in \mathbb{K}\langle\langle x^* \rangle\rangle$ is \mathbb{K} -rational if and only if there exist for any $p \in \mathbb{N}$ a set of \mathbb{K} -rational power series S_0, S_1, \dots, S_{p-1} and their merge is S . Moreover, if \mathbb{K} is commutative and $\lambda_0, \dots, \lambda_r$ are the roots of S with multiplicities m_0, \dots, m_r , then every of the series S_j has the following properties: The roots μ_0, \dots, μ_s ($s \leq r$) of S_j are among the numbers $\lambda_0^p, \dots, \lambda_r^p$, and any root μ_l of S_j has the multiplicity $m'_l \leq \max_{0 \leq i \leq r} \{m_i : \lambda_i^p = \mu_l\}$.

Proof: We are proving the first part of the theorem in both directions:

\implies : Let S be \mathbb{K} -rational. According to Theorem 2.2.4 S has a linear representation $S = \sum_{n=0}^{\infty} (\alpha \mu(x^n) \beta) x^n$. Since $S_i = \sum_{n=0}^{\infty} s_{i+np} x^n$ we have

$$S_i = \sum_{n=0}^{\infty} (\alpha \mu(x^{i+np}) \beta) x^n = \sum_{n=0}^{\infty} (\alpha \mu^i(x) \mu^p(x^n) \beta).$$

Thus S_i has the linear representation $(\alpha\mu^i(x), \mu^p, \beta)$ and is \mathbb{K} -rational.

\Leftarrow : Let S_i be \mathbb{K} -rational. First we observe that the image $S_i^{(p)}$ under the morphism $x \mapsto x^p$ is again \mathbb{K} -rational: Imagine S_i to be given as an expression of elements in $\mathbb{K}\langle x^* \rangle$, linked by rational operations. Now replace each occurrence of x by x^p . It is obvious that the resulting expression is \mathbb{K} -rational again. But then also the sum of products

$$\sum_{i=0}^{p-1} x^i S_i^{(p)} = S$$

is \mathbb{K} -rational.

We now come to the second part of the theorem concerning the roots of the series S_i :

Let $S = \sum_{n=0}^{\infty} s_n x^n$ be \mathbb{K} -rational, hence its coefficients are $s_n = \sum_{i=0}^r P_i(n) \lambda_i^n$ for large n (Theorem 2.4.5). Therefore S_j can be written as follows:

$$S_j = \sum_{n=0}^{\infty} s_{j+np} x^n \text{ with } s_{j+np} = \sum_{i=0}^r P_i(j+np) \lambda_i^j (\lambda_i^p)^n = \sum_{l=0}^s Q_l(n) \mu_l^n,$$

where $Q_l(n) = \sum_{\substack{0 \leq i \leq r \\ \lambda_i^p = \mu_l}} P_i(j+np) \lambda_i^j$. Of course $\deg P_i(n) = \deg P_i(j+np)$ and

we have:

$$m'_l = \deg Q_l + 1 \leq \max_{\substack{0 \leq i \leq r \\ \lambda_i^p = \mu_l}} \{\deg P_i\} + 1 = \max_{0 \leq i \leq r} \{m_i : \lambda_i^p = \mu_l\}.$$

This is always true because of the uniqueness of the exponential polynomial (see Theorem 2.4.5). \square

Lemma 2.5.9 *Let $S = \sum_{n=0}^{\infty} s_n x^n \in \mathbb{R}^{\text{rat}}\langle\langle x^* \rangle\rangle$ be a rational power series with different nonzero roots $\lambda_0, \dots, \lambda_r$, and λ_0 is dominating with multiplicity m . Then there exists a number $c > 0$ and an integer p such that if we decompose S in p power series S_0, \dots, S_{p-1} ($S_i = \sum_{n=0}^{\infty} s_{i+np} x^n$), the following statements hold:*

- *There exists a polynomial $R^{(p)} = 1 - r_1 x - \dots - r_k x^k \in \mathbb{R}\langle x^* \rangle$ with simple root λ_0^{-p} (which is the dominating root of S_i , again with multiplicity m , see Theorem 2.5.8), and its coefficients satisfy the following*

inequalities:

$$\begin{aligned} r_1 - c &=: \gamma_1 &> 0 \\ r_2 + cr_1 - c^2 &=: \gamma_2 &> 0 \\ &\dots & \\ r_k + cr_{k-1} + \dots + c^{k-1}r_1 - c^k &=: \gamma_k &> 0 \end{aligned}$$

and

$$\frac{r_1}{\lambda_0} + 2\frac{r_2}{\lambda_0^2} + \dots + k\frac{r_k}{\lambda_0^k} > 0.$$

- If $m = 1$ then $R^{(p)}$ is a denominator of S_i .
- The expression $\frac{1}{R^{(p)}}$ is \mathbb{R}_+ -rational.

Proof: We define a polynomial R as follows: If λ_0 is a simple root of S (thus $m = 1$) then let R be the minimal denominator of S . If otherwise $m \geq 2$ we choose R to be the minimal polynomial of λ_0^{-1} (or any other polynomial having the root λ_0^{-1} with multiplicity 1 and having real coefficients). Note that R in both cases divides the denominator of S . We denote the roots of R with $\mu_0^{-1}, \dots, \mu_{k-1}^{-1}$. Without loss of generality let $\mu_0 = \lambda_0$ and $\mu_1, \dots, \mu_{k-1} \in \{\lambda_i : 1 \leq i \leq r\}$. The μ_i^{-1} 's should be simple roots of R ; if we have multiple roots we have to denote them with several μ_i^{-1} 's (according to the multiplicity of this root). Hence R has degree k .

We choose a real number \tilde{c} in the same manner as in Lemma 2.5.4. Then the polynomial

$$\tilde{R} := 1 - \frac{r_1}{\tilde{c}}x - \frac{r_2}{\tilde{c}^2}x^2 - \dots - \frac{r_k}{\tilde{c}^k}x^k = 1 - \tilde{r}_1x - \tilde{r}_2x^2 - \dots - \tilde{r}_kx^k$$

has the roots $\frac{\mu_0^{-1}}{\tilde{c}}, \dots, \frac{\mu_{k-1}^{-1}}{\tilde{c}}$ and the first one is strictly dominating. We denote the roots of \tilde{R} by $\tilde{\mu}_0^{-1}, \dots, \tilde{\mu}_{k-1}^{-1}$. Let now $\tilde{R}^{(p)} = 1 - \tilde{r}_1^{(p)}x - \dots - \tilde{r}_k^{(p)}x^k$ be that polynomial, which has the p^{th} powers of the roots of \tilde{R} as roots (thus $\tilde{\mu}_0^{-p}, \dots, \tilde{\mu}_{k-1}^{-p}$) and again all of them with multiplicity 1. We write $\tilde{R}^{(p)}$ as $(1 - \tilde{\mu}_0^p x) \cdot \dots \cdot (1 - \tilde{\mu}_{k-1}^p x)$ and expand. Thus we get:

$$\tilde{R}^{(p)} = 1 + \underbrace{\sum_{n=1}^k (-1)^n \left(\sum_{0 \leq i_1 < \dots < i_n < k} \tilde{\mu}_{i_1}^p \cdot \dots \cdot \tilde{\mu}_{i_n}^p \right)}_{= -\tilde{r}_n^{(p)}} x^n$$

Therefore the coefficients $\tilde{r}_n^{(p)}$ satisfy the following inequalities:

$$\bullet \quad \tilde{r}_1^{(p)} = \sum_{i=0}^{k-1} \tilde{\mu}_i^p = \tilde{\mu}_0^p + \sum_{i=1}^{k-1} \tilde{\mu}_i^p \geq \tilde{\mu}_0^p - \left| \sum_{i=1}^{k-1} \tilde{\mu}_i^p \right| \geq \tilde{\mu}_0^p - \sum_{i=1}^{k-1} |\tilde{\mu}_i^p| \geq \tilde{\mu}_0^p - k + 1,$$

because from $\tilde{\mu}_0 > 1$ and $|\tilde{\mu}_0 \tilde{\mu}_i| < 1$ (remember that $\tilde{\mu}_0$ is strictly dominating) follows that $|\tilde{\mu}_i^p| < |\tilde{\mu}_i| < 1$ (for $1 \leq i < k$).

- Otherwise we have for $n \geq 2$:

$$|\tilde{r}_n^{(p)}| = \left| \sum_{0 \leq i_1 < \dots < i_n < k} \tilde{\mu}_{i_1}^p \cdot \dots \cdot \tilde{\mu}_{i_n}^p \right| \leq \sum_{0 \leq i_1 < \dots < i_n < k} |\tilde{\mu}_{i_1}^p \cdot \dots \cdot \tilde{\mu}_{i_n}^p| \leq \binom{k}{n},$$

since for every monomial $|\tilde{\mu}_{i_1}^p \cdot \dots \cdot \tilde{\mu}_{i_n}^p| < 1$ holds.

Now it is clear that $\tilde{r}_1^{(p)}$ grows for increasing p whereas the $\tilde{r}_2^{(p)}, \dots, \tilde{r}_k^{(p)}$ are bounded. We choose p such that $\tilde{r}_1^{(p)} \geq \sum_{i=2}^k |\tilde{r}_i^{(p)}| + 1$ holds. Then

$$\tilde{r}_1^{(p)} - \sum_{i=2}^j |\tilde{r}_i^{(p)}| \geq 1 \implies \tilde{r}_1^{(p)} + \dots + \tilde{r}_j^{(p)} \geq 1 \quad \text{for } 1 \leq j \leq k. \quad (*)$$

We set now $c := \tilde{c}^p$ and define analogously to $\tilde{R}^{(p)}$ the polynomial $R^{(p)} = 1 - r_1^{(p)}x - \dots - r_k^{(p)}x^k$ which has the roots $\mu_0^p, \dots, \mu_{k-1}^p$ and which will fulfill all postulated conditions. With the above considerations about the coefficients of $\tilde{R}^{(p)}$ it is clear that

$$\tilde{r}_n^{(p)} = (-1)^{n-1} \sum_{0 \leq i_1 < \dots < i_n < k} \tilde{\mu}_{i_1}^p \cdot \dots \cdot \tilde{\mu}_{i_n}^p = (-1)^{n-1} \sum_{0 \leq i_1 < \dots < i_n < k} \frac{\mu_{i_1}^p}{\tilde{c}^p} \cdot \dots \cdot \frac{\mu_{i_n}^p}{\tilde{c}^p} = \frac{r_n^{(p)}}{c^n}$$

From the inequalities (*) we can now conclude that all γ_i 's are nonnegative.

Furthermore we pay attention that $\tilde{r}_1^{(p)} > \frac{1}{\tilde{\mu}_0^p} \sum_{i=2}^k i \tilde{c}^{2p(i-1)} \binom{k}{i}$ holds. Then

$$\begin{aligned} 0 &< \frac{\tilde{c}^{2p} \tilde{r}_1^{(p)}}{\tilde{\mu}_0^p} - \frac{1}{\tilde{\mu}_0^{2p}} \sum_{i=2}^k i \tilde{c}^{2ip} \binom{k}{i} \\ &\leq \frac{\tilde{c}^{2p} \tilde{r}_1^{(p)}}{\tilde{\mu}_0^p} - \sum_{i=2}^k i \tilde{c}^{2ip} \binom{k}{i} \tilde{\mu}_0^{-ip} \\ &\leq \frac{\tilde{c}^{2p} \tilde{r}_1^{(p)}}{\tilde{\mu}_0^p} - \sum_{i=2}^k i \tilde{c}^{2ip} \cdot |\tilde{r}_i^{(p)}| \tilde{\mu}_0^{-ip} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\tilde{c}^{2p} \tilde{r}_1^{(p)}}{\tilde{\mu}_0^p} + 2 \frac{\tilde{r}_2^{(p)} \tilde{c}^{4p}}{\tilde{\mu}_0^{2p}} \dots + k \frac{r_k^{(p)} \tilde{c}^{2kp}}{\tilde{\mu}_0^{kp}} \\
&\implies \frac{r_1^{(p)}}{\mu_0^p} + 2 \frac{r_2^{(p)}}{\mu_0^{2p}} + \dots + k \frac{r_k^{(p)}}{\mu_0^{kp}} > 0.
\end{aligned}$$

Hereby the last inequality is satisfied and the polynomial $R^{(p)}$ with its coefficients $r_i^{(p)}$ obeys all postulated conditions.

If $m = 1$, then we have chosen R to be the minimal denominator of S , hence $R^{(p)}$ is a multiple of the minimal denominator of S_i (see Theorem 2.5.8).

