Numbers and Such

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Overview

- The Natural Numbers
- Minimum and Maximum
- Sum and Product
- Binomials
- Matrix Operations

The Natural Numbers

Natural Numbers

- The numbers of counting distinct objects.

  no object, one object, two objects, . . .

- Axiomatic characterization.

  1. Describe properties of natural numbers.
  2. Peano axioms.

- Set-theoretic construction.

  1. Numbers are defined as sets.
  2. Definition satisfies Peano laws.

Two ways to introduce the natural numbers.

Peano Arithmetic

- Theory of natural numbers.
  - Object constant 0 (zero).
  - Unary function constant ' (successor).
- Axioms
  1. 0 is not the successor of any natural number:
  $$\forall x : x' \neq 0.$$
  2. Different natural numbers have different successors:
  $$\forall x, y : x' = y' \Rightarrow x = y.$$
  3. F holds for every number, if F holds for 0 and with every number also for its successor:
  $$(F[x \leftarrow 0] \wedge (\forall x : F \Rightarrow F[x \leftarrow x + 1])) \Rightarrow \forall x : F.$$

Illustration

The natural numbers are a single infinite chain

$$0 \longrightarrow 0' \longrightarrow 0'' \longrightarrow 0''' \longrightarrow \ldots$$

1. Chain starts with 0.
2. Every application of ' yields a new natural number.
3. Every natural number is captured by the chain.

Construction from Sets

$$0 := \emptyset;$$
$$x' := x \cup \{x\}.$$

Proof of first Peano law:

We prove $\forall x : x' \neq 0$. Take arbitrary x. By definition of 0 and $'$, we have to prove

$$x \cup \{x\} \neq \emptyset$$

which is true because $x \in (x \cup \{x\})$ but $x \notin \emptyset$.

Proof of second Peano law: see lecture notes.

Set of Natural Numbers

Definition: $\mathbb{N}$, the set of natural numbers

(omitted)

Proposition: $\mathbb{N}$ is the smallest set that satisfies the properties:

$$
\begin{array}{l}
0 \in \mathbb{N}; \\
\forall x \in \mathbb{N} : x' \in \mathbb{N}; \\
\forall F : \\
\quad (F(0) \wedge \forall x \in \mathbb{N} : F(x) \Rightarrow F(x+1)) \Rightarrow \forall x \in \mathbb{N} : F(x).
\end{array}
$$

Peano laws are a consequence of this proposition.

Auxiliary Notions

All further definitions work for both constructions of the naturals.

- Subsets of the natural numbers:

$$\begin{aligned} \mathbb{N}_{>0} &:= \{x \in \mathbb{N} : x > 0\}; \\ \mathbb{N}_n &:= \{x \in \mathbb{N} : x < n\}. \end{aligned}$$

  e.g. $\mathbb{N}_3 = \{0, 1, 2\}$.

- Predecessor function:

$$x^- := \textbf{such } y : x = y'.$$

  e.g. $3^- = 2;\ 0^- = ?$

Natural Number Arithmetic

Constants

$$1 := 0',\ 2 := 1';$$

Addition

$$x + y := \textbf{if } y = 0 \textbf{ then } x \textbf{ else } (x + y^-)'$$

Multiplication

$$x * y := \textbf{if } y = 0 \textbf{ then } 0 \textbf{ else } x + (x * y^-)$$

Total Order

$$\begin{aligned} & x \leq y :\Leftrightarrow \\ & \quad \textbf{if } x = 0 \textbf{ then } \mathrm{T} \\ & \quad \textbf{else if } y = 0 \textbf{ then } \mathrm{F} \\ & \quad \textbf{else } x^- \leq y^- \end{aligned}$$

Termination function $r(x, y) := y$ for recursive definitions.

