A SIMPLIFIED PROOF OF THE CHARACTERIZATION THEOREM FOR GRÖBNER BASES

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Abstract

In /2/ a certain type of bases ("Gröbner-Bases") for polynomial ideals has been introduced whose usefulness stems from the fact that a number of important computability problems in the theory of polynomial ideals are reducible to the construction of bases of this type. The key to an algorithmic construction of Gröbner-bases is a characterization theorem for Gröbner-bases whose proof in /2/ is rather complex.

In this paper a simplified proof is given. The simplification is based on two new lemmas that are of some interest in themselves. The first lemma characterizes the congruence relation modulo a polynomial ideal as the reflexive-transitive closure of a particular reduction relation ("N-reduction") used in the definition of Gröbner-bases and its inverse. The second lemma is a lemma on general reduction relations, which allows to guarantee the Church-Rosser property under very weak assumptions.

1. Introduction

Gröbner-bases for polynomial ideals are defined as follows:

Definition:

A sequence F of polynomials from \( \mathbb{K}[x_1, \ldots, x_n] \) is called a
Gröbner-basis (for the ideal generated by F) if
(G1) \( g \in \text{Ideal}(F) \Rightarrow g \Rightarrow_\text{F} 0 \).

\( \Rightarrow_\text{F} \) is a certain reduction relation defined on \( \mathbb{K}[x_1, \ldots, x_n] \) that depends on F. For the detailed definition of \( \Rightarrow_\text{F} \) and for the definition of all auxiliary notions as well as for the motivation for dealing with Gröbner-bases see /2/ and /3/.

The following characterization theorem provides an algorithmic test for the property of being a Gröbner-basis and allows an algorithmic construction of Gröbner-bases C that generate the same polynomial ideal as a given basis F, see /1/. In fact, the Knuth-Bendix algorithm and the extended Knuth-Bendix algorithm /6/ and the above algorithms have a very similar structure (see also section 4.).

Characterization Theorem /2/:

The following statements are equivalent:

(G1) F is a Gröbner-basis

(G2) for \( 1 \leq i \leq \text{length of F} \):
the \( S^2 \)-polynomial of \( F_i \) and \( F_j \) \( \Rightarrow_\text{F} \) 0

(G3) \( \Rightarrow_\text{F} \) is a reduction relation with respect to the reduction relation \( \Rightarrow \) and any \( \Rightarrow_\text{F} \) is used whenever \( \Rightarrow \) is clear from the context.

It should be mentioned that, in /2/,
(G1) has been presented in the equivalent form

\( g \in \text{Ideal}(F) \Rightarrow g = 0 \)

(see (G6) in /2/), and that (G3) is equivalent to:

(G3') \( \Rightarrow_\text{F} \) is a reduction relation with respect to the Church-Rosser property for \( \Rightarrow \).

The equivalence of (G3) and (G3') is a general result on noetherian relations, see /5/. In fact, various other equivalent formulations of (G1) and (G3) may be proven easily.
In /2/ a complex proof is necessary for obtaining \(((G_2) \Rightarrow (G_3))\) and an easier, but still tedious, proof establishes \(((G_3) \Rightarrow (G_1))\), \(((G_1) \Rightarrow (G_2))\) is immediate.

The above-mentioned algorithms are based on \((G_2)\), which only requires to reduce the "S-polynomials" of \(F_i\) and \(F_j\) (a certain type of "least common multiple" of \(F_i\) and \(F_j\)) for finitely many index pairs \((i,j)\) in order to test a given \(F\) for being a Gröbner-basis. The complexity of the algorithms may be drastically decreased, /3/,

by exploiting the following refinement of the characterization theorem:

**Theorem /7/:**

\((G_1)\) is equivalent to

\((G_8)\) for all \(1 \leq i \leq k\) such that \(H(u_1, \ldots, u_k) = H(i,j)\) and for all pairs \((u_n, u_{n+1})\) such that \(n < k\):

the S-polynomial of \(F_n\) and \(F_{n+1}\)

This means that it suffices to test whether all pairs \((i,j)\) may be interconnected by certain "chains" of indices \(u_1, \ldots, u_k\) such that the corresponding S-polynomials \(SP(F_{i}, F_{j})\) reduce to 0.

It is clear that \(((G_2) \Rightarrow (G_8))\). Thus, the interesting implications are \(((G_8) \Rightarrow (G_3))\) and \(((G_3) \Rightarrow (G_1))\).

In Sections 2 and 3 simplified proofs of \(((G_3) \Rightarrow (G_1))\) and \(((G_8) \Rightarrow (G_3))\), respectively, are given.