Last we show that $\frac{1}{R^{(p)}}$ is a \mathbb{R}_+ -rational expression:

$$\begin{aligned}
R^{(p)}(cx)^* &= (1 - r_1^{(p)}x - r_2^{(p)}x^2 - \dots - r_k^{(p)}x^k)(1 + cx + c^2x^2 + \dots) \\
&= 1 - \underbrace{(r_1^{(p)} - c)}_{=: \gamma_1}x - \underbrace{(r_2^{(p)} + cr_1^{(p)} - c^2)}_{=: \gamma_2}x^2 - \dots \\
&\quad - \underbrace{(r_k^{(p)} + \dots + c^{k-1}r_1 - c^k)}_{=: \gamma_k}x^k (cx)^* \\
&= 1 - \gamma_1x - \gamma_2x^2 - \dots - \gamma_{k-1}x^{k-1} - \gamma_kx^k (cx)^*.
\end{aligned}$$

The γ_i 's are nonnegative. Therefore

$$\begin{aligned}
\frac{1}{R^{(p)}} &= (cx)^* \frac{1}{(cx)^* R^{(p)}} = \frac{(cx)^*}{1 - (\gamma_1x + \gamma_2x^2 + \dots + \gamma_kx^k (cx)^*)} \\
&= (cx)^* (\gamma_1x + \gamma_2x^2 + \dots + \gamma_kx^k (cx)^*)^*
\end{aligned}$$

is \mathbb{R}_+ -rational. □

Lemma 2.5.10 *Let $S \in \mathbb{R}_+ \langle\langle x^* \rangle\rangle$ be a rational series having the dominating root λ_0 with multiplicity $m > 1$ and $R = 1 - r_1x - \dots - r_kx^k \in \mathbb{R} \langle x^* \rangle$ a polynomial with simple root λ_0^{-1} and with all the properties given in Lemma 2.5.9. Then the series $T = S \cdot R$ has the dominating root λ_0 with multiplicity $m - 1$ and for its coefficients $t_n \geq 0$ holds for large n .*

Proof: We write the coefficients of T as follows:

$$t_n = s_n - r_1s_{n-1} - \dots - r_k s_{n-k} = \sum_{i=0}^r P_i(n) \lambda_i^n - \sum_{j=1}^k r_j \sum_{i=0}^r P_i(n-j) \lambda_i^{n-j}.$$

The exponential polynomial of T being $\sum_{i=0}^r Q_i(n)\lambda_i^n$, we get a representation of the polynomial $Q_0(n)$:

$$Q_0(n) = P_0(n) - \sum_{j=1}^k \frac{r_j}{\lambda_0^j} P_0(n-j).$$

Let $P_0(n) = an^{m-1} + bn^{m-2} + \widehat{P}_0(n)$, with $a \in \mathbb{R}_+^0$ (since $s_n \geq 0 \forall n \in \mathbb{N}$), $b \in \mathbb{C}$, and $\deg \widehat{P}_0 = m-3$. Therefore $Q_0(n) =$

$$\begin{aligned} &= an^{m-1} + bn^{m-2} + \widehat{P}_0(n) \\ &\quad - \sum_{j=1}^k \frac{r_j}{\lambda_0^j} \left(a(n-j)^{m-1} + b(n-j)^{m-2} + \widehat{P}_0(n-j) \right) \\ &= (an^{m-1} + bn^{m-2}) \left(1 - \sum_{j=1}^k \frac{r_j}{\lambda_0^j} \right) - \sum_{j=1}^k \frac{r_j}{\lambda_0^j} an^{m-2} (-j)(m-1) + r(n) \\ &= (an^{m-1} + bn^{m-2}) \underbrace{R\left(\frac{1}{\lambda_0}\right)}_{=0} + a(m-1)n^{m-2} \sum_{j=1}^k j \frac{r_j}{\lambda_0^j} + r(n) \\ &= a(m-1)n^{m-2} \left(\frac{r_1}{\lambda_0} + 2\frac{r_2}{\lambda_0^2} + \dots + k\frac{r_k}{\lambda_0^k} \right) + r(n) \end{aligned}$$

We collected the highest powers of n such that the degree of the residue $r(n)$ is not greater than $m-3$. Now we see that $\deg Q_0 = m-2$. Therefore the root λ_0 of T has the multiplicity $m-1$. Furthermore $t_n \geq 0$ for large n , since from the above follows:

$$t_n \sim a(m-1) \underbrace{\left(\frac{r_1}{\lambda_0} + 2\frac{r_2}{\lambda_0^2} + \dots + k\frac{r_k}{\lambda_0^k} \right)}_{>0 \text{ with 2.5.9}} \lambda_0^n$$

□

Theorem 2.5.11 *Let $S \in \mathbb{R}_+ \langle\langle x^* \rangle\rangle$ be rational with dominating root λ_0 (let m be its multiplicity). Then S is \mathbb{R}_+ -rational.*

Proof: Lemma 2.5.9 guarantees that there is a decomposition of S into p series such that a number c and a polynomial $R^{(p)}$ exist with the specified properties. It suffices to prove the \mathbb{R}_+ -rationality for each of the decomposed series S_0, \dots, S_{p-1} . Theorem 2.5.8 ensures that then also S is \mathbb{R}_+ -rational.

Without loss of generality let S be either any of the series S_i or the series S itself (if no decomposition is necessary to fulfill the criteria of Lemma 2.5.9). In the following steps we will show that $S = s_0 + s_1x + \dots + s_hx^h + S^{[h]}$ is \mathbb{R}_+ -rational. For this purpose $S^{[h]}$ is represented by means of \mathbb{R}_+ -rational expressions.

We define:

$$U := (1 - cx)S = \underbrace{s_0}_{=:u_0} + \underbrace{(s_1 - cs_0)}_{=:u_1}x + \underbrace{(s_2 - cs_1)}_{=:u_2}x^2 + \dots = \sum_{n=0}^{\infty} u_nx^n.$$

We have to make sure that h is large enough such that all u_i 's are positive for $i \geq h - k + 2$. This being possible is assured by Lemma 2.5.5.

We multiply S with the polynomial $R^{(p)}$ and obtain the coefficients of the series $T := S \cdot R^{(p)}$:

$$\begin{aligned} t_n &= s_n - r_1^{(p)}s_{n-1} - \dots - r_{k-1}^{(p)}s_{n-k+1} - r_k^{(p)}s_{n-k} \\ &= s_n - (\gamma_1 + c)s_{n-1} - (\gamma_2 - c\gamma_1)s_{n-2} - \dots - (\gamma_k - c\gamma_{k-1})s_{n-k} \\ &= (s_n - cs_{n-1}) - \gamma_1(s_{n-1} - cs_{n-2}) - \dots - \gamma_{k-1}(s_{n-k+1} - cs_{n-k}) - \gamma_k s_{n-k} \\ &= u_n - \gamma_1 u_{n-1} - \dots - \gamma_{k-1} u_{n-k+1} - \gamma_k s_{n-k} \end{aligned}$$

The γ_i are the same as in Lemma 2.5.9.

If $n > h + k$ ($h \geq k - 1$) we replace s_{n-k} by

$$u_{n-k} + cu_{n-k-1} + c^2u_{n-k-2} + \dots + c^{n-k-h-1}u_{h+1} + c^{n-k-h}s_h.$$

Now we sum all this up:

$$\begin{aligned} T^{[h]} &= \sum_{n>h} t_nx^n \\ &= \left(\sum_{n>h} u_nx^n \right) \left(1 - \gamma_1x - \dots - \gamma_k(x^k + cx^{k+1} + c^2x^{k+2} + \dots) \right) \\ &\quad - \gamma_k s_h x^{h+k} (1 + cx + c^2x^2 + \dots) - z(x) \\ &= U^{[h]} \underbrace{\left(1 - \gamma_1x - \dots - \gamma_{k-1}x^{k-1} - \gamma_kx^k(cx)^* \right)}_{=(cx)^*R^{(p)}} - \gamma_k s_h x^{h+k} (cx)^* - z(x) \end{aligned}$$

where

$$z(x) = \sum_{j=1}^{k-1} x^{h+j} \left(\gamma_k s_{h+j-k} + \sum_{i=j}^{k-1} \gamma_i u_{h+j-i} \right)$$

Note that z is a polynomial with nonnegative coefficients. Solving for $U^{[h]}$ provides:

$$U^{[h]} = \frac{1}{(cx)^* R^{(p)}} (T^{[h]} + \gamma_k s_h x^{h+k} (cx)^* + z(x)).$$

Since

$$\begin{aligned} U^{[h]} &= u_{h+1}x^{h+1} + u_{h+2}x^{h+2} + \dots \\ &= (s_{h+1} - cs_h)x^{h+1} + (s_{h+2} - cs_{h+1})x^{h+2} + \dots \\ &= (1 - cx)S^{[h]} - cs_h x^{h+1} \end{aligned}$$

we obtain:

$$\begin{aligned} S^{[h]} &= \frac{1}{1 - cx}(U^{[h]} + cs_h x^{h+1}) \\ &= (cx)^* \left(\frac{1}{(cx)^* R^{(p)}} (T^{[h]} + \gamma_k s_h x^{h+k} (cx)^* + z(x)) + cs_h x^{h+1} \right) \\ &= \frac{1}{R^{(p)}} (T^{[h]} + \gamma_k s_h x^{h+k} (cx)^* + z(x)) + cs_h x^{h+1} (cx)^* \end{aligned}$$

Last we have to check if all expressions in the last formula are \mathbb{R}_+ -rational. We showed already that $1/R^{(p)}$ is \mathbb{R}_+ -rational, $\gamma_k s_h x^{h+k} (cx)^*$ is obvious since $\gamma_k \geq 0$ and $s_h \geq 0$. The same holds for $cs_h x^{h+1} (cx)^*$ and $z(x)$. It remains to analyze $T^{[h]}$. We argue by induction over the multiplicity of S 's dominating root:

The case $m = 1$ is clear, because $T = R^{(p)} \cdot S$ then is a polynomial since $R^{(p)}$ is the minimal denominator of S . For h large enough, $T^{[h]}$ vanishes and doesn't cause any further problems. We assume now that a rational power series in $\mathbb{R}_+ \langle\langle x^* \rangle\rangle$ with a dominating root of multiplicity $m - 1$ is \mathbb{R}_+ -rational. According to Lemma 2.5.10 $T^{[h]}$ has positive coefficients and the same dominating root as S , but with multiplicity $m - 1$. Thus T is \mathbb{R}_+ -rational by the induction hypothesis. \square

Theorem 2.5.12 *Let $\mathbb{K} = \mathbb{R}_+$ or $\mathbb{K} = \mathbb{N}$. A series $S \in \mathbb{K} \langle\langle x^* \rangle\rangle$ is \mathbb{K} -rational if and only if it is a merge of rational series each of them having a dominating root.*

Proof: \implies : Let S be \mathbb{K} -rational with roots $\lambda_0, \dots, \lambda_r$ and $\varrho := \max_{0 \leq i \leq r} |\lambda_i|$. Then according to Theorem 2.5.2 all roots of modulus ϱ have the form $\varrho \vartheta_j$, and the ϑ_j 's are complex roots of unity. Let now p be a common order of these roots of unity, such that $\vartheta_j^p = 1$. We decompose S in p series S_0, \dots, S_{p-1} which are rational by Theorem 2.5.8. Each of these series has the roots

$\lambda_0^p, \dots, \lambda_r^p$, i.e., all roots of S of the form ϱ^ϑ are mapped on ϱ^p . Since ϱ is the maximum of the moduli of the roots of S , ϱ^p is the dominating root of each S_i .

\Leftarrow : Theorem 2.5.11 ensures that all S_i 's are \mathbb{K} -rational. For $\mathbb{K} = \mathbb{N}$ this is not clear at first glance. But if we take a closer look at the preceding lemmata and theorems we see that they also hold for the more special case of series over \mathbb{N} . There is only one point where we have to be careful: This is in Lemma 2.5.9 when we choose the number c : Now it must be an integer. In general the number p of subseries can be higher than in the case of \mathbb{R}_+ , since at least one integer must lie between λ_0^p and $\max_{1 \leq i \leq r} |\lambda_i^p|$. Thus by increasing p we make sure that c can be chosen from \mathbb{N} .

With Theorem 2.5.8 it is clear that also the merge S is \mathbb{K} -rational. \square

Chapter 3

Realization with Maple

This chapter provides a detailed description of how the inverse methodology of SCHÜTZENBERGER can be realized using the computer algebra system Maple. Many of the formulas and concepts of Chapter 2 are implemented, in order to obtain a regular expression for a given power series.

We use the notations that have been introduced in Chapter 2: Given a rational series S , its generating function is denoted by $f(x) = p(x)/q(x)$, and its roots by $\lambda_0, \dots, \lambda_r$. If not indicated differently, we assume that λ_0 is the dominating root (in case of existence).

3.1 Getting the Roots

First of all, we need a procedure for determining all (different) roots of a polynomial (or a rational function). The multiplicities of the roots need not be respected. Instead of solving the equation $f(x) = 0$ by using the Maple command `solve`, which can lead to time-consuming computations and unwieldy results (think of the general case of a polynomial with degree 4), we use without exception Maple's `RootOf` expressions. For this purpose the polynomial is made squarefree, then factorized, and for each factor f_i the expressions

$$\begin{aligned} &\text{RootOf}(f_i, \text{index}=1), \\ &\quad \vdots \\ &\text{RootOf}(f_i, \text{index}=d), \end{aligned}$$

$d = \deg f_i$, are created. The task is performed by the procedures `getRoots` (for polynomials) and `getRootsRat` (for rational functions); both return the roots as an unsorted list. The latter does not compute the roots of \bar{q} , the reciprocal polynomial of f 's denominator) but calls `getRoots(q)` and then

inverts these values. If we did not do it this way, then Maple would get problems in handling expressions that occur later on:

Example 3.1.1 *Let p be the polynomial $x^5 + 2x^4 + 3$; the following expression*

$$\text{RootOf}(p, \text{index}=1) * \text{RootOf}(\text{reciprocal}(p), \text{index}=5)$$

can not be simplified, although it equals 1.

3.2 Computing the Coefficients

We need a tool for efficiently computing the coefficients of a series. For this purpose the procedures `getCoefficients` and `coefficient` are programmed.

The first one computes the beginning s_0, \dots, s_n of a series S by comparing coefficients in the equation

$$p(x) = (q_0 - q_1x - \dots - q_kx^k) \cdot (s_0 + s_1x + s_2x^2 + \dots).$$

Hence the formulae

$$s_0 = \frac{p_0}{q_0}, s_1 = \frac{p_1 + s_0q_1}{q_0}, s_2 = \frac{p_2 + s_1q_1 + s_0q_2}{q_0}, \dots$$

are obtained. Note that these represent the general case; we restricted our rational functions to be normalized, i.e., esp. $q_0 = 1$.

