Hoare Calculus and Predicate Transformers

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1. The Hoare Calculus for Non-Loop Programs

2. Predicate Transformers

3. Partial Correctness of Loop Programs

4. Total Correctness of Loop Programs

5. Abortion

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The Hoare Calculus

Calculus for reasoning about imperative programs.

- **“Hoare triple”**: \( \{ P \} \ c \ \{ Q \} \)
  - Logical propositions \( P \) and \( Q \), program command \( c \).
  - The Hoare triple is itself a logical proposition.
  - The Hoare calculus gives rules for constructing true Hoare triples.

- **Partial correctness** interpretation of \( \{ P \} \ c \ \{ Q \} \):
  - “If \( c \) is executed in a state in which \( P \) holds, then it terminates in a state in which \( Q \) holds unless it aborts or runs forever.”
  - Program does not produce wrong result.
  - But program also need not produce any result.
    - Abortion and non-termination are not ruled out.

- **Total correctness** interpretation of \( \{ P \} \ c \ \{ Q \} \):
  - “If \( c \) is executed in a state in which \( P \) holds, then it terminates in a state in which \( Q \) holds.
  - Program produces the correct result.

We will use the partial correctness interpretation for the moment.
General Rules

\[ P \Rightarrow Q \quad \frac{\{P\}}{\{Q\}} \quad P \Rightarrow P' \quad \{P'\} \ c \ {\{Q'\}} \quad Q' \Rightarrow Q \quad \frac{\{P\} \ c \ {\{Q\}}}{\} \]

- **Logical derivation:** \[ \frac{A_1 \ A_2}{B} \]
  - **Forward:** If we have shown \( A_1 \) and \( A_2 \), then we have also shown \( B \).
  - **Backward:** To show \( B \), it suffices to show \( A_1 \) and \( A_2 \).

- **Interpretation of above sentences:**
  - To show that, if \( P \) holds in a state, then \( Q \) holds in the same state (no command is executed), it suffices to show \( P \) implies \( Q \).
    - Hoare triples are ultimately reduced to classical logic.
  - To show that, if \( P \) holds, then \( Q \) holds after executing \( c \), it suffices to show this for a \( P' \) weaker than \( P \) and a \( Q' \) stronger than \( Q \).
    - Precondition may be weakened, postcondition may be strengthened.
Special Commands

Commands modeling “emptiness” and abortion.

\[
\{P\} \textbf{skip} \{P\} \quad \{\text{true}\} \textbf{abort} \{\text{false}\}
\]

- The \textbf{skip} command does not change the state; if \(P\) holds before its execution, then \(P\) thus holds afterwards as well.
- The \textbf{abort} command aborts execution and thus trivially satisfies partial correctness.
  - Axiom implies \(\{P\} \textbf{abort} \{Q\}\) for arbitrary \(P, Q\).

Useful commands for reasoning and program transformations.
Scalar Assignments

\[ \{Q[e/x]\} \ x := e \ \{Q\} \]

- **Syntax**
  - Variable \(x\), expression \(e\).
  - \(Q[e/x] \ldots Q\) where every free occurrence of \(x\) is replaced by \(e\).

- **Interpretation**
  - To make sure that \(Q\) holds for \(x\) after the assignment of \(e\) to \(x\), it suffices to make sure that \(Q\) holds for \(e\) before the assignment.

- **Partial correctness**
  - Evaluation of \(e\) may abort.

\[
\begin{align*}
\{x + 3 < 5\} & \quad x := x + 3 \quad \{x < 5\} \\
\{x < 2\} & \quad x := x + 3 \quad \{x < 5\}
\end{align*}
\]
Array Assignments

\[
\{Q[a[i \mapsto e]/a]\} \quad a[i] := e \quad \{Q\}
\]

- An array is modelled as a function \( a : I \rightarrow V \)
  - Index set \( I \), value set \( V \).
  - \( a[i] = e \ldots \) a holds at index \( i \) the value \( e \).
- Updated array \( a[i \mapsto e] \)
  - Array that is constructed from \( a \) by mapping index \( i \) to value \( e \).
  - Axioms (for all \( a : I \rightarrow V, i \in I, j \in I, e \in V \)):
    \[
    i = j \Rightarrow a[i \mapsto e][j] = e
    \]
    \[
    i \neq j \Rightarrow a[i \mapsto e][j] = a[j]
    \]
    - Index violations and pointer semantics of arrays not yet considered.