Example

$$
\begin{aligned}
1+2 &= \text{(definitions of 1 and 2)} \\
0'+0'' &= \text{(definition of +)} \\
(\textbf{if } 0''=0 \textbf{ then } 0' \textbf{ else } (0'+0''^{-})') &= \text{(second Peano law)} \\
(0'+0''^{-})' &= \text{(definition of } {}^{-}\text{)} \\
(0'+0')' &= \text{(definition of +)} \\
(\textbf{if } 0'=0 \textbf{ then } 0' \textbf{ else } (0'+0'^{-})')' &= \text{(second Peano law)} \\
(0'+0'^{-})'' &= \text{(definition of } {}^{-}\text{)} \\
(0'+0)'' &= \text{(definition of +)} \\
(\textbf{if } 0=0 \textbf{ then } 0' \textbf{ else } (0'+0^{-})')'' &= \text{(reflexivity of } =\text{)} \\
0''' &= \text{(definition of 3)} \\
3. &
\end{aligned}
$$

Example

$$3 := 2'; 4 := 3'; 5 := 4'$$

$$\begin{aligned}
&3 < 5 :\Leftrightarrow \\
&\quad 3^- < 5^- \Leftrightarrow 2 < 4 \Leftrightarrow \\
&\quad 2^- < 4^- \Leftrightarrow 1 < 3 \Leftrightarrow \\
&\quad 1^- < 3^- \Leftrightarrow 0 < 2 \Leftrightarrow \\
&\quad \mathrm{T}
\end{aligned}$$

Recursive unfolding of definitions.

Natural Number Laws

For all natural numbers x and y, we have:

Addition

$$\begin{aligned} x + 0 &= x, \\ x + y' &= (x + y)'; \end{aligned}$$

Multiplication

$$\begin{aligned} x * 0 &= 0, \\ x * y' &= x + (x * y); \end{aligned}$$

Total Order

$$\begin{aligned} 0 \leq x &\Leftrightarrow \mathrm{T}, \\ x \leq 0 &\Leftrightarrow x = 0, \\ x' \leq y' &\Leftrightarrow x \leq y. \end{aligned}$$

Natural Number Laws

For all natural numbers x, y, z, we have:

$$\begin{aligned}
x + 0 &= x, \\
x * 1 &= x, \\
x + y &= y + x, \\
x * y &= y * x, \\
x + (y + z) &= (x + y) + z, \\
x * (y * z) &= (x * y) * z, \\
x * (y + z) &= (x * y) + (x * z), \\
x &\leq x, \\
(x \leq y \wedge y \leq x) &\Rightarrow x = y, \\
(x \leq y \wedge y \leq z) &\Rightarrow x \leq z.
\end{aligned}$$

Order Predicates

In every domain with a binary relation $\leq$:

$$x < y :\Leftrightarrow x \leq y \wedge x \neq y;$$
$$x > y :\Leftrightarrow x \not\leq y;$$
$$x \geq y :\Leftrightarrow x \not< y.$$

We often write $a \leq x < b$ to denote $a \leq x \wedge x < b$ and similar for all other combinations of the order predicates.

Logic Evaluator

```
pred N(x) <=> Nat(x);

fun N0 = 0;
fun '(x: N) = +(x, 1);
fun ^-(x: N) = such(n in nat(0, x): =(x, '(n)), n);

fun N1 = '(N0);
fun N2 = '(N1);

fun +N(x: N, y: N) recursive y =
  if(=(y, N0), x, '(+N(x, ^-(y))));
fun *N(x: N, y: N) recursive y =
  if(=(y, N0), N0, +N(x, *N(x, ^-(y))));
pred <=N(x: N, y: N) recursive y <=>
  if(=(x, N0), true, if(=(y, N0), false, <=N(^-(x), ^-(y))));
```

Difference

Definition: z is a difference of x and y if $x = y + z$.

$$x - y := \textbf{such } z : x = z + y.$$

- Difference is not defined for every x and y:

  There is no z with $1 = z + 2$, thus $1 - 2$ is undefined.

- If a difference exists, it is unique:

$$\forall x, y, z_0, z_1 : (x = z_0 + y \wedge x = z_1 + y) \Rightarrow z_0 = z_1.$$

- If $x \geq y$, the difference of x and y is defined:

$$\forall x, y : x \geq y \Rightarrow x = (x - y) + y.$$

Quotient and Remainder

Definition: quotient and remainder

$$x \text{ div } y := \textbf{such } q : \exists r : r < y \land x = (q * y) + r;$$
$$x \text{ mod } y := \textbf{such } r : \exists q : r < y \land x = (q * y) + r.$$

Examples:

- $5 \text{ div } 3 = 1$, $5 \text{ mod } 3 = 2$.
- $15 \text{ div } 6 = 2$, $15 \text{ mod } 6 = 3$.
- $1 \text{ div } 3 = 0$, $1 \text{ mod } 3 = 1$.
- $0 \text{ div } 3 = 0$, $0 \text{ mod } 3 = 0$.