If we want to compute a coefficient with large index, the above described procedure is not applicable. To develop an alternative, we make a short excursus on the interrelation between linear recurrences and matrices.

Consider a series that is defined by some initial values s_0, \dots, s_{k-1} and a recurrence relation:

$$q_0s_n = q_1s_{n-1} + \dots + q_k s_{n-k}.$$

This recurrence can be expressed by a $k \times k$ matrix

$$A = \begin{bmatrix} q_1/q_0 & q_2/q_0 & \cdots & q_k/q_0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let v_i denote the vector $v_i := (s_i, s_{i-1}, \dots, s_{i-k+1})^T$; hence v_{k-1} is the vector that contains the initial values (in descending order). Then it is obvious that $Av_{k-1} = v_k$. Generally the n^{th} coefficient can be computed by

$$s_n = (1, 0, \dots, 0) \cdot A^{n-k+1} v_{k-1}.$$

The eigenvalues of A turn out to be the roots of the underlying generating function $p(x)/q(x)$ ¹. The procedure `coefficient` uses the matrix formula to compute the n^{th} coefficient of a given series. But pay attention to the following fact: If $\deg p \geq \deg q$ then only the coefficients s_j with $j \geq \deg p$ satisfy the recurrence relation. So, in this case we have to start with the vector $v_{\deg p}$ instead of v_{k-1} . The initial values s_0, \dots, s_m , $m = \max\{\deg p, \deg q - 1\}$, are obtained by the procedure `getCoefficients`.

3.3 Decomposition

Consider the case that a given series S has no dominating root, but several different roots with maximal modulus; we denote these roots by $\varrho\vartheta_0, \dots, \varrho\vartheta_k$, where ϱ is a positive real number, and the ϑ_i 's are complex numbers with $|\vartheta_i| = 1$. To decide if S is \mathbb{N} -rational we must find an integer p , such that each of the subseries S_0, \dots, S_{p-1} (whose merge is S) has a dominating root (see Theorem 2.5.12); this is fulfilled by all numbers p for which $\vartheta_0^p = \dots = \vartheta_k^p = 1$ holds. By Theorem 2.5.2 we know that if S is \mathbb{N} -rational then the ϑ_i 's are complex roots of unity. So, how can the number p be found? The answer is given in the following two subsections. In Section 3.3.3 it is explained how the decomposition itself can be computed.

3.3.1 The Symmetric Polynomial

For identifying the roots of unity we define the symmetric polynomial

$$R(x) := \prod_{\substack{0 \leq i, j \leq r \\ i \neq j}} (\lambda_i - \lambda_j x).$$

In Lemma 3.3.1 we will show that $R(x)$ has integral coefficients. R has the roots λ_i/λ_j ($0 \leq i, j \leq r$), and in the case of \mathbb{N} -rationality among them the roots of unity $\vartheta_0, \dots, \vartheta_k$, since then ϱ itself is a root of S . To make clear what reason we compute $R(x)$ for, we make a short excursus on roots of unity.

Let ϑ be an n^{th} root of unity. Then the minimal polynomial of ϑ is called the n^{th} *cyclotomic polynomial*, denoted by Φ_n . The roots of Φ_n are all n^{th} primitive roots of unity, hence $\deg \Phi_n = \varphi(n)$, where φ denotes EULER's totient function²: $\varphi(n) = \prod_{d|n} (1 - \frac{1}{d})$. Additionally it is known that the

cyclotomic polynomials are monic and irreducible over \mathbb{Q} and have integral

¹An elaborate illustration of this topic is given in [CFR05].

²For more information about this function see e.g., [GKP89, Sect. 4.9].

coefficients³.

Now back to our polynomial $R(x)$: The above considerations show that if an n^{th} root of unity ϑ_i is a root of $R(x)$, then $R(x)$ must be divisible by the n^{th} cyclotomic polynomial $\Phi_n(x)$.

In the procedure `commonUnityRoots` the polynomial $R(x)$ is factorized over \mathbb{Z} : $R(x) = R_1(x) \cdot \dots \cdot R_m(x)$ where the R_i 's are irreducible polynomials. Each factor R_i is examined if it equals to any cyclotomic polynomial. Let $d = \deg R_i$. Then for all n with $\varphi(n) = d$ we must test if $\Phi_n(x) = R_i(x)$. For this purpose the Maple function `numtheory[invphi]` is used, which returns a list of the desired numbers n . Eventually let $\Phi_{n_1}, \dots, \Phi_{n_j}$ be the cyclotomic polynomials that divide $R(x)$. Then we set

$$p := \text{lcm}(n_1, \dots, n_j)$$

and have reached our goal.

We are now left with the question how to compute the symmetric polynomial. The following lemma gives the answer:

Lemma 3.3.1 *Let $S \in \mathbb{Z}^{\text{rat}}\langle\langle x^* \rangle\rangle$ be a rational series with generating function f and roots $\lambda_0, \dots, \lambda_r$. Then the symmetric polynomial $R(x)$ has integer coefficients and can be computed by means of a resultant:*

$$R(x) = \frac{\text{Res}(\hat{q}, \hat{q}_x, y)}{(1-x)^{r+1}} \left(\prod_{i=0}^r \lambda_i \right)^{-1}.$$

Herein $\hat{q}(y) := (y - \lambda_0) \cdot \dots \cdot (y - \lambda_r)$ denotes the squarefree part of \bar{q} , the reciprocal polynomial of f 's denominator. \hat{q}_x analogously is defined by $\hat{q}_x(y) := (y - \lambda_0 x) \cdot \dots \cdot (y - \lambda_r x)$.

Proof: We use the following identity for resultants⁴. Let $f(x)$ and $g(x)$ be polynomials, $\alpha_1, \dots, \alpha_n$ the roots of f and β_1, \dots, β_m the roots of g . Then the resultant $\text{Res}(f, g, x)$ can be written as follows:

$$\text{Res}(f, g, x) = f_n^m g_m^n \prod_{i=1}^n \prod_{j=1}^m (\beta_j - \alpha_i).$$

As usually, f_n and g_m denote the leading coefficients of f and g . In the following we can ignore them, since the polynomials \hat{q} and \hat{q}_x are monic. We get

$$\text{Res}(\hat{q}_x, \hat{q}, y) = \prod_{0 \leq i, j \leq r} (\lambda_i - \lambda_j x) = \left(\prod_{i=0}^r (\lambda_i - \lambda_i x) \right) R(x),$$

³see [LP84, p. 151]

⁴see [CLO98]

and this is nothing else but the claimed formula.

To show that $R(x)$ has integer coefficients, we remember that the resultant can be obtained by computing the determinant of the SYLVESTER matrix:

$$\text{Res}(p, q, x) = \begin{vmatrix} p_0 & p_1 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & p_0 & p_1 & \cdots & p_n & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_0 & p_1 & \cdots & p_n \\ q_0 & \cdots & q_m & 0 & 0 & \cdots & 0 \\ 0 & q_0 & \cdots & q_m & & & \vdots \\ \vdots & & & \ddots & \ddots & & 0 \\ 0 & \cdots & 0 & 0 & q_0 & \cdots & q_m \end{vmatrix}.$$

Since in our case both polynomials \hat{q}_x and \hat{q} have integer coefficients, it is clear that the determinant has this property, too. \square

3.3.2 Numerical Computation of $R(x)$

Another possibility is to compute the symmetric polynomial numerically. Since it is known that all its coefficients are integers, we need an estimate for the number of digits we have to use in our computation, such that the error in the end does not exceed 0.5.

Let us assume that the roots $\lambda_0, \dots, \lambda_r$ are sorted by descending modulus ($|\lambda_0| \geq |\lambda_1| \geq \dots \geq |\lambda_r|$) and that every root λ_i is given numerically with an error $\tilde{\varepsilon}_i \in \mathbb{C}$. We set $N := r(r+1)$ to be the degree of the symmetric polynomial $R(x)$. Then

$$\tilde{R}(x) = \prod_{\substack{0 \leq i, j \leq r \\ i \neq j}} \left((\lambda_i + \tilde{\varepsilon}_i) - (\lambda_j + \tilde{\varepsilon}_j)x \right) =: h_1 \cdot h_2 \cdot \dots \cdot h_N,$$

where the N factors of this product are denoted by the h_m 's. To get the n^{th} coefficient \tilde{r}_n of $\tilde{R}(x)$, we sum up all products where we take from each h_m either the first or the second part, but totally n times the second part (with the x), hence the power of x is always n . Let v be a binary vector and let $v_k = 0$ (resp. 1) indicate that we take the first part (second part respectively) of h_k , then we can write \tilde{r}_n as

$$\tilde{r}_n = \sum_{v \in \mathcal{B}_n^N} \prod_{k=1}^N (\lambda_{\zeta(k, v_k)} + \tilde{\varepsilon}_{\zeta(k, v_k)}),$$

where \mathcal{B}_n^N denotes the set of binary vectors of dimension N and hamming weight n , and the function $\zeta(k, v_k)$ gives the index of the first ($v_k = 0$) or second ($v_k = 1$) λ contained in h_k .

We define an upper bound M_k for the modulus of a product of arbitrary λ_i 's (their number being k). A rough estimate would be to define $M_k := |\lambda_0|^k$. But if we have a closer look at how $R(x)$ is defined, then we see that in any summand of the above formula, the factor λ_0 cannot occur more than $2r$ times. So, of course, $M_k = |\lambda_0|^k$ for $k \leq 2r$. But for $k > 2r$ we can enhance the estimate M_k by using the root λ_1 . Since we "used" already two factors where λ_1 occurs ($\lambda_0 - \lambda_1 x$ and $\lambda_1 - \lambda_0 x$), there remain $2r - 2$ occurrences of λ_1 and so on. Therefore we get the following formula for M_k :

$$M_k := |\lambda_0^{2r} \cdot \lambda_1^{2r-2} \cdot \dots \cdot \lambda_i^{2(r-i)} \cdot \dots \cdot \lambda_j^l| \text{ with } k = 2r + (2r-2) + \dots + l, l \leq 2(r-j),$$

$$\begin{aligned} \text{e.g., } M_0 &= 1 \\ M_1 &= |\lambda_0| \\ M_2 &= |\lambda_0^2| \\ &\vdots \\ M_{2r} &= |\lambda_0^{2r}| \\ M_{2r+1} &= |\lambda_0^{2r} \lambda_1| \\ &\vdots \\ M_N &= \left| \prod_{j=0}^{r-1} \lambda_j^{2(r-j)} \right|. \end{aligned}$$

By r_n denoting the n^{th} coefficient of the exact polynomial $R(x)$, and by defining

$$C := \binom{N}{\frac{N}{2}},$$

we can estimate the total error γ :

$$\begin{aligned} \gamma &= |\tilde{r}_n - r_n| = \left| \sum_{v \in \mathcal{B}_n^N} \prod_{k=1}^N (\lambda_{\zeta(k, v_k)} + \tilde{\varepsilon}_{\zeta(k, v_k)}) - r_n \right| \\ &\leq \binom{N}{n} \cdot \sum_{k=0}^{N-1} \binom{N}{k} M_k \varepsilon^{N-k} \\ &\leq C \cdot \left(\varepsilon N M_{N-1} + \sum_{k=0}^{N-2} C M_k \varepsilon^{N-k} \right) \end{aligned}$$

$$\begin{aligned}
&\leq C \cdot \left(\varepsilon N M_{N-1} + C \varepsilon^2 M_{N-2} \sum_{k=0}^{N-2} \left(\frac{\varepsilon}{\lambda_{r-1}} \right)^k \right) \\
&= C \cdot \left(\varepsilon N M_{N-1} + C \varepsilon^2 M_{N-2} \frac{1 - \left(\frac{\varepsilon}{\lambda_{r-1}} \right)^{N-1}}{1 - \left(\frac{\varepsilon}{\lambda_{r-1}} \right)} \right) \\
&= C \cdot \left(\varepsilon N M_{N-1} + \frac{C \varepsilon^2 M_{N-2}}{\lambda_{r-1}^{N-2}} \cdot \frac{\lambda_{r-1}^{N-1} - \varepsilon^{N-1}}{\lambda_{r-1} - \varepsilon} \right) \\
&\leq C (\varepsilon N M_{N-1} + 2C \varepsilon^2 M_{N-2})
\end{aligned}$$

For the last step we assumed that $\lambda_{r-1} - \varepsilon \geq \frac{\lambda_{r-1}}{2}$.

For computing the polynomial $R(x)$ correctly, the error γ must be smaller than 0.5. Hence we solve the inequation

$$\varepsilon^2 \cdot (2C M_{N-2}) + \varepsilon \cdot N M_{N-1} - \frac{1}{2C} < 0$$

for ε :

$$\begin{aligned}
\varepsilon &< \frac{\sqrt{N^2 M_{N-1}^2 + 4M_{N-2}} - N M_{N-1}}{4C M_{N-2}} \\
&= \frac{N M_{N-1}}{4C M_{N-2}} \cdot \left(\sqrt{1 + \frac{4M_{N-2}}{N^2 M_{N-1}^2}} - 1 \right) \\
&\approx \frac{N M_{N-1}}{4C M_{N-2}} \cdot \left(1 + \frac{4M_{N-2}}{2N^2 M_{N-1}^2} - 1 \right) \\
&= \frac{1}{2CN M_{N-1}}
\end{aligned}$$

In step “ \approx ”, we used the formula $\sqrt{1 + \delta} \approx 1 + \delta/2$ which is a good approximation for small δ . The effect on our result is negligible. Some examples may illustrate the validity of this estimate.