Proof of \(((G_3) \Rightarrow (G_1))\):

This proof is based on the following new lemma:

**Lemma 1:**

Let \(F\) be an arbitrary sequence of polynomials (not necessarily a Gröbner-basis), then \(\forall F \quad f \equiv_F g \iff f \text{ vvv } g\).

Here \(\equiv_F\) is the congruence relation modulo the ideal generated by \(F\), i.e.

\[ L(F) \]

\[ f \equiv_F g \iff f g + \sum_{i=1}^{\ell} h_i F_i \]

for certain polynomials \(h_i, \ldots, h_{L(F)}\).

L(F) ... length of F.

vvv denotes the reflexive-transitive closure of the reduction relation \(\Rightarrow_F\) and its inverse, i.e.

\[ f \text{ vvv } g \iff \exists h_i, h_k \text{ such that } f = h_i g = h_k \text{ and for some } i < k \text{ or } h_i \Rightarrow_F h_{i+1} \text{ or } h_{i+1} \Rightarrow_F h_i.\]

We again use \(\equiv\) and \(\text{ vvv }\) instead of \(\equiv_F\) and \(\Rightarrow_F\), resp., if \(F\) is clear from the context.

**Lemma 1 so far has escaped our attention, although it turns out to be an easy consequence of property (R1) in /2/.

Lemma 1 establishes an easy connection between those formulations of the concept of Gröbner-basis using the ideal-theoretic notion of congruence and those using the notion of M-reduction.

The reader is advised to carefully examine the definition of the relation \(\Rightarrow_F\) in order to see, why the lemma is non-trivial.

**Proof of Lemma 1:**

\[\equiv: \text{ Immediate from the definitions (see \((E5)\) in /2/).}\]

\[\Rightarrow: \text{ We show by induction on } m \text{ that } f = g \Rightarrow f \text{ vvv } g\text{.}\]

\[f = g + \sum_{j=1}^{m} a_j t_j F_{i_j}\]

\[(a_1, \ldots, a_m) \neq K, t_1, \ldots, t_m \text{ terms}\]

implies \(f \text{ vvv } g\).

From this \(f \equiv g \Rightarrow f \text{ vvv } g\) may be concluded because if \(f \equiv g\), then

\[f = g + \sum_{j=1}^{m} a_j t_j F_{i_j}\]

for certain \(a_1, \ldots, a_m \in K\) and terms \(t_1, \ldots, t_m\).

\[m=1: \text{ Let } f = g + a_1 t_1 F_{i_1}. \text{ It is clear that } a_1 t_1 F_{i_1} \Rightarrow 0\]

(subtract \(a_1 t_1 F_{i_1}\) from \(a_1 t_1 F_{i_1}\):...
In order to make the presentation more readable and to single out the essential points of the simplication a simplified proof of \((G2) \Rightarrow (G3))\) is presented first. Of course, logically, this proof will be superseded by the subsequent proof of \((G8) \Rightarrow (G3))\).

The essential simplification in the proof of \((G2) \Rightarrow (G3))\) consists in the application of a general lemma on noetherian reduction relations (see for instance /5/) showing that a certain "local" Church-Rosser property implies the global Church-Rosser property.

Analogously, a new lemma on arbitrary reduction relations, showing that the Church-Rosser property may be asserted under weaker assumptions, allows a simplification in the structure of the proof of \((G8) \Rightarrow (G3))\). In essence, the new general lemma arises from the lemma in /5/ by a refinement analogous to that by which condition \((G8))\) arises from \((G2))\). Thus, the results presented in this section may also be viewed as a means of exploiting the refined method developed in /7/ and /3/ for polynomial reductions for the case of arbitrary noetherian reduction relations. This method could prove useful, for instance, in various term algebras in which the Knuth-Bendix algorithm is applied.

In order to make the presentation self-contained the following notations and results are resumed from /5/.

Let \(M\) be an arbitrary set and \(\rightarrow\) a reduction relation on \(M\), \(\rightarrow\) denotes the transitive closure of \(\rightarrow\), \(\sim\) denotes the transitive-reflexive closure of \(\rightarrow\).

**Definition:**

\(\sim\) is noetherian iff there is no infinite sequence \(x_1 \rightarrow x_2 \sim \cdots \rightarrow x_n \sim \cdots\)

\(\forall (x) := \exists * \forall \{x \rightarrow z \wedge y \rightarrow z\} \sim x \rightarrow y \sim (\forall \{y \rightarrow z\} \sim y \rightarrow z)\)

\(\sim\) is CR (Church-Rosser) if \(x \sim y \sim x \rightarrow z \sim y \rightarrow z\).