Properties of Quotient and Remainder

- Quotient and remainder are not defined for every x and y:

  $(x \text{ div } 0)$ and $(x \text{ mod } 0)$ are undefined for every x.

- If quotient respectively remainder exist, they are unique.

$$\forall x, y, q_0, q_1, r_0 < y, r_1 < y :$$
$$(x = (q_0 * y) + r_0 \wedge x = (q_1 * y) + r_1) \Rightarrow (q_0 = q_1 \wedge r_0 = r_1).$$

- If the divisor is not 0, quotient and remainder exist:

$$\forall x, y \neq 0 : (\exists q, r : r < y \wedge x = (q * y) + r).$$

- We thus have the following relationship:

$$\forall x, y \neq 0 : x = (x \text{ div } y) * y + (x \text{ mod } y).$$

Exponentiation

$$\cdot^\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$x^n := \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } x * x^{n^-}.$$

Termination function: $r(x, n) := n$

Example:

$\underline{5^3} = 5*(\underline{5^2}) = 5*(5*(\underline{5^1})) = 5*(5*(5*(\underline{5^0}))) = 5*(5*(5*(1))) =$
$5 * 5 * 5.$

More Notions

- x divides y if $x * z = y$ for some z:

$$x|y :\Leftrightarrow \exists z : x * z = y.$$

- The greatest common divisor of x and y is the largest number that divides both x and y:

$$\gcd(x, y) := \textbf{such } z : z|x \wedge z|y \wedge (\forall w : (w|x \wedge w|y) \Rightarrow w \leq z).$$

- The least common multiple of x and y is the smallest number that both x and y divide:

$$\mathrm{lcm}(x, y) := \textbf{such } z : x|z \wedge y|z \wedge (\forall w : (x|w \wedge y|w) \Rightarrow z \leq w).$$

Examples

- $1|18$, $2|18$, $3|18$, $6|18$, $9|18$, $18|18$.
- $1|24$, $2|24$, $3|24$, $4|24$, $6|24$, $8|24$, $12|24$, $24|24$.
- $\gcd(18, 24) = 6$.
- $\gcd(16, 27) = 1$.
- $\mathrm{lcm}(4, 6) = 12$.
- $\mathrm{lcm}(8, 12) = 24$.

More Notions

- Two numbers are relatively prime if their gcd is 1:

$$x \text{ and } y \text{ are relatively prime} :\Leftrightarrow \gcd(x, y) = 1.$$

- A number greater than 1 is prime if its only divisors are 1 and itself:

$$x \text{ is prime } :\Leftrightarrow x > 1 \wedge (\forall y : y|x \Rightarrow (y = 1 \vee y = x)).$$

- 16 and 27 are relatively prime.
- (Only) the underlined numbers are prime:

$$0, 1, \underline{2}, \underline{3}, 4, \underline{5}, 6, \underline{7}, 8, 9, 10, \underline{11}, 12, \underline{13}, 14, 15, 16, \underline{17}, \ldots$$

Logic Evaluator

```
pred divides(x, y) <=> exists(z in nat(N0, y): =(*N(x, z), y));

fun gcd(x, y) =
  let(m = if(=(x, N0), y, x):
    such(z in nat(N0, m):
      and(divides(z, x), divides(z, y),
        forall(w in nat(+N(z, N1), m):
          or(not(divides(w, x)), not(divides(w, y))))),
      z));

pred isprime(x) <=>
  and(not(<=N(x, N1)),
    forall(y in nat(N0, x):
      implies(divides(y, x), or(=(y, N1), =(y, x)))));
```

Minimum and Maximum

Minimum and Maximum Quantifier

Definition: If x is a variable and F is a formula, then the following are terms with bound variable x:

$$\min_x F;$$
$$\max_x F.$$

The value of the first term is the smallest value of x such that F holds; the value of the second term is the largest such value:

$$\min_x F := \textbf{such } x : F \wedge (\forall y : F[x \leftarrow y] \Rightarrow x \leq y);$$
$$\max_x F := \textbf{such } x : F \wedge (\forall y : F[x \leftarrow y] \Rightarrow x \geq y).$$

Quantifiers for every domain with a binary predicate $\leq$.