\[
\begin{align*}
\{&a[i \mapsto x][1] > 0\} & a[i] := x & \{a[1] > 0\} \\
\{(i = 1 \Rightarrow x > 0) \land (i \neq 1 \Rightarrow a[1] > 0)\} & a[i] := x & \{a[1] > 0\}
\end{align*}
\]
Command Sequences

\[
\begin{align*}
\{P\} & \quad c_1 \quad \{R_1\} \quad R_1 \Rightarrow R_2 \quad \{R_2\} \quad c_2 \quad \{Q\} \\
\{P\} & \quad c_1; \quad c_2 \quad \{Q\}
\end{align*}
\]

■ Interpretation

■ To show that, if \( P \) holds before the execution of \( c_1; c_2 \), then \( Q \) holds afterwards, it suffices to show for some \( R_1 \) and \( R_2 \) with \( R_1 \Rightarrow R_2 \) that
  ■ if \( P \) holds before \( c_1 \), that \( R_1 \) holds afterwards, and that
  ■ if \( R_2 \) holds before \( c_2 \), then \( Q \) holds afterwards.

■ Problem: find suitable \( R_1 \) and \( R_2 \)
  ■ Easy in many cases (see later).

\[
\begin{align*}
\{x + y - 1 > 0\} & \quad y := y - 1 \quad \{x + y > 0\} \quad \{x + y > 0\} \quad x := x + y \quad \{x > 0\} \\
\{x + y - 1 > 0\} & \quad y := y - 1; \quad x := x + y \quad \{x > 0\}
\end{align*}
\]
Conditionals

\[
\begin{align*}
\{P \land b\} & \quad c_1 \quad \{Q\} \quad \{P \land \neg b\} & \quad c_2 \quad \{Q\} \\
\{P\} & \quad \text{if } b \text{ then } c_1 \quad \text{else } c_2 \quad \{Q\}
\end{align*}
\]

\[
\begin{align*}
\{P \land b\} & \quad c \quad \{Q\} \quad (P \land \neg b) \implies Q \\
\{P\} & \quad \text{if } b \text{ then } c \quad \{Q\}
\end{align*}
\]

**Interpretation**

- To show that, if \( P \) holds before the execution of the conditional, then \( Q \) holds afterwards,
- it suffices to show that the same is true for each conditional branch, under the additional assumption that this branch is executed.

\[
\begin{align*}
\{x \neq 0 \land x \geq 0\} \quad & y := x \quad \{y > 0\} \quad \{x \neq 0 \land x \not\geq 0\} \quad & y := -x \quad \{y > 0\} \\
\{x \neq 0\} \quad & \text{if } x \geq 0 \text{ then } y := x \quad \text{else } y := -x \quad \{y > 0\}
\end{align*}
\]
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Backward Reasoning

Implication of rule for command sequences and rule for assignments:

\[
\begin{align*}
\{P\} & \quad c \quad \{Q[e/x]\} \\
\{P\} & \quad c ; \quad x := e \quad \{Q\}
\end{align*}
\]

- **Interpretation**
  - If the last command of a sequence is an assignment, we can remove the assignment from the proof obligation.
  - By multiple application, assignment sequences can be removed from the back to the front.

\[
\begin{align*}
\{P\} & \quad x := x+1; \quad y := 2*x; \quad z := x+y \quad \{z = 15\} \\
\{P\} & \quad x := x+1; \quad y := 2*x; \quad \{x + y = 15\} \\
\{P\} & \quad x := x+1; \quad \{x + 2x = 15\} \quad \{x + 1 = 5\} \quad (\Leftrightarrow x = 4) \\
\{P\} & \quad \{x + 1 = 5\} \quad \{x = 5\} \quad (\Leftrightarrow x = 4) \\
\{P\} & \quad \{x = 4\} \quad (\Leftrightarrow 3x = 15) \\
\{P\} & \quad \{x = 4\} \quad (\Leftrightarrow x = 5) \\
P & \Rightarrow x = 4
\end{align*}
\]
Weakest Preconditions

A calculus for “backward reasoning”.