Example 3.3.2 *The following polynomials may serve to verify the described estimate:*

$$p_1(x) = x^5 - 6x^4 + 4x^3 + 7x^2 - 12x + 10$$

$$p_2(x) = (x^7 - x^6 + 23)(x - 2344)(x - 3456)(x - 5432)(x - 3421)(x - 4444)$$

$$p_3(x) = \Phi_{17} = \sum_{k=0}^{16} x^k$$

The table below gives a comparison of the number of digits suggested by our estimate, with the minimal number of digits for which the numerical computation of the symmetric polynomial leads to the correct result (i.e., the numerical error is smaller than 0.5):

	<i>estimate</i>	<i>minimal</i>
p_1	14	10
p_2	373	341
p_3	74	20

It shows that in some cases (p_2) our estimate works very well, while other cases (p_3) are farly overrated.

Nevertheless, to compute the symmetric polynomial numerically costs much more computation time than by using the resultant. Therefore we do not employ the numerical approach in the implementation.

3.3.3 Computing the Decomposition

At this point we know into how many subseries we must decompose the series S . But we need the subseries S_0, \dots, S_{p-1} in an explicit form (i.e., given by their generating functions). The same task arises when we want to compute a regular expression. It is carried out by the procedure `decomposition` which computes the generating functions of the subseries by means of the multisection formula⁵.

Theorem 3.3.3 *Given a series S by its generating function $f(x)$, an integer p , and S_0, \dots, S_{p-1} denotes the decomposition of S . Then*

$$f_i(x) = \frac{1}{px^{i/p}} \sum_{j=1}^p s^{p-ij} f(s^j x^{1/p}), \quad s = e^{2\pi i/p}$$

is the generating function for the subseries S_i .

Proof: We know that the merge of the subseries S_0, \dots, S_{p-1} must again give the original series S :

$$S = \sum_{i=0}^{p-1} x^i S'_i,$$

⁵The p -section formula cited here, and an overview over the decomposition (or *multisection*) of series can be found in [Rio58, Chap. 4].

where S'_i denotes the series which is obtained from S_i by applying the morphism $x \mapsto x^p$. Therefore we have to test if we get f by merging the f_i 's.

$$\begin{aligned} \sum_{i=0}^{p-1} x^i f_i(x^p) &= \sum_{i=0}^{p-1} \frac{1}{p} \sum_{j=1}^p s^{p-ij} f(s^j x) \\ &= \frac{1}{p} \left[\underbrace{\sum_{i=0}^{p-1} s^{p-ip} f(s^p x)}_{=1} + \sum_{j=1}^{p-1} \underbrace{\left(\sum_{i=0}^{p-1} s^{p-ij} \right)}_{=0} f(s^j x) \right] \\ &= f(x) \end{aligned}$$

Why does $\sum_{i=0}^{p-1} s^{p-ij} = 0$ hold for $1 \leq j < p$? Let $k = \gcd(p, j)$. Then in this expression all the p/k^{th} roots of unity are summed up k times, which gives 0. \square

Concerning the implementation, there arise some problems: If the above formula for the f_i 's is fed one-to-one into Maple, then in many cases the system does not succeed in simplifying the resulting expression. And anyway, the computation is very slow. A first improvement to speed up the computation is to substitute $x^{1/p}$ by a new variable y . But it remains the problem that we often get a result that fills pages instead of a nice rational function: Maple sometimes fails in simplifying expressions that contain roots of unity.

In his paper *A Story About Computing with Roots of Unity*⁶ François Bergeron gives a hint on how to handle this difficulty: Consider an expression containing several p^{th} roots of unity (let s be a primitive one). Thus our computations take place in the field $\mathbb{Q}[s]$. This field is isomorphic to the field $\mathbb{Q}[x]/\langle \Phi_p(x) \rangle$ of polynomials in x modulo $\Phi_p(x)$, where $\Phi_p(x)$ again is the p^{th} cyclotomic polynomial⁷. Recall that the cyclotomic polynomials are irreducible over \mathbb{Q} and $\Phi_p(x)$ is the minimal polynomial of any p^{th} primitive root of unity.

For our purposes this means that we first introduce a new variable s that represents the root of unity $e^{2\pi i/p}$. After each step of our computation we reduce modulo $\Phi_p(s)$. Thanks to the above isomorphism we obtain the correct result in a fraction of computation time compared to before.

⁶[Ber89]

⁷A good presentation of field isomorphisms can be found in [Chi79, Chapter 18]

3.4 Testing for \mathbb{N} -Rationality

To find out if a given series is \mathbb{N} -rational, two properties must be verified (according to Theorem 2.5.11): The existence of a dominating root and the nonnegativeness of all coefficients of the series.

3.4.1 Existence of a Dominating Root

In Definition 2.5.3 we introduced the notion of a dominating root; therefore $\lambda_0 > \max_{1 \leq i \leq r} |\lambda_i|$ must hold. The problem of deciding if $|\lambda_i| = |\lambda_j|$ looks easier than it is, because in general Maple is not capable to maintain this task by symbolic computation.

Example 3.4.1 Let $p(x) = x^5 + 2x^4 + 3$ be the polynomial from Example 3.1.1. Then we define

```
lambda1:= RootOf(p, index=1);
lambda2:= RootOf(p, index=5);
```

Although $\lambda_1 = \overline{\lambda_2}$ holds, the expression

```
evalb(abs(lambda1)=abs(lambda2));
```

is evaluated to false.

So, let's try to compute the roots numerically and compare them. But what do we do if the moduli of two roots lie closer together than our numerical precision? If we get the result that $|\lambda_i| \doteq |\lambda_j|$ (where \doteq denotes the numerical equality) then we only know that either $|\lambda_i| = |\lambda_j|$ or $|\lambda_i| - |\lambda_j| < 10^{1-d+\lceil \log_{10} |\lambda_i| \rceil}$ where d is the number of digits. A result from XAVIER GOURDON and BRUNO SALVY⁸ is helpful (it is cited here without proof):

Theorem 3.4.2 Let p be a polynomial with integer coefficients, $\alpha_1, \dots, \alpha_n$ its roots and thus $\deg p = n > 0$ its degree. Define $\kappa(p)$ to be the following quantity

$$\kappa(p) = \frac{\sqrt{3}}{2} \left(\frac{n(n+1)}{2} \right)^{-\left(\frac{1}{4}n(n+1)+1\right)} \cdot M(p)^{-\frac{1}{2}n(n^2+2n-1)},$$

then $|\alpha_i| \neq |\alpha_j| \implies \left| |\alpha_i| - |\alpha_j| \right| \geq \kappa(p)$ and $|\operatorname{Im}(\alpha_i)|$ is either 0 or larger than $\kappa(p)$. Herein $M(p)$ is defined by

$$M(p) := |p_n| \prod_{i=1}^n \max\{1, |\alpha_i|\}.$$

⁸see [GS96]

Proof: See [GS96, p. 9] □

This formula is implemented in the procedure `kappa`, but it has to be used carefully: Consider the generating function from Section 4.3 (p. 63): Applying Theorem 3.4.2 we get

$$\kappa(q) \doteq 2.159917528 \cdot 10^{-287579}.$$

Thus we had to compute with a precision of 287580 digits! But the dominating root (we will see that there is one) differs already in the second digit from the moduli of all other roots. This gives reason for proceeding in the following way: In a first step all roots are computed numerically (using the command `evalf`) with “low” precision (20 digits). In most cases where a dominating root exists it should be possible to see this fact now. Only in the cases where either there are several roots having the same maximal modulus or the difference between the dominating root and the maximal modulus of the other roots is extremely small, it is necessary to increase the precision up to $1 - \lfloor \log_{10} \kappa \rfloor + \lfloor \log_{10} \lambda_0 \rfloor$ digits. In a second step we have to examine these roots that seemed to have the same modulus by computing with low precision.

This algorithm is implemented in the procedure `hasDominatingRoot`. For finding the elements having maximal modulus with respect to a given precision, the procedure `maximalModulus` is used.

3.4.2 The Exponential Polynomial

In this section we describe how the exponential polynomial from Theorem 2.4.5 is obtained. It will be needed in the subsequent section where we want to determine whether all coefficients of a series are nonnegative.

Recall that the coefficients of a rational series can be written as

$$s_n = \sum_{i=0}^r P_i(n) \lambda_i^n$$

for large n . We have to compute the polynomials P_i which is done by the procedure `exponentialPolynomial`. It is sufficient to have the P_i 's numerically since they are only used for an estimate.

The command `gfun[ratpolytocoef]`⁹ is used to get an expression of the form

$$\sum_j \sum_{\alpha = \text{RootOf}(q_j)} \frac{s_j(\alpha) \cdot \alpha^{-n}}{\alpha}$$

⁹see [SZ92]

where the polynomials q_j are the irreducible factors of q and the s_j 's are polynomials over \mathbb{Q} . From this we have to extract the P_i 's. Therefore the sums are expanded:

$$\sum_{\alpha=\text{RootOf}(q_j)} \frac{s_j(\alpha) \cdot \alpha^{-n}}{\alpha} = \frac{s_j(\alpha_1) \cdot \alpha_1^{-n}}{\alpha_1} + \dots + \frac{s_j(\alpha_m) \cdot \alpha_m^{-n}}{\alpha_m}.$$

Herein $\alpha_1, \dots, \alpha_m$ are the roots of q_j . They are obtained by using the subroutine `expandRootOf` which only adds the option `index` to a `RootOf` expression (for all indices $1, \dots, m = \deg q_j$). In this context it is very advantageous that we represented the roots as `RootOf` expressions.

3.4.3 Nonnegative Coefficients

The second important property we have to verify is that all coefficients of the series S are nonnegative. If the series is finite (i.e., is a polynomial) there is not much to do: We can look at each coefficient and test whether it is negative or not. This is no longer possible when the series is infinite. In this case we need a more elaborate method. Therefore we first realize an estimate for a boundary n_0 which makes sure that all coefficients s_n with $n > n_0$ are nonnegative. The remaining coefficients s_0, \dots, s_{n_0} are tested one by one. In the following the implementation of the procedure `boundaryForNonnegCoeffs` is described that computes this boundary n_0 , whereat it is assumed that the given rational function f has a dominating root. The following cases have to be distinguished:

- A: f is a polynomial.
- B: f is a rational function that has exactly one root (but this root can have a multiplicity greater than 1).
- C: f is a rational function and has several different roots.

For the cases B and C we have to compute the exponential polynomial (see Theorem 2.4.5). In both cases we will look if the leading coefficient of the polynomial $P_0(n)$, which corresponds to the dominating root, is negative. If this is the case then no boundary n_0 can be found and the procedure returns `false`.

Case A Beginning with the leading coefficient we look for the highest negative coefficient; it is the wanted boundary. In the case that no negative coefficient is found, -1 is returned.

Case B If f has only one root then the coefficients of the series can be written as

$$s_n = P_0(n)\lambda_0^n.$$

Since λ_0 is the dominating root it must be positive. Thus we have to choose n_0 such that $P_0(n) > 0$ for all $n > n_0$. Therefore the greatest real root of P_0 is computed and is taken as n_0 . If no such root exists, $n_0 = -1$ is returned, which means that all coefficients are nonnegative. Note that we already checked that the leading coefficient of P_0 is positive.

Case C This case is the most complicated one. In [BLFR01] the following estimate is given: Let m_i be the multiplicity of λ_i . Thus $\deg P_i = m_i - 1$. We choose real numbers a and b such that:

$$\lambda_0 > a > b > \max_{1 \leq i \leq r} |\lambda_i|.$$

Moreover we have to find a number c , that satisfies

$$c \cdot n^{m_0-1} \leq P_0(n).$$

This is done by setting $c := \frac{1}{2}p_{0,m_0-1}$, where p_{0,m_0-1} is the leading coefficient of P_0 . But we have to be careful: The number c chosen in this manner needs not necessarily fulfill the above inequation for all n , but for all $n > \tilde{n}_0$. \tilde{n}_0 is computed and to assure that the boundary n_0 in the end is valid, care is taken that n_0 is not smaller than \tilde{n}_0 .

And last we define

$$d := \max_{1 \leq i \leq r} m_i \quad \text{and} \quad e := \max_{\substack{1 \leq i \leq r \\ 0 \leq j \leq m_i}} |p_{ij}|,$$

where the p_{ij} 's denote the coefficients of the polynomials $P_i(n)$.

For computing our boundary we look at the following fraction: If it is smaller than 1 then the coefficient s_n must be positive. Hence we can determine n_0 .

$$\begin{aligned} \left| \frac{\sum_{i=1}^r P_i(n)\lambda_i^n}{P_0(n)\lambda_0^n} \right| &\leq \frac{\sum_{i=1}^r |P_i(n)| \cdot |\lambda_i^n|}{c \cdot n^{m_0-1}\lambda_0^n} \\ &\leq \frac{\sum_{i=1}^r |p_{i,m_i-1}n^{m_i-1} + \dots + p_{i,1}n + p_{i,0}| b^n}{c \cdot n^{m_0-1}a^n} \\ &\leq \frac{\sum_{i=1}^r |en^{d-1} + \dots + en + e| b^n}{c \cdot n^{m_0-1}a^n} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{r \cdot d \cdot e \cdot n^{d-1} b^n}{c \cdot n^{m_0-1} a^n} \\
&= \underbrace{\frac{r \cdot d \cdot e}{c}}_{=: \varrho} \cdot n^{d-m_0} \cdot \left(\frac{b}{a}\right)^n.
\end{aligned}$$

It remains to solve the inequality

$$g(n) := n^{d-m_0} \cdot \left(\frac{b}{a}\right)^n < \frac{1}{\varrho}$$

for n . Depending on $d - m_0$ three cases have to be distinguished:

i) $d - m_0 = 0$: The inequality is easily solved: $n_0 = \left\lceil \frac{\ln \varrho}{\ln a - \ln b} \right\rceil$.

ii) $d - m_0 < 0$: The function $g(n)$ is strictly monotonic decreasing for $n > 0$ (since $b/a < 1$) and goes to 0 for $n \rightarrow \infty$. Therefore we get the correct solution by using `fsolve` and restricting the range for n to the interval $(0, \infty)$.

iii) $d - m_0 > 0$: In this case $g(n)$ has a maximum at

$$n_{\max} = \frac{m_0 - d}{\ln b - \ln a}.$$

We compute $g(n_{\max})$: If it is smaller than ϱ^{-1} then the inequality holds for all n . Otherwise we use again `fsolve` to determine the point beyond the maximum, from where the function is smaller than ϱ^{-1} . Note that $g(n)$ is strictly monotonic decreasing in the interval (n_{\max}, ∞) .