Minimum and Maximum Function

$$\begin{aligned}\min(S) &:= \min_x\ x \in S;\\ \max(S) &:= \max_x\ x \in S;\end{aligned}$$

Examples:

- We have

$$\max_x(\text{isprime}(x) \wedge x|100) = 5.$$

- The value of

$$\min(\{1/x : x \in \mathbb{N}_{>0}\})$$

  is undefined, because for every $z{=}1/x$ in $\{1/1, 1/2, 1/3, 1/4, \ldots\}$ there is always a y in this set with $y < z$, namely $1/(x+1)$.

Sum and Product

Sum Quantifier

Definition: If x is a variable, F is a formula and T is a term, then the following is a term with bound variable x:

$$\sum_{x,F} T.$$

The value of this term is 0, if F does not hold for any x; otherwise it is, for every x that satisfies F, the sum of the value of T and of the value of the term for all other x:

$$\begin{aligned}(\forall x : \neg F) &\Rightarrow \textstyle\sum_{x,F} T = 0;\\ (\forall y : F[x \leftarrow y] &\Rightarrow \textstyle\sum_{x,F} T = T[x \leftarrow y] + \sum_{x,F \wedge x \neq y} T).\end{aligned}$$

Examples

$$\sum_{1 \leq i \leq n} i^2 = \sum_{i, (i \in \mathbb{N} \wedge 1 \leq i \wedge i \leq n)} i^2;$$

$$\sum_{1 \leq i \leq 0} i^2 = 0;$$

$$\sum_{1 \leq i \leq 5} i^2 = 1^2 + \sum_{2 \leq i \leq 5} i^2;$$

$$\sum_{1 \leq i \leq 5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2.$$

Examples

$$\sum_{1\leq i\leq 9} (x-i)^2 = (x-1)^2 + \sum_{2\leq i\leq 9} (x-i)^2;$$

$$\sum_{1\leq i\leq n} x^i = \sum_{1\leq i\leq n \wedge \mathrm{iseven}(i)} x^i + \sum_{1\leq i\leq n \wedge \mathrm{isodd}(i)} x^i.$$

Identities which are true for every x.

Example: Decimal Number Representation

Let $a := [3, 1, 2, 9, 0, 7]$. We have

$$\sum_{0 \leq i \leq 5} a_i * 10^i = 709213.$$

In general, for any finite sequence d of "decimal digits" the term

$$\sum_{0 \leq i < \text{length}(d)} d_i * 10^i$$

denotes the value of this sequence in the decimal number system.

Example: Binary Number Representation

Likewise, for any finite sequence b of binary digits 0 and 1, the value

$$\sum_{0 \leq i < \mathrm{length}(b)} b_i * 2^i$$

denotes the value of this sequence in the binary number system, e.g., the value of $[0, 1, 1, 0, 1]$ is

$$1 * 2^4 + 0 * 2^3 + 1 * 2^2 + 1 * 2^1 + 0 * 2^0 = 22.$$

Generalization to any number base.

Multiple Variable Bindings

$$\sum_{1 \leq i \leq 5, 1 \leq j \leq 3} i * j = 1 * 1 + 1 * 2 + 1 * 3 + \sum_{2 \leq i \leq 5, 1 \leq j \leq 3} i * j;$$

$$\sum_{1 \leq i \leq 3, 1 \leq j \leq i} i * j = 1 * 1 + 2 * 1 + 2 * 2 + 3 * 1 + 3 * 2 + 3 * 3.$$

Bound variables sometimes have to be deduced from context.

Sum Identities

For all vars i and j, and formulas F (in which j does not occur freely), G (in which i does not occur freely) and H, and terms T and U:

$$\sum_{i,F} T * \sum_{j,G} U = \sum_{i,F} \sum_{j,G} T * U;$$

$$\sum_{i,F} \sum_{j,G} T = \sum_{j,G} \sum_{i,F} T = \sum_{i,j,F \wedge G} T;$$

$$\sum_{i,F} T + \sum_{i,H} T = \sum_{i,F \vee H} T + \sum_{i,F \wedge H} T.$$

Sum Identities

Furthermore, if C is a term in which i does not occur freely:

$$\sum_{i,F} C * T = C * \sum_{i,F} T;$$

$$\sum_{i,F} C = n * C$$

(where n is the number of i's for which F holds).