**Definition:**

\(\sim\) is locally CR if \(x \sim y \sim x \rightarrow z \sim y \rightarrow z\).
Lemma 2. /5/: 

A noetherian reduction relation is CR iff it is locally CR.

In the sequel the following elementary properties of polynomial M-reduction are used (Proofs may be found in /2/).

(El) \( f \xrightarrow{\text{term}} g \), \( \text{Hterm}(f) \xrightarrow{\text{t}} \text{Hterm}(g) \)

(\( f \rightarrow g \), \( g \rightarrow h \)).

(E2) \( g \rightarrow h \), \( \bigwedge (\text{Coeff}(t,i)=0 \Rightarrow t \rightarrow \text{Hterm}(g)) \)

(\( f \rightarrow g \)).

(E3) \( f \rightarrow g \rightarrow a \cdot t \rightarrow a \cdot g \) (a=K, t a term).

(E4) \( f \rightarrow g \rightarrow f \rightarrow h \). \( g \rightarrow h \bigwedge \text{succ}(g) \).

(E5) \( f \rightarrow g \rightarrow 0 \Rightarrow f \rightarrow h \).

(E6) \( \Rightarrow \) is a noetherian reduction relation.

Proof. \( \Rightarrow \) (G2) \( \Rightarrow \) (G3) by Lemma 2.

Sketch: We show that if \( \Rightarrow \) satisfies (G2) then \( \Rightarrow \) is locally CR. The assertion then follows from Lemma 2 and (E6).

Details:

Assume (G2), i.e.

(1) \( \bigwedge (f \rightarrow g, f \rightarrow h \Rightarrow g \rightarrow h) \).

Let \( f, g, h, t, s, i, j \) be such that

(iii) \( f \rightarrow g, f \rightarrow h \).

Without loss of generality we may assume \( g \rightarrow s \). We distinguish the cases \( s=t \) and \( s \neq t \).

Case 1: \( s=t \)

There are polynomials \( f_1, f_2, g, h_1 \) such that \( f \rightarrow f_1 + a \cdot t \rightarrow f_2 \) and

(iv) \( t \rightarrow \text{Hterm}(f_2) \).

(v) \( f \rightarrow \text{Hterm}(t) \).

(vi) \( f_1 \rightarrow g \).

(vii) \( f_2 \rightarrow h_1 \).

From (vi), (vii), (El) and (E2) we easily deduce

(viii) \( g \rightarrow f_1 + g_1 + f_2 \).

(ix) \( h \rightarrow f_1 + a \cdot t + h_1 \).

Furthermore (vi), (El), (E2) yield

(x) \( h \rightarrow t + f_1 + g_1 + h_1 \).

(vii) and (E4) imply

(xi) \( g \rightarrow f_1 + g_1 + f_2 \bigwedge g \rightarrow h \).

Thus, from (x) and (xi)

(xii) \( g \rightarrow h \).

Case 2: \( s \neq t \)

Then \( g \) and \( h \) are such that

(xiii) \( g \rightarrow f \cdot \text{Coeff}(t, f) \cdot \text{Hterm}(f_1) \cdot f_1 \).

(xiv) \( h \rightarrow f \cdot \text{Coeff}(t, f) \cdot \text{Hterm}(f_1) \cdot f_1 \).

Let the term \( t' \) be such that

(xv) \( t \rightarrow t' \cdot \text{Lcm(Hterm}(f_1), \text{Hterm}(f_1)) \).

In order to show \( g \rightarrow h \) we observe that

(xvi) \( g \rightarrow h \text{Coeff}(t, f) \cdot \text{Hterm}(f_1) \cdot f_1 \).

From (1), (xvi), and (E3) we deduce

(xvii) \( g \rightarrow h \).

(E5), then, yields the assertion (xvii).

We now present the above-mentioned refinement of Lemma 2.

Definition:

Let \( \rightarrow \) be a reduction relation on M. \( \rightarrow \) is locally pseudo-CR iff

\( \bigwedge (x \rightarrow y, x \rightarrow z \Rightarrow y \rightarrow u_1, u_1 \rightarrow u_n \), \( 1 \leq k < n \).

Lemma 3:

A noetherian reduction relation is CR iff it is locally pseudo-CR.

Proof:

(\( \Rightarrow \)): trivial.

(\( \Leftarrow \)): Assume \( \rightarrow \) is a noetherian reduction relation and is locally pseudo-CR, i.e.
Details:

Assume (G8), i.e.

\[
(i) \quad (x \cdot y, x \cdot z \Rightarrow y \cdot z).
\]

(a variant of the CR property).