The last step that the procedure `boundaryForNonnegCoeffs` executes is to check the coefficients with index $\leq n_0$ (in decreasing order) to find out whether there are negative coefficients or not. Hence the returned boundary is always minimal.

3.5 Regular Expressions

For computing the regular expression the formula from Theorem 2.5.11 can be applied:

$$S = \frac{1}{R^{(p)}} \left(T^{[h]} + \gamma_k s_h x^{h+k} (cx)^* + z(x) \right) + c s_h x^{h+1} (cx)^* + \sum_{n=0}^h s_n x^n.$$

According to Lemma 2.5.9 it may be necessary to decompose the series S . Recall that the original series S may be decomposed in order to have a dominating root. Then for each of the subseries a regular expression is computed separately, where the subseries may be decomposed further.

The computation of a regular expression consists of four steps:

1. Determine an integer p such that the subseries S_0, \dots, S_{p-1} have the properties described in Lemma 2.5.9.
2. Compute the polynomial $R^{(p)}$ (it depends on the multiplicity of the dominating root).
3. Find an integer c such that the γ_i 's defined in Lemma 2.5.9 are non-negative.
4. Apply the above formula to obtain a regular expression for S .

Step 1 This step is carried out by the procedure `pForRegExp`. In [BLFR01] there is given an estimate for p : Let

$$\frac{p_i(x)}{q_i(x)} = \frac{p_i(x)}{1 - q_{i,1}x - \dots - q_{i,k+1}x^{k+1}}$$

be the generating function of one of the subseries S_0, \dots, S_{p-1} , further let μ_0, \dots, μ_k be its roots (each counted with its multiplicity), and $\mu_0 = \lambda_0^p$. Define

$$u_j := \sum_{1 \leq m_1 < \dots < m_j \leq k} \mu_{m_1} \cdot \dots \cdot \mu_{m_j}$$

and $\bar{\lambda} := \max_{1 \leq j \leq r} |\lambda_j|$. Then γ_j (see Lemma 2.5.9) can be written as

$$\begin{aligned} \gamma_j &= -c^j + q_{i,1}c^{j-1} + q_{i,2}c^{j-2} + \dots + q_{i,j} \\ &= \mu_0 (c^{j-1} - u_1c^{j-2} + \dots + (-1)^{j-1}u_{j-1}) - c^j + u_1c^{j-1} - \dots + (-1)^{j-1}u_j \\ &\geq \lambda_0^p c^{j-1} \left(1 - \frac{k\bar{\lambda}^p}{c} - \frac{k^2\bar{\lambda}^{2p}}{c^2} - \dots - \frac{k^{j-1}\bar{\lambda}^{(j-1)p}}{c^{j-1}} \right) \\ &\quad - c^j \left(1 + \frac{k\bar{\lambda}^p}{c} + \frac{k^2\bar{\lambda}^{2p}}{c^2} + \dots + \frac{k^j\bar{\lambda}^{jp}}{c^j} \right) \\ &\geq \lambda_0^p c^{j-1} \left(1 - \sum_{m=1}^{\infty} \left(\frac{k\bar{\lambda}^p}{c} \right)^m \right) - c^j \sum_{m=0}^{\infty} \left(\frac{k\bar{\lambda}^p}{c} \right)^m \\ &\geq 0 \end{aligned}$$

We choose $c \geq 6k\bar{\lambda}^p$ (for making the sums small enough) and get the inequality

$$\frac{\lambda_0^p}{6k\bar{\lambda}^p} \left(1 - \frac{1}{5}\right) - \frac{6}{5} \geq 0$$

which leads to¹⁰

$$p \geq \frac{\ln(9k)}{\ln \lambda_0 - \ln \bar{\lambda}}.$$

Unfortunately, this estimate is very rough in many cases: E.g., for the function from Section 4.3 it gives $p \geq 56$ but we already succeed with $p = 8$. Since for growing p the subsequent computations become larger and larger very quickly (recall that the roots of the decomposed subseries grow exponentially in p), we perform a binary search for p (with the above estimate as an upper bound) until we have found the minimal one. We should also bear in mind that the size of the resulting regular expression depends on p .

Step 2 In the proof of Lemma 2.5.9 we described how to determine the polynomial $R^{(p)}$ depending on m_0 , the multiplicity of the dominating root λ_0 : If $m_0 = 1$ then we set $R^{(p)} := q$ where q is the denominator of the generating function. If $m_0 > 1$ then $R^{(p)}$ is chosen to be the squarefree part of q which is obtained by:

$$R^{(p)} := \frac{q}{\gcd(q, \frac{d}{dx}q)}.$$

Exactly this rule is implemented in the procedure `getRp`. But how can we find the multiplicity of λ_0 ? We will give two methods for maintaining this task. The first one is very easy while the second one is more elaborate and is based on a suggestion from YVAN LE BORGNE.

The naive way is to compute $\tilde{q} := \gcd(q, \frac{d}{dx}q)$ and test whether the maximal modulus of its roots (denoted by $\tilde{\lambda}$) equals to the dominating root λ_0 . If this is the case, then we know that $m_0 > 1$ (we are not interested in the actual value), since obviously λ_0 is a root of \tilde{q} and therefore a multiple root of q . Otherwise $m_0 = 1$. For comparing λ_0 and $\tilde{\lambda}$ we proceed in the same manner as described in Section 3.4.1: We first evaluate both expressions with a relatively low precision (20 digits). In the worst case (if we can not decide whether $\lambda_0 \neq \tilde{\lambda}$), we have to increase the number of digits up to the boundary given by Theorem 3.4.2.

The second method uses the following algorithm for computing the multiplicity of the dominating root. In Lemma 2.5.5 we have seen that $s_n \sim p_{0,d} n^d \lambda_0^n$.

¹⁰Note that the formula in [BLFR01, p. 143] contains a mistake.

If we think of the coefficients s_n as a continuous function $s(n)$ we get:

$$\begin{aligned}\ln s(n) &= \ln p_{0,d} + d \ln n + n \ln \lambda_0 \\ (\ln s(n))' &= \frac{d}{n} + \ln \lambda_0 \\ (\ln s(n))'' &= -\frac{d}{n^2}\end{aligned}$$

Hence d can be computed by

$$d = -n^2 (\ln s(n))'' = -n^2 \left(\frac{s''(n)}{s(n)} - \left(\frac{s'(n)}{s(n)} \right)^2 \right)$$

To apply this formula to the discrete case, we substitute $s'(n)$ by $s_{n+1} - s_n$ and $s''(n)$ by $s_{n+2} - 2s_{n+1} + s_n$. The obtained formula

$$\begin{aligned}& \lim_{n \rightarrow \infty} -n^2 \left(\frac{s_{n+2} - 2s_{n+1} + s_n}{s_n} - \frac{(s_{n+1} - s_n)^2}{s_n^2} \right) \\ &= \lim_{n \rightarrow \infty} -\frac{n^2}{s_n^2} (s_n s_{n+2} - s_{n+1}^2)\end{aligned}$$

seems to be an approximation for d , but its correctness has to be verified.

By substituting $s_n = p_{0,d} n^d \lambda_0^n$ we get:

$$\begin{aligned}& \lim_{n \rightarrow \infty} -\frac{n^2}{p_{0,d}^2 n^{2d} \lambda_0^{2n}} \left(p_{0,d}^2 \lambda_0^{2n+2} (n^d (n+2)^d - (n+1)^{2d}) \right) \\ &= \lim_{n \rightarrow \infty} -\frac{\lambda_0^2}{n^{2d-2}} (n^d (n+2)^d - (n+1)^{2d}) \\ &= \lim_{n \rightarrow \infty} -\frac{\lambda_0^2}{n^{2d-2}} \left(n^{2d} + 2dn^{2d-1} + 4\binom{d}{2}n^{2d-2} + \dots \right. \\ & \quad \left. - (n^{2d} + 2dn^{2d-1} + \binom{2d}{2}n^{2d-2} + \dots) \right) \\ &= \lim_{n \rightarrow \infty} -\frac{\lambda_0^2}{n^{2d-2}} (-dn^{2d-2}) = \lambda_0^2 d\end{aligned}$$

We know that

$$\lim_{n \rightarrow \infty} \frac{s_{n+1}}{s_n} = \lambda_0.$$

Thus we divide the above by λ_0^2 and replace it:

$$\begin{aligned}d &= \lim_{n \rightarrow \infty} -\frac{n^2}{s_n^2} (s_n s_{n+2} - s_{n+1}^2) \frac{s_n^2}{s_{n+1}^2} \\ &= \lim_{n \rightarrow \infty} n^2 \left(1 - \frac{s_n s_{n+2}}{s_{n+1}^2} \right).\end{aligned}$$

As we know, the multiplicity of the dominating root is $d+1$. We demonstrate this method by means of a few examples.

Example 3.5.1 *We examine the three rational functions*

$$f_1(x) = \frac{1}{(1-2x)^3},$$

$$f_2(x) = \frac{1}{(1-10x)^7(1-7x)^3(1+x)},$$

$$f_3(x) = \frac{1+x-x^2-10x^3}{7x^4-9x^3-2x^2+1}.$$

Let $d_i(n) = n^2(1 - s_{i,n}s_{i,n+2}s_{i,n+1}^{-2})$, where $s_{i,n}$ is the n^{th} coefficient of the series generated by f_i . The following table illustrates the convergence behaviour:

n	$d_1(n)$	$d_2(n)$	$d_3(n)$
10	1.282051282	3.579173958	-656.7874688
20	1.581027668	4.984035014	233.2570689
50	1.814223512	5.956757803	1738.540736
100	1.903674091	6.111329271	-27395.38088
1000	1.990037870	6.026971366	$3.936031443 \cdot 10^5$
10000	1.999000380	6.002871415	8.734245105
15000	1.999333502	6.001918607	3.323418783
20000	1.999500095	6.001440580	-0.01009164225
30000	1.999666709	6.000961469	$1.336160035 \cdot 10^{-7}$
\vdots	\vdots	\vdots	\vdots
$\rightarrow \infty$	2	6	0

For f_1 and f_2 this method produces a good approximation for reasonable values of n , whereas it converges slowly for f_3 . The reason is that the moduli of certain roots of f_3 lie very close together: The dominating root is $\lambda_0 \doteq 2.160148917$, and there are two other roots with modulus 2.157485778.

These Examples show that the method described above can not be employed without further researches on the convergence behaviour. Therefore in the implementation of the procedure `getRp`, the first method (the “naive way”) is used.

Step 3 Given the polynomial $R^{(p)}$ the procedure `findc` checks if a constant c exists (see Lemma 2.5.9), and in the case of success computes this constant. Recall that c must fulfill

$$\lambda_0 > c > \max_{1 \leq i \leq r} |\lambda_i|$$

. It is not a good idea to test for all possible integers c whether the corresponding γ_i 's are nonnegative: If $R^{(p)}$ belongs to a decomposed series then the range for c can be very large. Consider for example a series with roots 7 and 1. Let us decompose this series into 8 subseries, each of them having the roots 7^8 and 1. Thus c had to be searched for in the interval $[1, 5764801]!$ For $1 \leq i \leq \deg R^{(p)}$ each polynomial

$$\gamma_i(c) = r_i + r_{i-1}c + r_{i-2}c^2 + \dots + r_1c^{i-1} - c^i$$

is examined in which intervals it is nonnegative. Therefore the Maple command `realroot` is used which returns a list of isolating intervals for all real roots using DESCARTES' rule of signs. The intervals are intersected for all i 's, and in the case that the intersection is not empty, the minimal c is chosen.

Step 4 The procedure `regularExpression` finally computes a regular expression for the given series. This is done exactly in the same manner described in Theorem 2.5.11.

First it is checked if the series is finite: In this case the series already has the desired form — if all its coefficients are positive. If the series is infinite, step 1 is performed. If the number p of subseries is greater than 1, then the decomposition is computed and `regularExpression` is recursively applied to each subseries. Since it is not necessary to perform step 1 for the subseries again, the parameter `checkDecomp` is introduced which indicates if step 1 has to be performed or not. Step 2 and 3 are easily done by calling the corresponding procedures (described above). Next the series $T = SR^{(p)}$ and $U = S(1 - cx)$ are computed; for determining the constant h the procedure `boundaryForNonnegCoeffs` is applied to T and U : Then h has to be the maximum of the two results. $T^{(h)}$ is computed by successively subtracting all monomials $t_n x^n$ with $n \leq h$ from T . $1/R^{(p)}$ is obtained by using the formula from Lemma 2.5.9 and after computing the polynomial $z(x)$ the regular expression can be assembled.