- **Predicate transformer wp**
  - Function “wp” that takes a command \( c \) and a postcondition \( Q \) and returns a precondition.
  - Read \( \text{wp}(c, Q) \) as “the weakest precondition of \( c \) w.r.t. \( Q \)”.
- \( \text{wp}(c, Q) \) is a precondition for \( c \) that ensures \( Q \) as a postcondition.
  - Must satisfy \( \{ \text{wp}(c, Q) \} \ c \ {Q} \).
- \( \text{wp}(c, Q) \) is the weakest such precondition.
  - Take any \( P \) such that \( \{ P \} \ c \ {Q} \).
  - Then \( P \Rightarrow \text{wp}(P, Q) \).
- Consequence: \( \{ P \} \ c \ {Q} \) iff \( (P \Rightarrow \text{wp}(c, Q)) \)
  - We want to prove \( \{ P \} \ c \ {Q} \).
  - We may prove \( P \Rightarrow \text{wp}(c, Q) \) instead.

Verification is reduced to the calculation of weakest preconditions.
Weakest Preconditions

The weakest precondition of each program construct.

\[
\begin{align*}
wp(\text{skip}, Q) & \iff Q \\
wp(\text{abort}, Q) & \iff \text{true} \\
wp(x := e, Q) & \iff Q[e/x] \\
wp(c_1; c_2, Q) & \iff wp(c_1, wp(c_2, Q)) \\
wp(\text{if } b \text{ then } c_1 \text{ else } c_2, Q) & \iff (b \Rightarrow wp(c_1, Q)) \land (\neg b \Rightarrow wp(c_2, Q)) \\
wp(\text{if } b \text{ then } c, Q) & \iff (b \Rightarrow wp(c, Q)) \land (\neg b \Rightarrow Q)
\end{align*}
\]

Alternative formulation of a program calculus.
Forward Reasoning

Sometimes, we want to derive a postcondition from a given precondition.

\[ \{ P \} \ x := e \ \{ \exists x_0 : P[x_0/x] \land x = e[x_0/x] \} \]

- **Forward Reasoning**

- What is the maximum we know about the post-state of an assignment \( x := e \), if the pre-state satisfies \( P \)?
- We know that \( P \) holds for some value \( x_0 \) (the value of \( x \) in the pre-state) and that \( x \) equals \( e[x_0/x] \).

\[
\{ x \geq 0 \land y = a \}
\]

\[
x := x + 1
\]

\[
\{ \exists x_0 : x_0 \geq 0 \land y = a \land x = x_0 + 1 \}
\]

\[
(\Leftrightarrow (\exists x_0 : x_0 \geq 0 \land x = x_0 + 1) \land y = a)
\]

\[
(\Leftrightarrow x > 0 \land y = a)
\]
A calculus for forward reasoning.

- **Predicate transformer sp**
  - Function “sp” that takes a precondition $P$ and a command $c$ and returns a postcondition.
  - Read $\text{sp}(P, c)$ as “the strongest postcondition of $c$ w.r.t. $P$”.
- $\text{sp}(P, c)$ is a **postcondition** for $c$ that is ensured by precondition $P$.
  - Must satisfy $\{P\} \ c \ \{\text{sp}(P, c)\}$.
- $\text{sp}(P, c)$ is the **strongest** such postcondition.
  - Take any $P, Q$ such that $\{P\} \ c \ \{Q\}$.
  - Then $\text{sp}(P, c) \Rightarrow Q$.
- **Consequence:**  $\{P\} \ c \ \{Q\}$ iff $(\text{sp}(P, c) \Rightarrow Q)$.
  - We want to prove $\{P\} \ c \ \{Q\}$.
  - We may prove $\text{sp}(P, c) \Rightarrow Q$ instead.