We give a proof by noetherian induction:

\[
\text{Induction hypothesis: for a fixed } \hat{x} \cdot \hat{y} \cdot \hat{z}:
\]

\[
(iii) \quad x \cdot y \cdot z \Rightarrow (x \cdot y, x \cdot z \Rightarrow y \cdot z).
\]

We shall show:

\[
(iv) \quad \hat{x} \cdot y \cdot \hat{z} \Rightarrow y \cdot z.
\]

Let \( y, z \) be such that

\[
(v) \quad \hat{x} \cdot y, \hat{x} \cdot \hat{z}.
\]

We distinguish the following cases:

Case 1: \( \hat{x} \cdot y \cdot \hat{z} \): trivial.

Case 2: \( \hat{x} \cdot y, \hat{z} \).

Then there exist \( y_1, z_1 \) such that

\[
(vi) \quad \hat{x} \cdot y_1, \hat{y}, \hat{z}_1.
\]

\[
(vii) \quad \hat{x} \cdot z_1 \cdot \hat{z}.
\]

By applying (i) to \( \hat{x}, y_1, z_1 \) we get

\[
u_1, \ldots, u_n \text{ such that }
\]

\[
(viii) \quad (y_1 \cdot u_1, u_n \cdot z_1) \Rightarrow (x \cdot u_k, u_k \cdot u_{k+1}^\prime).
\]

Now let \( v_1, \ldots, v_{n-1} \) be such that

\[
(ix) \quad (u_k \cdot v_k, u_{k+1} \cdot v_{k+1}).
\]

Then

\[
(x) \quad (u_1 \cdot y, u_1 \cdot v_1).
\]

\[
(xi) \quad (v_1 \cdot u_k, u_{k+1} \cdot v_{k+1}).
\]

\[
(xii) \quad u_n \cdot v_{n-1}, u_n \cdot z.
\]

From (x), (xi), (xii) and induction hypothesis (iii) we obtain (using (viii)) \( y \cdot u_1 = \ldots = v_{n-1} \cdot z \), which concludes the proof.

\[
\text{Proof of } (G8) \Rightarrow (G3) \text{ by applying Lemma 3:}
\]

Sketch:

We show that \( \Rightarrow \) satisfies (G8) then \( \Rightarrow \) is locally pseudo-CR. The assertion then follows from Lemma 3 and (E6).
We know from (i), (ii), (vii) and (viii):
\[ q = u_k f_n, f_{u_k} = 1 \]

We shall show success for:
\[ \begin{array}{c}
1 \leq k < n \\
\end{array} \]
\[ f_{u_k} \lor f_{u_{k+1}} \]

An easy calculation shows (see also the above proof of \((G2) \Rightarrow (G3)\))
\[ \begin{array}{c}
1 \leq k < n \\
\end{array} \]
\[ b_k t_k f_{u_k+1} = f_{u_k} + f_{u_{k+1}} \]
\[ -(f_1 g_{u_k+1} + f_2) = g_{u_k+1} - g_{u_k} = b_k t_k \cdot \text{SP}(F_{u_k}, F_{u_{k+1}}) \]

Therefore by (E3) and (i)
\[ \begin{array}{c}
1 \leq k < n \\
\end{array} \]
\[ f_{u_k+1} - f_{u_k} > 0 \]

Thus, by (E5), we obtain (ix) which completes the proof.

4. Conclusion

Loos \cite{Loos87} conjectures that the algorithm in \cite{Loos87} based on (G2) (not the refinement in \cite{Loos87} based on (G8)) may be viewed as a special case of the Knuth-Bendix algorithm by a suitable interpretation of the notion of term in the Knuth-Bendix algorithm. We remark that (G2) may be replaced by

\[ \begin{array}{c}
1 \leq i < j \leq \text{L}(F) \\
\end{array} \]
\[ g_{i,j} \lor g_{i,j}' \]
\[ g_{i,j} = H_p(i,j) - H_p(i) \cdot P_1, \]
\[ g_{i,j}' = H_p(i,j) - H_p(j) \cdot P_1 \]

(see \cite{Loos87}). This means that the check, whether the difference of \[ g_{i,j} \] and \[ g_{i,j}' \] (=S-polynomial of \( F_1 \) and \( F_2 \)) may be replaced by the check, whether \[ g_{i,j} \] and \[ g_{i,j}' \] have a common successor.

This observation and the deduction of both the Knuth-Bendix criterion and our (G2') from the same Lemma 2 (compare the presentation in \cite{Loos87}) shows that the conjecture is reasonable.

It should be also noted that property (E5) (=R3) in \cite{Loos87} can be eliminated from the proof of \((G2') \Rightarrow (G3)\), such that (E4) (=R1) in \cite{Loos87} seems to be the property of \( \ast \) that is central to the above proofs. This gives some hints how to characterize those domains to which our own algorithm may be generalized.

References

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