A final note on regular expressions Of course, what we have computed until now is not a regular expression in the narrower sense of the definition, but an expression composed of monomials with positive coefficients using the operations sum, product, and star, e.g.,

$$2(x^*)^3 \cdot x(x^*x^2 + 1)$$

This can be translated into a classical regular expression using the procedure *ren*¹¹. First of all, an alphabet Σ is initialized with the empty set.

$$\begin{aligned} \text{ren}(1) &= \lambda \\ \text{ren}(a) &= \begin{cases} a & \text{if } a \notin \Sigma. \text{ Then set } \Sigma := \Sigma \cup \{a\} \\ b, b \notin \Sigma & \text{if } a \in \Sigma. \text{ Then set } \Sigma := \Sigma \cup \{b\} \end{cases} \\ \text{ren}(X + Y) &= \text{ren}(X) \vee \text{ren}(Y) \\ \text{ren}(X \cdot Y) &= \text{ren}(X)\text{ren}(Y) \\ \text{ren}(X^*) &= (\text{ren}(X))^* \end{aligned}$$

Herein X and Y denote arbitrary “regular expressions”. Applying this procedure to the above example leads to

$$\begin{aligned} &\text{ren} [2(x^*)^3 \cdot x(x^*x^2 + 1)] \\ &= \text{ren} [(x^*x^*x^* + x^*x^*x^*)] \text{ren} [x(x^*xx + 1)] \\ &= [\text{ren}(x^*x^*x^*) \vee \text{ren}(x^*x^*x^*)] \text{ren}(x) [\text{ren}(x^*xx) \vee \lambda] \\ &= (a^*b^*c^* \vee d^*e^*f^*) g(h^*ij \vee \lambda) \end{aligned}$$

and $\Sigma = \{a, b, c, d, e, f, g, h, i, j\}$.

Since in many cases the coefficients in our resulting expression are not as small as in the example above, the procedure *ren* is not implemented. Otherwise we would get extremely huge regular expressions and alphabets containing thousands of letters. Consider for example the harmless expression

$$24(x^*)^2(24xx^*)^*x + xx^* + 1,$$

which would be written by about 100 letters.

¹¹proposed in [BLFR01, p. 133]

Chapter 4

Examples

At the beginning of this chapter we present an example which illustrates the functionality of our Maple program. The following examples are taken from combinatorics and show the applicability of our method.

4.1 An Introductory Example

We use our Maple program to analyze the rational function

$$f(x) = \frac{1}{(1-2x)^2(1-10x^2)}.$$

Therefore we input

```
> analyze(1/(1-2*x)^2/(1-10*x^2));
```

First the system prints out the expanded denominator and its reciprocal polynomial:

$$\begin{aligned}q(x) &= -40x^4 + 40x^3 - 6x^2 - 4x + 1 \\ \bar{q}(x) &= x^4 - 4x^3 - 6x^2 + 40x - 40\end{aligned}$$

Next the roots of $f(x)$ are computed:

$$\lambda = \left[2, \frac{1}{\text{RootOf}(-1 + 10_Z^2, \text{index} = 1)}, \frac{1}{\text{RootOf}(-1 + 10_Z^2, \text{index} = 2)} \right].$$

We see that the second and the third root both have maximal modulus. To verify this, Maple has to compute the quantity κ (formula from Theorem 3.4.2)

$$\kappa = \frac{\sqrt{3}}{2} \cdot 10^{-6} \cdot 20^{-46} \approx 1.230696 \cdot 10^{-66}$$

and then evaluates the roots with a precision of 68 digits. Now it must be checked if these roots are multiples of roots of unity. Therefore the symmetric polynomial $R(x)$ is computed:

$$R(x) = -8000x^6 - 16000x^5 + 15200x^4 + 46400x^3 + 15200x^2 - 16000x - 8000.$$

Note that the factor $\prod \lambda_i$ is disregarded since it is of no relevance for the following computation. We find that solely the cyclotomic polynomial $\Phi_2(x) = x + 1$ divides $R(x)$, thus $f(x)$ has to be decomposed into 2 subseries:

$$\begin{aligned} f_1(x) &= \frac{4x + 1}{1 - 18x + 96x^2 - 160x^3} \\ f_2(x) &= \frac{4}{1 - 18x + 96x^2 - 160x^3} \end{aligned}$$

For each of the subseries we have to check if it has a dominating root and if it has nonnegative coefficients. Here we exemplify this only for the first case. The roots of $f_1(x)$ are 10 and 4, so obviously it has a dominating root. To determine if all coefficients are nonnegative, the exponential polynomial is computed:

$$\frac{35}{9} \cdot 10^n - \left(\frac{4}{3}n + \frac{26}{9} \right) 4^n$$

According to case C of Section 3.4.3 the variables a, b, c, d, e are assigned the following values:

$$\begin{aligned} a &= 9.94 \\ b &= 4.06 \\ c &= 35/18 \\ d &= 2 \\ e &= 26/9 \end{aligned}$$

We get the following inequality:

$$\frac{208}{35}n \cdot \left(\frac{4.06}{9.94} \right)^n < 1$$

which is satisfied for all $n \geq 4$. So, by estimation of the exponential polynomial we can be sure that all coefficients $s_n, n \geq 4$ are nonnegative. It rests to compute the coefficients having smaller index:

$$s_0 = 1, s_1 = 22, s_2 = 300, s_3 = 3448.$$

Hence, all coefficients are positive and $f_1(x)$ is \mathbb{N} -rational. The same holds for $f_2(x)$ and so we can conclude by Theorem 2.5.12 that f is \mathbb{N} -rational.

Now a regular expression has to be derived which is done separately for f_1 and f_2 (again only the first one is presented here). In order to find a constant c that fulfills the inequalities from Lemma 2.5.9, we have to perform a further decomposition into 2 sub-subseries. This information is obtained by the procedure `pForRegExp`, and indeed, it can be seen that no c ($4 \leq c \leq 10$) can be found such that all γ_i 's are nonnegative, if we do not decompose:

$$\begin{aligned} c = 4 & : \gamma = [14, -40, 0] \\ c = 5 & : \gamma = [13, -31, 5] \\ c = 6 & : \gamma = [12, -24, 16] \\ c = 7 & : \gamma = [11, -19, 27] \\ c = 8 & : \gamma = [10, -16, 32] \\ c = 9 & : \gamma = [9, -15, 25] \\ c = 10 & : \gamma = [8, -16, 0] \end{aligned}$$

So, the decomposition is performed; the first of the resulting sub-subseries is

$$f_{11} = \frac{1 + 168x + 640x^2}{1 - 132x + 3456x^2 - 25600x^3}.$$

The polynomial $R^{(p)}$ is simply the denominator of f_{11} since the dominating root has multiplicity 1. The constant c is found to be 36, and the γ_i 's are computed:

$$\begin{aligned} \gamma_1 & = 132 - 36 = 96 \\ \gamma_2 & = -3456 + 36 \cdot 132 - 36^2 = 0 \\ \gamma_3 & = 25600 - 3456 \cdot 36 + 132 \cdot 36^2 - 36^3 = 25600 \end{aligned}$$

They serve to get the inverse of $R^{(p)}$:

$$\frac{1}{R^{(p)}} = (36x)^* (96x + 25600x^3(36x)^*)^*$$

Now the series T and U have to be defined:

$$\begin{aligned} T & = 1 + 168x + 640x^2 \\ U & = \frac{(1 + 168x + 640x^2)(-1 + 36x)}{-1 + 132x - 3456x^2 + 25600x^3} \end{aligned}$$

The number h is chosen to be 0. Finally a regular expression for f_{11} is obtained by the formula from Theorem 2.5.11:

$$\begin{aligned}
& 25600 ((36x)^*)^2 (96x + 25600x^3(36x)^*)^* x^3 \\
& + 264(36x)^* (96x + 25600x^3(36x)^*)^* x \\
& + 640(36x)^* (96x + 25600x^3(36x)^*)^* x^2 + 36x(36x)^* + 1
\end{aligned}$$

The same procedure has to be carried out for the remaining sub-subseries f_{12} , f_{21} , and f_{22} . These results are put together and provide a regular expression for f .

4.2 The MIU System

In [Hof79] Douglas Hofstadter introduces the famous MIU system. This formal system defines a language L_{MIU} over the alphabet $\Sigma = \{M, I, U\}$. Its words can be obtained by starting with the axiom MI and by applying the following rules:

1. $wI \rightarrow wIU$
2. $Mw \rightarrow Mww$
3. $III \rightarrow U$
4. $UU \rightarrow \lambda$

where w denotes an arbitrary word $w \in \Sigma^*$. The language L_{MIU} turns out to be regular, since every word begins with an M, followed by a string containing only I's and U's, where the number of I's is not divisible by 3:

$$w \in L_{MIU} \iff w = Mw', w' \in \{I, U\}^*, \#_I(w') \not\equiv 0 \pmod{3}.$$

By analyzing the finite automaton that accepts L_{MIU} , we find that

$$x \mapsto \frac{x^2}{1 - 3x + 3x^2 - 2x^3}$$

is the generating function of the corresponding power series¹

$$x^2 + 3x^3 + 6x^4 + 11x^5 + 21x^6 + 42x^7 + 85x^8 + \dots$$

We feed our Maple program with this function and obtain the following regular expression:

¹see [Slo, A024495]

> analyze(x^2/(1-3*x+3*x^2-2*x^3);

$$\begin{aligned} & \text{star}(x^2)\text{star}(2x^2 + 5x^4 + 9x^6\text{star}(x^2))x^2(1 + 3x^2) + \\ & x^3\text{star}(x^2)\text{star}(2x^2 + 5x^4 + 9x^6\text{star}(x^2))(3 + 2x^2) \end{aligned}$$

By factoring and replacing the inconvenient *star* notation, this simplifies to

$$(x^2)^* (x^2 (2 + 5x^2 + 9x^4(x^2)^*))^* x^2(2x + 1)(x^2 + x + 1)$$

4.3 Look and Say

A very interesting sequence discovered and examined by JOHN CONWAY² is the so-called *Look and Say Sequence*. It starts with 1, and every subsequent element is the “description” of the previous one. The elements are considered to be strings over the alphabet of digits (it turns out that solely the digits 1, 2, and 3 appear). Then the “description” of an element can be written by the rule

$$x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} \rightarrow m_1 x_1 m_2 x_2 \cdots m_k x_k.$$

So, the initial string 1 can be described by 11 (“one one”), which itself can be described by 21 (“two ones”). Hence, the first elements of the Look and Say Sequence³ are

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, \dots$$

CONWAY built up a strange theory concerning its marvellous properties – the so-called *Cosmological Theorem*: Examining repeating patterns in the sequence, he found out that there are 92 strings into which the elements can be split and each of the fractions develops without influencing the others any more. In analogy to the 92 atomic elements up to Uranium, he called this process the “audioactive decay”⁴.

We are now interested in the formal power series S_{LS} obtained by writing down the lengths of the words in the Look and Say Sequence⁵:

$$S_{LS} = 1 + 2x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 8x^6 + 10x^7 + \dots$$

Taking in account the splitting process of the words, it would appear that its lengths must obey some linear recurrence. Indeed, the series S_{LS} is generated by the rational function $f_{LS} = p/q$ where

²see [Con87]

³[Slo, A005150]

⁴A list of the audioactive elements can be found under <http://mathworld.wolfram.com/CosmologicalTheorem.html>

⁵[Slo, A005341]

$$\begin{aligned}
p(x) = & -12x^{78} + 18x^{77} - 18x^{76} + 18x^{75} - 18x^{74} + 20x^{73} + 22x^{72} - 31x^{71} - 15x^{70} + \\
& 4x^{69} + 4x^{68} + 19x^{67} - 62x^{66} + 50x^{65} + 21x^{64} + 11x^{63} - 41x^{62} - 54x^{61} + 56x^{60} + \\
& 44x^{59} - 15x^{58} + 27x^{57} + 15x^{56} - 45x^{55} + 8x^{54} - 89x^{53} + 64x^{52} + 66x^{51} + 25x^{50} - \\
& 38x^{49} - 126x^{48} + 39x^{47} + 32x^{46} + 33x^{45} + 65x^{44} - 107x^{43} - 14x^{42} - 16x^{41} + 13x^{40} + \\
& 79x^{39} - 7x^{38} - 42x^{37} - 12x^{36} - 8x^{35} + 26x^{34} + 9x^{33} - 35x^{32} + 23x^{31} + 20x^{30} + 30x^{29} - \\
& 34x^{28} - 58x^{27} + x^{26} + 20x^{25} + 36x^{24} + 6x^{23} - 13x^{22} - 8x^{21} - 6x^{20} - 3x^{19} + x^{18} + \\
& 4x^{17} + x^{16} + 4x^{15} + 5x^{14} + x^{13} - 8x^{12} - 6x^{11} + 6x^9 + 4x^8 - x^7 - x^5 - x^4 - x^3 - x^2 + x + 1
\end{aligned}$$

and

$$\begin{aligned}
q(x) = & 6x^{72} - 9x^{71} + 9x^{70} - 18x^{69} + 16x^{68} - 11x^{67} + 14x^{66} - 8x^{65} + x^{64} - 5x^{63} + \\
& 7x^{62} + 2x^{61} + 8x^{60} - 14x^{59} - 5x^{58} - 5x^{57} + 19x^{56} + 3x^{55} - 6x^{54} - 7x^{53} - 6x^{52} + \\
& 16x^{51} - 7x^{50} + 8x^{49} - 22x^{48} + 17x^{47} - 12x^{46} + 7x^{45} + 5x^{44} + 7x^{43} - 8x^{42} + \\
& 4x^{41} - 7x^{40} - 9x^{39} + 13x^{38} - 4x^{37} - 6x^{36} + 14x^{35} - 14x^{34} + 19x^{33} - 7x^{32} - \\
& 13x^{31} + 2x^{30} - 4x^{29} + 18x^{28} - x^{26} - 4x^{25} - 12x^{24} + 8x^{23} - 5x^{22} + 8x^{20} + x^{19} + \\
& 7x^{18} - 8x^{17} - 5x^{16} - 2x^{15} + 3x^{14} + 3x^{13} - 2x^8 - x^7 + 3x^5 + x^4 - x^3 - x^2 - x + 1
\end{aligned}$$

This monstrous function is quite a challenge for our program! By means of our procedure `hasDominatingRoot` we find out that it has a dominating root:

```
> Digits:= 20: evalf(op(1, getRootsRatSort(f[71])));
```

1.3035772690342963913

This number (we denote it by γ) is known as CONWAY's constant and gives the order of growth for the word lengths in the Look and Say Sequence:

$$s_{LS,n} = C\gamma^n$$

where C can be computed by the procedure `exponentialPolynomial`⁶:

```
> op(1,exponentialPolynomial(f[71],getRootsRatSort(f[71])));
```

2.042160065

We now try to determine a regular expression for the series S_{LS} . Indeed, after a few hours of computation time, we get a result that fills lots of pages, not least because between γ and the maximal modulus of the remaining roots (≈ 1.161124201) there is only a small gap. For finding the constant c the series has to be decomposed into 8 subseries which inflates the length of the result by the factor 8. We can verify the correctness of the regular expression by assigning the function $x \mapsto 1/(1-x)$ to the star symbol and by subsequent simplifying: Voilà, we obtain the original function f_{LS} !