Verification is reduced to the calculation of strongest postconditions.
The strongest postcondition of each program construct.

\[
\begin{align*}
\text{sp}(P, \text{skip}) & \iff P \\
\text{sp}(P, \text{abort}) & \iff \text{false} \\
\text{sp}(P, x := e) & \iff \exists x_0 : P[x_0/x] \land x = e[x_0/x] \\
\text{sp}(P, c_1; c_2) & \iff \text{sp}(\text{sp}(P, c_1), c_2) \\
\text{sp}(P, \text{if } b \text{ then } c_1 \text{ else } c_2) & \iff (b \Rightarrow \text{sp}(P, c_1)) \land (\neg b \Rightarrow \text{sp}(P, c_2)) \\
\text{sp}(P, \text{if } b \text{ then } c) & \iff (b \Rightarrow \text{sp}(P, c)) \land (\neg b \Rightarrow P)
\end{align*}
\]

The use of predicate transformers is an alternative/supplement to the Hoare calculus; this view is due to Dijkstra.
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The Hoare Calculus and Loops

\{\text{true}\} \text{ loop } \{\text{false}\}

\[
P \Rightarrow I \quad \{I \land b\} \quad c \quad \{I\} \quad (I \land \neg b) \Rightarrow Q
\]

\[
\{P\} \text{ while } b \text{ do } c \quad \{Q\}
\]

\textbf{Interpretation:}

- The \texttt{loop} command does not terminate and thus trivially satisfies partial correctness.
- Axiom implies \{P\} \texttt{ loop } \{Q\} for arbitrary \(P, Q\).
- To show that, if before the execution of a \texttt{while}-loop the property \(P\) holds, after its termination the property \(Q\) holds, it suffices to show for some property \(I\) (the \textit{loop invariant}) that
  - \(I\) holds before the loop is executed (i.e. that \(P\) implies \(I\)),
  - if \(I\) holds when the loop body is entered (i.e. if also \(b\) holds), that after the execution of the loop body \(I\) still holds,
  - when the loop terminates (i.e. if \(b\) does not hold), \(I\) implies \(Q\).

\textbf{Problem:} find appropriate loop invariant \(I\).

- Strongest relationship between all variables modified in loop body.
Example

\[ l \iff (n \geq 0 \Rightarrow 1 \leq i \leq n + 1) \land s = \sum_{j=1}^{i-1} j \]

\[(i = 1 \land s = 0) \Rightarrow l\]

\{l \land i \leq 0\} \quad s := s + i; \quad i := i + 1 \quad \{l\}

\[(l \land i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j\]

\{i = 1 \land s = 0\} \quad \textbf{while} \quad i \leq n \quad \textbf{do} \quad (s := s + i; \quad i := i + 1) \quad \{s = \sum_{j=1}^{n} j\}

The invariant captures the “essence” of a loop; only by giving its invariant, a true understanding of a loop is demonstrated.
We want to verify the following program:

\[
\{ P \} \; c_1 ; \textbf{while} \; b \; \textbf{do} \; c ; \; c_2 \; \{ Q \}
\]

- Assume \( c_1 \) and \( c_2 \) do not contain loop commands.
- It suffices to prove

\[
\{ \text{sp}(P, c_1) \} \; \textbf{while} \; b \; \textbf{do} \; c \; \{ \text{wp}(c_2, Q) \}
\]

Verification of loops is the core of most program verifications.
Weakest Liberal Preconditions for Loops

\[
\begin{align*}
wp(\text{\texttt{loop}}, Q) & \iff \text{true} \\
wp(\text{while } b \text{ do } c, Q) & \iff \forall i \in \mathbb{N} : L_i(Q)
\end{align*}
\]

\[
\begin{align*}
L_0(Q) & \iff \text{true} \\
L_{i+1}(Q) & \iff (\neg b \Rightarrow Q) \land (b \Rightarrow wp(c, L_i(Q)))
\end{align*}
\]

- **Interpretation**
  - Weakest precondition that ensures that loops stops in a state satisfying \( Q \), unless it aborts or runs forever.

- **Infinite sequence of predicates \( L_i(Q) \):**
  - Weakest precondition that ensures that loops stops after less than \( i \) iterations in a state satisfying \( Q \), unless it aborts or runs forever.