Examples

$$\sum_{1 \le i \le n} x^i * \sum_{1 \le j \le m} x^j = \sum_{1 \le i \le n} \sum_{1 \le j \le m} x^{i+j} = \sum_{1 \le i \le n \wedge 1 \le j \le m} x^{i+j};$$

$$\sum_{1 \le i \le n} \sum_{1 \le j \le m} i * x^j = \sum_{1 \le i \le n} (i * \sum_{1 \le j \le m} x^j).$$

Many more identities can be deduced from basic definition.

Product Quantifier

If x is a variable, F is a formula and T is a term, then the following is a term with bound variable x:

$$\prod_{x,F} T.$$

The value of this term is 1, if F does not hold for any x; otherwise it is, for every x that satisfies F, the product of the value of T and of the value of the term for all other x:

$$(\forall x : \neg F) \Rightarrow \textstyle\prod_{x,F} T = 1;$$
$$(\forall y : F[x \leftarrow y] \Rightarrow \textstyle\prod_{x,F} T = T[x \leftarrow y] * \prod_{x,F \wedge x \neq y} T).$$

Example

$$\prod_{1 \le i \le n} i^2 = \prod_{i,(i \in \mathbb{N} \wedge 1 \le i \wedge i \le n)} i^2;$$

$$\prod_{1 \le i \le 0} i^2 = 1$$

$$\prod_{1 \le i \le 5} i^2 = 1^2 * \prod_{2 \le i \le 5} i^2;$$

$$\prod_{1 \le i \le 5} i^2 = 1^2 * 2^2 * 3^2 * 4^2 * 5^2.$$

Product Identities

For all vars i and j, and formulas F (in which j does not occur freely), G (in which i does not occur freely) and H, and terms T and U:

$$\prod_{i,F}\prod_{j,G} T = \prod_{j,G}\prod_{i,F} T = \prod_{i,j,F\wedge G} T;$$

$$\prod_{i,F} T * \prod_{i,H} T = \prod_{i,F\vee H} T * \prod_{i,F\wedge H} T.$$

Product Identities

Furthermore, if C is a term in which i does not occur freely:

$$\prod_{i,F} C * T = C^n \prod_{i,F} T;$$

$$\prod_{i,F} C = C^n$$

(where n is the number of i's for which F holds).

Prime Number Factorization

For every natural number $n \neq 0$, there exists a unique prime number factorization:

$$\forall n \in \mathbb{N}_{>0} : (\exists p : \mathrm{pf}(n,p) \wedge (\forall q : \mathrm{pf}(n,q) \Rightarrow p = q)).$$

A prime number factorization of n is a sequence of pairs (p, e) ordered by the primes p such that n is the product of all p^e.

$$\begin{aligned}
&\mathrm{pf}(n,p) :\Leftrightarrow \\
&\quad p : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \wedge \\
&\quad (\forall i \in \mathbb{N} : p(i)_0 \text{ is prime } \wedge p(i)_0 < p(i+1)_0) \wedge \\
&\quad (\exists k \in \mathbb{N} : n = \textstyle\prod_{0 \le i \le k} p(i)_0{}^{p(i)_1} \wedge \forall i > k : p(i)_1 = 0).
\end{aligned}$$

Example

Because $300 = 2^2 * 3^1 * 5^2$, we have for 300 the prime number factorization

$$[\langle 2,2\rangle, \langle 3,1\rangle, \langle 5,2\rangle, \langle 7,0\rangle, \langle 11,0\rangle, \langle 13,0\rangle, \ldots].$$

Likewise, 1 has the prime number factorization

$$[\langle 2,0\rangle, \langle 3,0\rangle, \langle 5,0\rangle, \langle 7,0\rangle, \langle 11,0\rangle, \langle 13,0\rangle, \ldots].$$

Binomials

Factorial

Definition: The factorial of a natural number n is the product of all non-zero numbers less than or equal to n:

$$n! := \prod_{1 \leq i \leq n} i.$$

Handy notation for a particular product.

Binomial

Definition: The binomial coefficient (Binomialkoeffizient) "n choose k" of two natural numbers n and k:

$$\binom{n}{k} := \textbf{if } 0 \leq k \leq n \textbf{ then } \frac{n!}{k! * (n-k)!} \textbf{ else } 0.$$

Proposition: We have for every n and k

$$\binom{n}{k} = \frac{\prod_{n-k+1 \leq i \leq n} i}{\prod_{1 \leq i \leq k} i}.$$

Important notion in combinatorics (the math of "counting things").