⁶Note that our result differs from the value ($C=1.567\dots$) given in Eric Weisstein's Math-World (<http://mathworld.wolfram.com/LookandSaySequence.html>), although it seems to be correct.

4.4 Rational Series with Positive Coefficients that are not \mathbb{N} -Rational

An interesting question is to examine rational series with nonnegative coefficients that are not \mathbb{N} -rational. From about 60 rational series with nonnegative coefficients taken from [Slo], not one single turned out to be not \mathbb{N} -rational. And also an exhaustive search with randomly generated rational functions, did not provide any result. So, we can ask: Are there at all some rational positive series that are not \mathbb{N} -rational? In Section 2.5 we already answered this question with “yes” and gave also an example:

$$f_1(x) = \frac{x + 5x^2}{1 + x - 5x^2 - 125x^3}.$$

Its roots are 5 and $3 \pm 4i$ which tells us instantly (bearing in mind Theorem 2.5.2) that this function is not \mathbb{N} -rational. But are really all its coefficients positive? We can not apply the procedure `hasPositiveCoefficients` since f_1 has no dominating root. And of course, it is dangerous to trust in testing the first coefficients: Look at the function

$$f_2(x) = \frac{1 + 9x}{1 - 15x + 57x^2 - 5x^3}.$$

Computing the first 81 coefficients shows that they are growing steadily, but suddenly the 82nd coefficient turns out to be negative!

n	$s_{2,n}$
0	1
1	24
2	303
3	3182
10	12125514603
20	11728000407381188379
30	8154598335301238519946423397
50	2356099125769316699176781703031384401195326103
80	18166319191809537974353717811220489771970865857972844177342 283118226121
81	25008178463392720970253380654955855226918683083994030578472 691521897824
82	-6382723572058113096288554398545045201151261630613072303232 11111914904697

Now back to the function $f_1(x)$. Its positiveness can be proved by examining the exponential polynomial:

$$s_{1,n} = \frac{1}{8} \cdot 5^n - \frac{1}{16} \cdot (3 + 4i)^n - \frac{1}{16} \cdot (3 - 4i)^n$$

Since $|3 \pm 4i| = 5$ we see that $s_{1,n} \geq 0$ must hold for all n .⁷

This leads to an algorithm for finding more series with positive coefficients that are not \mathbb{N} -rational: Choose two integers c and $d < c$ such that $d \pm i\sqrt{c^2 - d^2}$ are no roots of unity. Then define the exponential polynomial

$$a_1 c^n + a_2 \left(d + i\sqrt{c^2 - d^2}\right)^n + a_3 \left(d - i\sqrt{c^2 - d^2}\right)^n,$$

where $a_1 \geq a_2 + a_3$. Then the resulting series has positive coefficients but is not \mathbb{N} -rational.

We choose for example $c = 3$ and $d = 1$. Further we set $a_1 = 2$ and $a_2 = a_3 = 1$. Then we get the rational function

$$\begin{aligned} f_3(x) &= \frac{2}{1 - 3x} + \frac{1}{1 - (1 + i2\sqrt{2})x} + \frac{1}{1 - (1 - i2\sqrt{2})x} \\ &= \frac{4 - 12x + 24x^2}{1 - 5x + 15x^2 - 27x^3} \end{aligned}$$

which has a series expansion with positive coefficients:

$$f_3(x) = 4 + 8x + 4x^2 + 8x^3 + 196x^4 + 968x^5 + 2116x^6 + 1352x^7 + 1156x^8 + \dots$$

⁷see also [Ges03]

Chapter 5

Manual Pages

This chapter contains a collection of manual pages for all implemented procedures, that have been mentioned in Chapter 3. While there the theoretical background of the implemented algorithms is described, the reader here is told about how these procedures can be applied.

The Maple worksheet is designed to print out some additional information concerning the computation and provisional results. By setting the global variable `myinfolevel`, the user can control the level of detail. The permissible values range from 0 through 5, where `myinfolevel:=0` causes that no additional information is given. To observe the main steps of the computation, it is sufficient to set `myinfolevel:=3`.

Many procedures taking a list of roots `lambda` as one of their arguments, only yield correct results if `lambda` is “partially sorted”, i.e., the root(s) having maximal modulus are on the head, and in the special case that there is a dominating root, it is followed by one of the roots having second greatest modulus. The examples on pages 79 and 82 may illustrate this.

5.1 analyze

Test for \mathbb{N} -rationality and compute a regular expression

Calling Sequence

`analyze(f)`

Parameters

`f` – rational function in `x` with integer coefficients

Description

The procedure `analyze` decides if the rational function `f` is \mathbb{N} -rational. In the affirmative case a corresponding regular expression is computed. It returns `false` if `f` is not \mathbb{N} -rational. Otherwise the regular expression is returned; herein the star operation x^* is denoted by the function `star`.

Examples

```
> analyze(x^2/(1-3*x+3*x^2-2*x^3));
```

$$\text{star}(x^2)\text{star}(2x^2 + 5x^4 + 9x^6\text{star}(x^2))x^2(1 + 3x^2) + \\ x^3\text{star}(x^2)\text{star}(2x^2 + 5x^4 + 9x^6\text{star}(x^2))(3 + 2x^2)$$

```
> star := x->1/(1-x);
```

$$\text{star} := x \rightarrow \frac{1}{1-x}$$

```
> simplify(%);
```

$$\frac{x^2}{2x^3 - 3x^2 + 3x - 1}$$

```
> analyze((x+5*x^2)/(1+x-5*x^2-125*x^3));
```

false

```
> analyze(1-x);
```

false

5.2 **boundaryForNonnegCoeffs**

Boundary from which on all coefficients of a series are nonnegative

Calling Sequence

`boundaryForNonnegCoeffs(f, lambda)`

Parameters

- `f` – rational function in `x` with integer coefficients and having a dominating root
- `lambda` – list of all roots of `f` (partially sorted)

Description

The procedure `boundaryForNonnegCoeffs` computes a boundary n_0 such that all coefficients with an index greater than n_0 are nonnegative. The boundary n_0 is minimal in the sense that either $n_0 = -1$ (i.e., all coefficients are nonnegative) or, if $n_0 \geq 0$, the coefficient with index n_0 itself is negative. In the case that no such boundary exists (because there are infinitely many negative coefficients), `false` is returned.

Examples

```
> f1:= (1+2*x+x^2)/(1-x+x^2-2*x^3):
> getCoefficients(f1, 10);
```

[1, 3, 3, 2, 5, 9, 8, 9, 19, 26, 25]

```
> boundaryForNonnegCoeffs(f1, getRootsRatSort(f1));
-1
```

```
> f2:= simplify(f1-10*x^7);
```

$$f2 := -\frac{1 + 2x + x^2 - 10x^7 + 10x^8 - 10x^9 + 20x^{10}}{-1 + x - x^2 + 2x^3}$$

```
> boundaryForNonnegCoeffs(f2, getRootsRatSort(f2));
7
```

```
> boundaryForNonnegCoeffs(-f1, getRootsRatSort(-f1));
false
```

5.3 coefficient

n^{th} coefficient of a series

Calling Sequence

`coefficient(f, n)`

Parameters

- `f` – rational function in x with integer coefficients
- `n` – nonnegative integer

Description

The procedure `coefficient` computes the n^{th} coefficient of the series given by the rational function `f` by means of matrix exponentiation.

Examples

```
> coefficient(x^4-4*x^3+6*x^2-4*x+1, 3);
```

-4

```
> coefficient(1/(1-x-x^2), 9);
```

55

```
> coefficient(1/(1-x-x^2)+x^9, 9);
```

56

```
> coefficient(1/(3-x), 3);
```

$\frac{1}{81}$

5.4 commonUnityRoots

Common order of unity roots in a polynomial

Calling Sequence

`commonUnityRoots(p)`

Parameters

`p` – polynomial in x with integer coefficients

Description

The procedure `commonUnityRoots` computes an integer n such that for all roots of p , which have the form $\rho\vartheta$ ($\rho \in \mathbb{R}_+$ and ϑ denoting a root of unity), $\vartheta^n = 1$ holds (see Theorem 2.5.12).

Examples

```
> commonUnityRoots(4*x^2-1);  
2  
> commonUnityRoots(x^3+4*x-5);  
1  
> commonUnityRoots(x^6+x^5-x-1);  
10
```

5.5 decomposition

Generating function for a subseries

Calling Sequence

`decomposition(f, k, i)`

Parameters

- `f` – rational function in x with integer coefficients
- `k` – number of subseries
- `i` – index for a certain subseries

Description

Let f_1, \dots, f_k be a decomposition of f into k subseries (see Definition 2.5.6). The procedure `decomposition` computes the generating function for the i^{th} subseries f_i . For this purpose the multisection formula from [Rio58, p. 131] is used.

Examples

`> f := (1+x)/(1-6*x+11*x^2-6*x^3);`

$$f := \frac{x + 1}{1 - 6x + 11x^2 - 6x^3}$$

`> getCoefficients(f, 10);`

[1, 7, 31, 115, 391, 1267, 3991, 12355, 37831, 115027, 348151]

`> d1 := decomposition(f, 3, 1);`

$$d1 := -\frac{1 + 102x^2 + 79x}{-1 + 216x^3 - 251x^2 + 36x}$$

`> d2 := decomposition(f, 3, 2);`

$$d2 := -\frac{139x + 36x^2 + 7}{-1 + 216x^3 - 251x^2 + 36x}$$

`> getCoefficients(d2, 5);`

[7, 391, 12355, 348151, 9516787, 257887111]

```
> d3:= decomposition(f, 3, 3);
```

$$d3 := -\frac{151x + 31}{-1 + 216x^3 - 251x^2 + 36x}$$

```
> simplify(subs(x=x^3,d1) + x*subs(x=x^3,d2) +  
x^2*subs(x=x^3,d3));
```

$$-\frac{x + 1}{6x^3 - 11x^2 + 6x - 1}$$

5.6 expandRootOf

Expand an unindexed RootOf expression

Calling Sequence

```
expandRootOf(expr)
```

Parameters

`expr` – a RootOf expression

Description

The expression `RootOf(p)`, `p` being a polynomial, denotes the set of `p`'s roots. The procedure `expandRootOf` determines all elements of this set by adding the parameter `index` to the `RootOf` expression. It returns a list containing all these elements, their number being `deg(p)`.

Examples

```
> r := RootOf(x^3+x^2+x-1);
```

$$r := \text{RootOf}(-Z^3 + Z^2 + Z - 1)$$

```
> expandRootOf(r);
```

$$\begin{aligned} & [\text{RootOf}(-Z^3 + Z^2 + Z - 1, \text{index} = 1), \\ & \text{RootOf}(-Z^3 + Z^2 + Z - 1, \text{index} = 2), \\ & \text{RootOf}(-Z^3 + Z^2 + Z - 1, \text{index} = 3)] \end{aligned}$$

5.7 exponentialPolynomial

Exponential polynomial for a series' coefficients

Calling Sequence

`exponentialPolynomial(f, lambda)`

Parameters

`f` – rational function in `x` with integer coefficients
`lambda` – list of all roots of `f`

Description

The coefficients of every rational series S can be represented by means of the exponential polynomial (see Theorem 2.4.5): $s_n = \sum P_i(n)\lambda_i^n$. The procedure `exponentialPolynomial` computes the polynomials $P_i(n)$ numerically (since they are only needed for the estimate in Section 3.4.3). It returns a list containing the P_i 's in the same order as the roots `lambda` were given.

Examples

```
> fib:= 1/(1-x-x^2): lambda:= getRootsRatSort(fib);
```

$$\lambda := \left[\frac{1}{\text{RootOf}(-1 + _Z + _Z^2, \text{index} = 1)}, \frac{1}{\text{RootOf}(-1 + _Z + _Z^2, \text{index} = 2)} \right]$$

```
> ep:= exponentialPolynomial(fib, lambda);
```

$$ep := [.7236067976, .2763932022]$$

```
> fn:= add(ep[i]*lambda[i]^n, i=1..2);
```

$$fn := .7236067976 \left(\frac{1}{\text{RootOf}(-1 + _Z + _Z^2, \text{index} = 1)} \right)^n + .2763932022 \left(\frac{1}{\text{RootOf}(-1 + _Z + _Z^2, \text{index} = 2)} \right)^n$$

```
> evalf(subs(n=7, fn));
```

21.00000001

5.8 findc

Calling Sequence

```
findc(Rp, lambda)
```

Parameters

`Rp` – polynomial in x with integer coefficients
`lambda` – list of all roots of $1/Rp$ (partially sorted)

Description

Given the polynomial Rp , the procedure `findc` computes the minimal integer c that fulfills the conditions of Lemma 2.5.9. If such a constant c does not exist then -1 is returned.