- **Alternative view:** \( L_i(Q) \iff wp(\text{if}_i, Q) \)
  - \( \text{if}_0 := \text{\texttt{loop}} \)
  - \( \text{if}_{i+1} := \text{if } b \text{ then } (c; \text{if}_i) \)
Example

$$\text{wp(while } i < n \text{ do } i := i + 1, Q)$$

\[ L_0(Q) \Leftrightarrow \text{true} \]
\[ L_1(Q) \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, \text{true})) \]
\[ \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{true}) \]
\[ \Leftrightarrow (i \not< n \Rightarrow Q) \]
\[ L_2(Q) \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, i \not< n \Rightarrow Q)) \]
\[ \Leftrightarrow (i \not< n \Rightarrow Q) \land \]
\[ (i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i])) \]
\[ L_3(Q) \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, \]
\[ (i \not< n \Rightarrow Q) \land (i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i]))) \]
\[ \Leftrightarrow (i \not< n \Rightarrow Q) \land \]
\[ (i < n \Rightarrow (((i + 1 \not< n \Rightarrow Q[i + 1/i]) \land \][
\[ (i + 1 < n \Rightarrow (i + 2 \not< n \Rightarrow Q[i + 2/i]))) \]
Weakest Liberal Preconditions for Loops

- Sequence $L_i(Q)$ is monotonically increasing in strength:
  - $\forall i \in \mathbb{N} : L_{i+1}(Q) \Rightarrow L_i(Q)$.

- The weakest precondition is the “lowest upper bound”:
  - $\text{wp(while } b \text{ do } c, Q) \Rightarrow \forall i \in \mathbb{N} : L_i(Q)$.
  - $\forall P : (P \Rightarrow \forall i \in \mathbb{N} : L_i(Q)) \Rightarrow (P \Rightarrow \text{wp(while } b \text{ do } c, Q))$.

- We can only compute weaker approximation $L_i(Q)$.
  - $\text{wp(while } b \text{ do } c, Q) \Rightarrow L_i(Q)$.

- We want to prove $\{P\} \text{ while } b \text{ do } c \{Q\}$.
  - This is equivalent to proving $P \Rightarrow \text{wp(while } b \text{ do } c, Q)$.
  - Thus $P \Rightarrow L_i(Q)$ must hold as well.

- If we can prove $\neg(P \Rightarrow L_i(Q))$, . . .
  - $\{P\} \text{ while } b \text{ do } c \{Q\}$ does not hold.
  - If we fail, we may try the easier proof $\neg(P \Rightarrow L_{i+1}(Q))$.

Falsification is possible by use of approximation $L_i$, but verification is not.
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Total Correctness of Loops

Hoare rules for loop and while are replaced as follows:

\[
\{\text{false}\} \text{ loop } \{\text{false}\}
\]

\[
\{l \land b \land t = N\} \; c \; \{l \land t < N\} \; (l \land \neg b) \Rightarrow Q
\]

\[
P \Rightarrow l \; l \land b \Rightarrow t > 0
\]

\[
\{P\} \text{ while } b \text{ do } c \; \{Q\}
\]

- New interpretation of \{P\} c \{Q\}.
  - If execution of c starts in a state where \(P\) holds, then execution terminates in a state where \(Q\) holds, unless it aborts.
  - Non-termination is ruled out, abortion not (yet).
  - The loop command thus does not satisfy total correctness.

- Termination term \(t\).
  - Denotes a natural number before and after every loop iteration.
  - If \(t = N\) before an iteration, then \(t < N\) after the iteration.
  - Consequently, if term denotes zero, loop must terminate.