Motivation

$\binom{n}{k}$ is the number of ways to choose a k-element set from an n-element set.

Example:

The set $\{0, 1, 2, 3\}$ has $6 = \binom{4}{2}$ subsets with 2 elements:

$$\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}.$$

Binomial Identities

For every $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $0 \leq k \leq n$:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1},$$

$$\binom{n}{k} = \binom{n}{n-k},$$

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Pascal's Triangle

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
=
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

....

Construction

This triangle is bounded by sides of length 1 and where every interior element is the sum of both parents:

$$\begin{array}{c} \binom{n}{k} \quad \binom{n}{k+1} \\ \cdots\cdots \binom{n+1}{k+1} \cdots\cdots \end{array}$$

Quick construction of binomial values.

Matrix Operations

Matrices

Matrices over the real numbers.

The domain of real matrices of dimension $m \times n$ is defined as follows:

$$\mathbb{M}_{m,n} := \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{R}.$$

Set of matrices of the same dimensions.

Matrix Operations

Null Matrix

$$\begin{aligned} &0 : \mathbb{M}_{m,n}, \\ &0_{i,j} := 0. \end{aligned}$$

Unity Matrix

$$\begin{aligned} &1 : \mathbb{M}_{n,n}, \\ &1_{i,j} := \textbf{if } i = j \textbf{ then } 1 \textbf{ else } 0. \end{aligned}$$

Addition

$$\begin{aligned} &+ : \mathbb{M}_{m,n} \times \mathbb{M}_{m,n} \to \mathbb{M}_{m,n} \\ &A + B := \textbf{ such } C \in \mathbb{M}_{m,n} : \\ &\quad \forall i \in \mathbb{N}_m, j \in \mathbb{N}_n : C_{i,j} = A_{i,j} + B_{i,j}. \end{aligned}$$

Matrix Operations

Scalar Product

$$
\begin{aligned}
&* : \mathbb{R} \times \mathbb{M}_{m,n} \to \mathbb{M}_{m,n} \\
&c * A := \textbf{ such } C \in \mathbb{M}_{m,n} : \\
&\quad \forall i \in \mathbb{N}_m, j \in \mathbb{N}_n : C_{i,j} = c * A_{i,j}.
\end{aligned}
$$

short: $(c * A)_{i,j} := c * A_{i,j}$.

Matrix Product

$$
\begin{aligned}
&* : \mathbb{M}_{m,n} \times \mathbb{M}_{n,p} \to \mathbb{M}_{m,p} \\
&A * B := \textbf{ such } C \in \mathbb{M}_{m,p} : \\
&\quad \forall i \in \mathbb{N}_m, j \in \mathbb{N}_p : C_{i,j} = \sum_{0 \le k < n} A_{i,k} * B_{k,j}.
\end{aligned}
$$

short: $(A * B)_{i,j} := \sum_{0 \le k < n} A_{i,k} * B_{k,j}$.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 17 & 19 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 14 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1a + 2c + 3e & 1b + 2d + 3f \\ 4a + 5c + 6e & 4b + 5d + 6f \end{bmatrix}$$

Matrix multiplication is not commutative.

Matrix Operations

Determinant

$$|\ .\ | : \mathbb{M}_{n,n} \rightarrow \mathbb{R},$$

$$\begin{aligned}
&\text{if } n = 1 : \\
&\quad |A| := A_{0,0}, \\
\\
&\text{if } n > 1 : \\
&\quad |A| := \textstyle\sum_{0 \leq j < n} A_{0,j} * (-1)^j * |B| \\
&\qquad \textbf{where } B = \textbf{ such } B \in \mathbb{M}_{n-1,n-1} : \\
&\qquad\quad \forall k \in \mathbb{N}_{n-1}, l \in \mathbb{N}_{n-1} : \\
&\qquad\qquad B_{k,l} = (\textbf{if } l < j \textbf{ then } A_{k+1,l} \textbf{ else } A_{k+1,l+1}).
\end{aligned}$$

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 * \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 * \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 * \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 * (-3) - 2 * (-6) + 3 * (-3) = 0;$$

$$\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 5 * 9 - 6 * 8 = -3;$$

$$\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 4 * 9 - 6 * 7 = -6;$$

$$\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 4 * 8 - 5 * 7 = -3.$$