Examples

```
> Rp:= 1-5*x+4*x^2-13*x^3-x^4;
```

$$Rp := 1 - 5x + 4x^2 - 13x^3 - x^4$$

```
> c:= findc(Rp, getRootsRatSort(1/Rp));
```

$$c := 2$$

```
> gamma[1]:= -coeff(Rp, x, 1) - c;
```

```
> for i from 2 to degree(Rp) do
```

```
>   gamma[i]:= -coeff(Rp, x, i) + c*gamma[i-1];
```

```
> od;
```

$$\gamma_1 := 3$$

$$\gamma_2 := 2$$

$$\gamma_3 := 17$$

$$\gamma_4 := 35$$

```
> Rp:= 1-3*x+4*x^2-x^3:
```

```
findc(Rp, getRootsRatSort(1/Rp));
```

-1

5.9 `getCoefficients`

The first n coefficients of a series

Calling Sequence

```
getCoefficients(f, n)
```

Parameters

- f – rational function in x with integer coefficients
- n – nonnegative integer

Description

Given a rational function f , the procedure `getCoefficients` returns a list containing the coefficients s_0, \dots, s_n of its MACLAURIN expansion, i.e., the first $n+1$ coefficients of the series generated by f .

Examples

```
> getCoefficients(x/(1-2*x), 7);
```

```
[0, 1, 2, 4, 8, 16, 32, 64]
```

```
> getCoefficients(x^2-1, 10);
```

```
[-1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0]
```

```
> getCoefficients(1/(2-x), 5);
```

```
[ $\frac{1}{2}$ ,  $\frac{1}{4}$ ,  $\frac{1}{8}$ ,  $\frac{1}{16}$ ,  $\frac{1}{32}$ ,  $\frac{1}{64}$ ]
```

5.10 getRoots

The (different) roots of a polynomial

Calling Sequence

```
getRoots(p)
```

Parameters

p – polynomial in x with integer coefficients

Description

The procedure `getRoots` computes the roots of `p` without respect to their multiplicity. Therefore `p` is made squarefree and then it is factorized. All non-integral roots are given as `RootOf` expressions. The roots are returned as an unsorted list.

Examples

```
> getRoots(x^3-5*x^2-2*x+24);
```

```
[3, 4, -2]
```

```
> getRoots(x^5+2*x^4+3*x^3+3*x^2+2*x+1);
```

```
[-1, RootOf(_Z^2 + 1, index = 1), RootOf(_Z^2 + 1, index = 2),
RootOf(_Z^2 + _Z + 1, index = 1), RootOf(_Z^2 + _Z + 1, index = 2)]
```

```
> getRoots(x^5+3*x^4+3*x^3+3*x^2+2*x+1);
```

```
[RootOf(_Z^5 + 3_Z^4 + 3_Z^3 + 3_Z^2 + 2_Z + 1, index = 1),
RootOf(_Z^5 + 3_Z^4 + 3_Z^3 + 3_Z^2 + 2_Z + 1, index = 2),
RootOf(_Z^5 + 3_Z^4 + 3_Z^3 + 3_Z^2 + 2_Z + 1, index = 3),
RootOf(_Z^5 + 3_Z^4 + 3_Z^3 + 3_Z^2 + 2_Z + 1, index = 4),
RootOf(_Z^5 + 3_Z^4 + 3_Z^3 + 3_Z^2 + 2_Z + 1, index = 5)]
```

5.11 **getRootsRat, getRootsRatSort**

The (different) roots of a rational function

Calling Sequence

```
getRootsRat(f)
getRootsRatSort(f)
```

Parameters

f – rational function in *x* with integer coefficients

Description

The procedures `getRootsRat` and `getRootsRatSort` compute the roots of *f* without respect to their multiplicities. The roots of a rational function are defined to be the roots of the denominator's reciprocal polynomial — or equivalently the reciprocal values of *f*'s poles.

The roots of *f*'s denominator are computed by `getRoots` (see p. 78); in order to obtain the roots of *f*, these values are inverted.

`getRootsRat` returns the roots as an unsorted list.

On the other hand, `getRootsRatSort` returns the roots as a partially sorted list: The head is made up of the root(s) with maximal modulus. In the special case that there is a dominating root, on the second position is a root with second greatest modulus. The rest of the list is unsorted. The sorting is performed by `hasDominatingRoot` (see p. 82).

Examples

```
> r := getRootsRat(1/(9*x^4-18*x^3+11*x^2-5*x+1));
```

$$r := \left[\frac{1}{\text{RootOf}(-1 + 2_Z - 5_Z^2 + 3_Z^3, \text{index} = 1)}, \frac{1}{\text{RootOf}(-1 + 2_Z - 5_Z^2 + 3_Z^3, \text{index} = 2)}, \frac{1}{\text{RootOf}(-1 + 2_Z - 5_Z^2 + 3_Z^3, \text{index} = 3)}, 3 \right]$$

```
> for i in r do evalf(i) od;
```

```
0.7372777205
0.6313611397 - 1.915830396I
0.6313611397 + 1.915830396I
3.
```

```
> getRootsRatSort(1/(9*x^4-18*x^3+11*x^2-5*x+1));
```

$$\left[3, \frac{1}{\text{RootOf}(3_Z^3 - 5_Z^2 + 2_Z - 1, \text{index} = 2)}, \frac{1}{\text{RootOf}(3_Z^3 - 5_Z^2 + 2_Z - 1, \text{index} = 1)}, \frac{1}{\text{RootOf}(3_Z^3 - 5_Z^2 + 2_Z - 1, \text{index} = 3)} \right]$$

5.12 *getRp*

Calling Sequence

```
getRp(f, lambda)
```

Parameters

- f* – rational function in *x* with integer coefficients and having a dominating root
- lambda* – list of all roots of *f* (partially sorted)

Description

When computing the regular expression of a series, its generating function has to be multiplied step by step by polynomials $R^{(p)}$ such that the multiplicity of the dominating root is decreased by 1 in each step. The proof of Lemma 2.5.9 gives instructions how $R^{(p)}$ can be chosen: If the dominating root is simple then $R^{(p)}$ is set to be the denominator of *f*. Otherwise $R^{(p)}$ is assigned the squarefree part of *f*'s denominator. The procedure *getRp* determines the multiplicity of the dominating root and depending on it computes the polynomial $R^{(p)}$.

Examples

```
> getRp(1/((1-3*x)*(1-4*x)^2*(1-6*x)), [6,4,3]);
```

$$(3x - 1)(-1 + 4x)^2(-1 + 6x)$$

```
> getRp(1/((1-3*x)*(1-4*x)^2*(1-6*x)^3), [6,4,3]);
```

$$-(3x - 1)(-1 + 4x)(-1 + 6x)$$

5.13 hasDominatingRoot

Check the existence of a dominating root

Calling Sequence

```
hasDominatingRoot(p, lambda, 'lambdaSorted')
```

Parameters

`p` – polynomial in x with integer coefficients
`lambda` – list of all roots of p
`'lambdaSorted'` – unevaluated name

Description

The procedure `hasDominatingRoot` checks if the polynomial p has a dominating root, i.e., this root is positive real and greater than the modulus of every other root of p . The return value is an integer s indicating the following cases:

- i) $s = 0$: p has a dominating root.
- ii) $s = 1$: None of the roots with maximal modulus is positive real.
- iii) $s = 2$: p has several roots with maximal modulus and one of them is positive real.

The third argument `'lambdaSorted'` will be assigned the list `lambda`, but partially sorted. In every case the root(s) having maximal modulus form the head (but not in a special order themselves), and the rest of the list is unsorted. In case i), the dominating root is followed by one of the roots having second greatest modulus.

Examples

```
> p:= x^3-7*x^2+14*x-8: lambda:= getRoots(p);
      lambda := [1, 2, 4]
> hasDominatingRoot(p, getRoots(p), 'lambdaSorted');
```

```

> lambdaSorted;

[4, 2, 1]

> p:= 3-4*x+2*x^2-x^3: lambda:= getRoots(p);

lambda :=
[1, RootOf(_Z^2 - _Z + 3, index = 1), RootOf(_Z^2 - _Z + 3, index = 2)]

> hasDominatingRoot(p, getRoots(p), 'lambdaSorted');

1

> lambdaSorted;

lambda :=
[RootOf(_Z^2 - _Z + 3, index = 1), RootOf(_Z^2 - _Z + 3, index = 2), 1]

> p:= (1-x)*(2-x)*(3-x)*(4-x)*(5-x)*(36-x^2):
> lambda:= getRoots(p);

lambda := [1, 2, 3, 4, 5, 6, -6]

> hasDominatingRoot(p, getRoots(p), 'lambdaSorted');

2

> lambdaSorted;

[6, -6, 1, 2, 3, 4, 5]

> p:= numtheory[cyclotomic](22, x)*(x-1);
p := (x10 - x9 + x8 - x7 + x6 - x5 + x4 - x3 + x2 - x + 1)(x - 1)

> hasDominatingRoot(p, getRoots(p), 'lambdaSorted');

2

```

5.14 hasPositiveCoefficients

Test if a series is positive

Calling Sequence

```
hasPositiveCoefficients(f, lambda)
```

Parameters

f – rational function in x with integer coefficients
and having a dominating root
lambda – list of all roots of **f** (partially sorted)

Description

The procedure `hasPositiveCoefficients` determines if all coefficients of the series given by **f** are positive (or vanishing). Therefore the procedure `boundaryForNonnegCoeffs` (see p. 69) is called.

Examples

```
> f:= x^2/(1-x-x^2):
hasPositiveCoefficients(f, getRootsRatSort(f));
```

true

```
> f:= (x^2-x)/(1-x-x^2):
hasPositiveCoefficients(f, getRootsRatSort(f));
```

false

```
> f:= 1-x-x^2:
hasPositiveCoefficients(f, getRootsRatSort(f));
```

false

5.15 *kappa*

Estimate for the minimal difference between two (different) moduli of a polynomial's roots

Calling Sequence

`kappa(p, lambda)`

Parameters

`p` – polynomial in x with integer coefficients
`lambda` – list of all roots of p

Description

The number κ is defined in [GS96, p. 9] (see also Theorem 3.4.2). If the moduli of two roots of p are distinct, their difference must be greater than κ . The imaginary part is either 0 or greater than κ . The procedure `kappa` computes a numerical value for this quantity.

Examples

```
> p:= x^3+3*x-1: kappa(p, getRoots(p));
          3.127140010 · 10-14
> p:= 56*x^3+63*x-1: kappa(p, getRoots(p));
          1.088260059 · 10-41
> p:= x^71+x^52+x^25+x^17+1: kappa(p, getRoots(p));
          7.408773040 · 10-49433
```

5.16 maximalModulus

Find elements with maximal modulus

Calling Sequence

```
maximalModulus(lambda, digits)
```

Parameters

`lambda` – list of expressions representing complex values
`digits` – number of digits

Description

Returns a list of indices indicating the elements having maximal modulus. The result depends on the precision given by `digits`: Moduli that lie very close together may seem to be equal if the number of digits is too small. In order to minimize the influence of numerical errors, the last digit is not taken into account.

Examples

```
> lambda:= getRoots(315*x^4-990*x^3-5761*x^2+15136*x+14448):

> for l in lambda do evalf(abs(l)); od;

          3.910100792
          0.7672436491
          3.910100880
          3.910100880

> maximalModulus(lambda, 1);

          [1, 2, 3, 4]

> maximalModulus(lambda, 8);

          [1, 3, 4]

> maximalModulus(lambda, 100);

          [3, 4]
```

5.17 pForRegExp

Order of decomposition for computing a regular expression

Calling Sequence

`pForRegExp(f)`

Parameters

`f` – \mathbb{N} -rational function in x with integer coefficients
and having a dominating root

Description

To an \mathbb{N} -rational function `f`, the procedure `pForRegExp` determines an integer `p` such that a decomposition into `p` subseries fulfills the statements given in Lemma 2.5.9.

Examples

```
> pForRegExp(1/(1-x));
```

1

```
> pForRegExp(1/(x^6+x^5-x^4-3*x^3-x^2+x+1));
```

9

5.18 reciprocal

Reciprocal polynomial

Calling Sequence

`reciprocal(p)`

Parameters

`p` – polynomial in x with integer coefficients

Description

Returns the reciprocal polynomial of p .

Examples

```
> reciprocal(x^3+2*x^2+3*x+4);
```

$$4x^3 + 3x^2 + 2x + 1$$

```
> reciprocal(1);
```

1

```
> reciprocal(0);
```

0

5.19 regularExpression

Regular expression of an N-rational series

Calling Sequence

`regularExpression(f, lambda, checkDecomp)`

Parameters

- `f` – N-rational function in x with integer coefficients and having a dominating root
- `lambda` – list of all roots of f (partially sorted)
- `checkDecomp` – indicates if a decomposition has to be considered

Description

Computes a regular expression for the given N-rational function f . The parameter `checkDecomp` must be 0 (it is only changed when the procedure calls itself recursively).

Examples

> `regularExpression(1/(1-x), [1], 0);`

$$x \text{ star}(x) + 1$$

> `regularExpression(x^7, [], 0);`

$$x^7$$

> `regularExpression(1/(1-2*x)^2, [2], 0);`

$$\text{star}(x) \text{ star}(x \text{ star}(x)) (x \text{ star}(x) + 2 \text{ star}(x) \text{ star}(x \text{ star}(x)) x) + x \text{ star}(x) + 1$$

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