Instead of the natural numbers, any well-founded ordering may be used for the domain of \(t\).
Example

\[ l \iff (n \geq 0 \Rightarrow 1 \leq i \leq n+1) \land s = \sum_{j=1}^{i-1} j \]

\[ (i = 1 \land s = 0) \Rightarrow l \land l \land i \leq n \Rightarrow n - i + 1 > 0 \]
\[ \{ l \land i \leq 0 \land n - i + 1 = N \} \ s := s + i; \ i := i + 1 \{ l \land n - i + 1 < N \} \]
\[ (l \land i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j \]
\[ \{ i = 1 \land s = 0 \} \textbf{ while } i \leq n \textbf{ do } (s := s + i; \ i := i + 1) \{ s = \sum_{j=1}^{n} j \} \]

In practice, termination is easy to show (compared to partial correctness).
Weakest Preconditions for Loops

\[
\text{wp(} \text{loop, } Q) \iff \text{false}
\]
\[
\text{wp(} \text{while } b \text{ do } c, Q) \iff \exists i \in \mathbb{N} : L_i(Q)
\]

\[
L_0(Q) \iff \text{false}
\]
\[
L_{i+1}(Q) \iff (\neg b \Rightarrow Q) \land (b \Rightarrow \text{wp}(c, L_i(Q)))
\]

- **New interpretation**
  - Weakest precondition that ensures that the loop terminates in a state in which \( Q \) holds, unless it aborts.

- **New interpretation of** \( L_i(Q) \)
  - Weakest precondition that ensures that the loop terminates after less than \( i \) iterations in a state in which \( Q \) holds, unless it aborts.

- Preserves property: \( \{P\} \ c \ \{Q\} \iff (P \Rightarrow \text{wp}(c, Q)) \)
  - Now for **total correctness** interpretation of Hoare calculus.

- Preserves alternative view: \( L_i(Q) \iff \text{wp}(\text{if } i, Q) \)
  
  \[
  \begin{align*}
  i_0 &:= \text{loop} \\
  i_{i+1} &:= \text{if } b \text{ then } (c; i)
  \end{align*}
  \]
Example

\( \text{wp(while } i < n \text{ do } i := i + 1, Q) \)

\( L_0(Q) :\Leftrightarrow \text{false} \)
\( L_1(Q) :\Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, L_0(Q))) \)
\( \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{false}) \)
\( \Leftrightarrow i \not< n \land Q \)
\( L_2(Q) :\Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, L_1(Q))) \)
\( \Leftrightarrow (i \not< n \Rightarrow Q) \land 
\quad i < n \Rightarrow (i + 1 \not< n \land Q[i + 1/i]) \)
\( L_3(Q) :\Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, L_2(Q))) \)
\( \Leftrightarrow (i \not< n \Rightarrow Q) \land 
\quad (i < n \Rightarrow ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \land 
\quad (i + 1 < n \Rightarrow (i + 2 \not< n \land Q[i + 2/i]))) \))

\[ \ldots \]
Weakest Preconditions for Loops

- Sequence $L_i(Q)$ is now monotonically decreasing in strength:
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow L_{i+1}(Q)$.

- The weakest precondition is the “greatest lower bound”:
  - $(\forall i \in \mathbb{N} : L_i(Q)) \Rightarrow \text{wp(while } b \text{ do } c, Q)$.
  - $\forall P : ((\forall i \in \mathbb{N} : L_i(Q)) \Rightarrow P) \Rightarrow (\text{wp(while } b \text{ do } c, Q) \Rightarrow P)$.

- We can only compute a stronger approximation $L_i(Q)$.
  - $L_i(Q) \Rightarrow \text{wp(while } b \text{ do } c, Q)$.

- We want to prove $\{P\} c \{Q\}$.
  - It suffices to prove $P \Rightarrow \text{wp(while } b \text{ do } c, Q)$.
  - It thus also suffices to prove $P \Rightarrow L_i(Q)$.
  - If proof fails, we may try the easier proof $P \Rightarrow L_{i+1}(Q)$

Verifications are typically not successful with finite approximation of weakest precondition.
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Abortion

New rules to prevent abortion.

\[
\begin{align*}
\{\text{false}\} & \textbf{abort} \ \{\text{true}\} \\
\{Q[e/x] \land D(e)\} & \ x := e \ \{Q\} \\
\{Q[a[i \mapsto e]/a] \land D(e) \land 0 \leq i < \text{length}(a)\} & \ a[i] := e \ \{Q\}
\end{align*}
\]

- New interpretation of \{P\} c \{Q\}.
  - If execution of c starts in a state, in which property P holds, then it does not abort and eventually terminates in a state in which Q holds.

- Sources of abortion.
  - Division by zero.
  - Index out of bounds exception.

\(D(e)\) makes sure that every subexpression of e is well defined.
Definedness of Expressions

\[ D(0) : \iff \text{true} \]
\[ D(1) : \iff \text{true} \]
\[ D(x) : \iff \text{true} \]
\[ D(a[i]) : \iff D(i) \land 0 \leq i < \text{length}(a) \]
\[ D(e_1 + e_2) : \iff D(e_1) \land D(e_2) \]
\[ D(e_1 \times e_2) : \iff D(e_1) \land D(e_2) \]
\[ D(e_1 / e_2) : \iff D(e_1) \land D(e_2) \land e_2 \neq 0 \]
\[ D(\text{true}) : \iff \text{true} \]
\[ D(\text{false}) : \iff \text{true} \]
\[ D(\neg b) : \iff D(b) \]
\[ D(b_1 \land b_2) : \iff D(b_1) \land D(b_2) \]
\[ D(b_1 \lor b_2) : \iff D(b_1) \land D(b_2) \]
\[ D(e_1 < e_2) : \iff D(e_1) \land D(e_2) \]
\[ D(e_1 \leq e_2) : \iff D(e_1) \land D(e_2) \]
\[ D(e_1 > e_2) : \iff D(e_1) \land D(e_2) \]
\[ D(e_1 \geq e_2) : \iff D(e_1) \land D(e_2) \]

Assumes that expressions have already been type-checked.
Slight modification of existing rules.

\[
\begin{array}{ll}
\{P \land b \land D(b)\} & c_1 \quad \{Q\} \\
\{P\} & \text{if } b \text{ then } c_1 \text{ else } c_2 \quad \{Q\}
\end{array}
\]

\[
\begin{array}{ll}
\{P \land b \land D(b)\} & c \quad \{Q\} \\
(P \land \neg b \land D(b)) & \Rightarrow \quad \{P\} \quad \text{if } b \text{ then } c \quad \{Q\}
\end{array}
\]

\[
\begin{array}{ll}
P \Rightarrow I \\
I \Rightarrow (T \in \mathbb{N} \land D(b)) \\
\{I \land b \land T = t\} & c \quad \{I \land T < t\} \\
(I \land \neg b) & \Rightarrow \quad \{P\} \quad \text{while } b \text{ do } c \quad \{Q\}
\end{array}
\]

Expressions must be defined in any context.
Abortion

Similar modifications of weakest preconditions.

\[ \text{wp}(\text{abort}, Q) \iff \text{false} \]
\[ \text{wp}(\text{x} := e, Q) \iff Q[e/x] \land D(e) \]
\[ \text{wp}(\text{if } b \text{ then } c_1 \text{ else } c_2, Q) \iff \]
\[ \quad D(b) \land (b \Rightarrow \text{wp}(c_1, Q)) \land (\neg b \Rightarrow \text{wp}(c_2, Q)) \]
\[ \text{wp}(\text{if } b \text{ then } c, Q) \iff D(b) \land (b \Rightarrow \text{wp}(c, Q)) \land (\neg b \Rightarrow Q) \]
\[ \text{wp}(\text{while } b \text{ do } c, Q) \iff \exists i \in \mathbb{N} : L_i(Q) \]

\[ L_0(Q) :\iff \text{false} \]
\[ L_{i+1}(Q) :\iff D(b) \land (\neg b \Rightarrow Q) \land (b \Rightarrow \text{wp}(c, L_i(Q))) \]

\( \text{wp}(c, Q) \) now makes sure that the execution of \( c \) does not abort but eventually terminates in a state in which \( Q \) holds.
1. The Hoare Calculus for Non-Loop Programs

2. Predicate Transformers

3. Partial Correctness of Loop Programs

4. Total Correctness of Loop Programs

5. Abortion

6. Procedures
Procedure Specifications

global $F$;
requires $Pre$;
ensures $Post$;
$p(i, t, o) \{ \ c \ \}$

- Specification of procedure $p(i, t, o)$.
  - Input parameter $i$, transient parameter $t$, output parameter $o$.
  - A call has form $p(e, x, y)$ for expression $e$ and variables $x$ and $y$.
  - Set of global variables (“frame”) $F$.
    - Those global variables that $p$ may read/write (in addition to $i, t, o$).
    - Let $f$ denote all variables in $F$; let $g$ denote all variables not in $F$.
  - Precondition $Pre$ (may refer to $i, t, f$).
  - Postcondition $Post$ (may refer to $i, t, t_0, f, f_0, o$).

- Proof obligation
  $\{ Pre \land i_0 = i \land t_0 = t \land f_0 = f \} \ c \ \{ Post[i_0/i]\}$
First let us give an alternative (equivalent) version of the assignment rule.

- **Original:**
  \[
  \{D(e) \land Q[e/x]\} \\
  x := e \\
  \{Q\}
  \]

- **Alternative:**
  \[
  \{D(e) \land \forall x' : x' = e \Rightarrow Q[x'/x]\} \\
  x := e \\
  \{Q\}
  \]

The new value of \(x\) is given name \(x'\) in the precondition.
Procedure Calls

From this, we can derive a rule for the correctness of procedure calls.

\[
\{ D(e) \land Pre[e/i, x/t] \land \forall x', y', f' : Post[e/i, x/t_0, x'/t, y'/o, f/f_0, f'/f] \Rightarrow Q[x'/x, y'/y, f'/f] \} \\
p(e, x, y) \\
\{ Q \}
\]

- \( Pre[e/i, x/t] \) refers to the values of the actual arguments \( e \) and \( x \) (rather than to the formal parameters \( i \) and \( t \)).
- \( x', y', f' \) denote the values of the vars \( x, y,\) and \( f \) after the call.
- \( Post[...] \) refers to the argument values before and after the call.
- \( Q[x'/x, y'/y, f'/f] \) refers to the argument values after the call.

Modular reasoning: rule only relies on the specification of \( p \), not on its implementation.
Corresponding Predicate Transformers

\[ \text{wp}(p(e, x, y), Q) \iff D(e) \land \text{Pre}[e/i, x/t] \land \forall x', y', f' : \]
\[ \text{Post}[e/i, x/t_0, x'/t, y'/o, f/f_0, f'/f] \implies Q[x'/x, y'/y, f'/f] \]

\[ \text{sp}(P, p(e, x, y)) \iff \exists x_0, y_0, f_0 : \]
\[ P[x_0/x, y_0/y, f_0/f] \land \]
\[ \text{Post}[e[x_0/x, y_0/y, f_0/f]/i, x_0/t_0, x/t, y/o] \]

Explicit naming of old/new values required.
Procedure Calls Example

- Procedure specification:
  
  global \( f \)
  
  requires \( f \geq 0 \land i > 0 \)
  
  ensures \( f_0 = f \cdot i + o \land 0 \leq o < i \)
  
  \( \text{dividesF}(i, o) \)

- Procedure call:
  
  \{ f \geq 0 \land f = N \land b \geq 0 \}
  
  \( \text{dividesF}(b + 1, y) \)
  
  \{ f \cdot (b + 1) \leq N < (f + 1) \cdot (b + 1) \}

- To be ultimately proved:
  
  \( f \geq 0 \land f = N \land b \geq 0 \Rightarrow \)
  
  \( D(b + 1) \land f \geq 0 \land b + 1 > 0 \land \)
  
  \( \forall y', f' : \)
  
  \( f = f' \cdot (b + 1) + y' \land 0 \leq y' < b + 1 \Rightarrow \)
  
  \( f' \cdot (b + 1) \leq N < (f' + 1) \cdot (b + 1) \)
Not Yet Covered

- Primitive data types.
  - `int` values are actually finite precision integers.
- More data and control structures.
  - `switch`, `do-while` (easy); `continue`, `break`, `return` (more complicated).
  - Records can be handled similar to arrays.
- Recursion.
  - Procedures may not terminate due to recursive calls.
- Exceptions and Exception Handling.
  - Short discussion in the context of ESC/Java2 later.
- Pointers and Objects.
  - Here reasoning gets complicated.
- ...

The more features are covered, the more complicated reasoning becomes.