Hoare Calculus and Predicate Transformers

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The Hoare Calculus

Calculus for reasoning about imperative programs.

- “Hoare triple”: \{P\} c \{Q\}
  - Logical propositions P and Q, program command c.
  - The Hoare triple is itself a logical proposition.
  - The Hoare calculus gives rules for constructing true Hoare triples.
- Partial correctness interpretation of \{P\} c \{Q\}:
  - “If c is executed in a state in which P holds, then it terminates in a state in which Q holds unless it aborts or runs forever.”
  - Program does not produce wrong result.
  - But program also need not produce any result.
  - Abortion and non-termination are not ruled out.
- Total correctness interpretation of \{P\} c \{Q\}:
  - “If c is executed in a state in which P holds, then it terminates in a state in which Q holds.
  - Program produces the correct result.

We will use the partial correctness interpretation for the moment.

1. The Hoare Calculus for Non-Loop Programs
2. Predicate Transformers
3. Partial Correctness of Loop Programs
4. Total Correctness of Loop Programs
5. Abortion
6. Procedures

General Rules

\[
\begin{align*}
P & \Rightarrow Q \\
\{P\} & \{Q\}
\end{align*}
\]

\[
\begin{align*}
P & \Rightarrow P' \\
\{P\} & c \{Q'\} \\
Q' & \Rightarrow Q
\end{align*}
\]

- Logical derivation: \[ A_1 \ A_2 \]
  - Forward: If we have shown \(A_1\) and \(A_2\), then we have also shown \(B\).
  - Backward: To show \(B\), it suffices to show \(A_1\) and \(A_2\).
- Interpretation of above sentences:
  - To show that, if \(P\) holds in a state, then \(Q\) holds in the same state (no command is executed), it suffices to show \(P\) implies \(Q\).
  - Hoare triples are ultimately reduced to classical logic.
  - To show that, if \(P\) holds, then \(Q\) holds after executing \(c\), it suffices to show this for a \(P'\) weaker than \(P\) and a \(Q'\) stronger than \(Q\).
  - Precondition may be weakened, postcondition may be strengthened.
### Special Commands

Commands modeling “emptiness” and abortion.

\[
\{ P \} \text{skip} \{ P \} \quad \{ \text{true} \} \text{abort} \{ \text{false} \}
\]

- The **skip** command does not change the state; if \( P \) holds before its execution, then \( P \) thus holds afterwards as well.
- The **abort** command aborts execution and thus trivially satisfies partial correctness.
- Axiom implies \( \{ P \} \text{abort} \{ Q \} \) for arbitrary \( P, Q \).

Useful commands for reasoning and program transformations.

### Scalar Assignments

Decomposition of \( Q[e/x] \) . . . \( Q \) where every free occurrence of \( x \) is replaced by \( e \).

- **Syntax**
  - Variable \( x \), expression \( e \).
  - \( Q[e/x] \) . . . \( Q \)

- **Interpretation**
  - To make sure that \( Q \) holds for \( x \) after the assignment of \( e \) to \( x \), it suffices to make sure that \( Q \) holds for \( e \) before the assignment.
  - **Partial correctness**
    - Evaluation of \( e \) may abort.

\[
\begin{align*}
\{ x + 3 < 5 \} & \quad x := x + 3 \quad \{ x < 5 \} \\
\{ x < 2 \} & \quad x := x + 3 \quad \{ x < 5 \}
\end{align*}
\]

### Array Assignments

- An array is modelled as a function \( a : I \rightarrow V \)
  - Index set \( I \), value set \( V \).
  - \( a[i] = e \ldots a \) holds at index \( i \) the value \( e \).
  - **Updated array** \( a[i] \mapsto e \)
  - Array that is constructed from \( a \) by mapping index \( i \) to value \( e \).
  - Axioms (for all \( a : I \rightarrow V, i, j \in I, e \in V \)):
    - \( i = j \Rightarrow a[i] \mapsto e[j] = e \)
    - \( i \neq j \Rightarrow a[i] \mapsto e[j] = a[j] \)
    - \( a[i] \mapsto e[1] > 0 \) \( a[i] := x \quad \{ a[1] > 0 \} \)
    - \( (i = 1 \Rightarrow x > 0) \wedge (i \neq 1 \Rightarrow a[1] > 0) \) \( a[i] := x \quad \{ a[1] > 0 \} \)

Index violations and pointer semantics of arrays not yet considered.

### Command Sequences

Decomposition of \( Q[a[i \mapsto e]/a] \) . . . \( Q \)

- **Syntax**
  - \( \{ P \} c_1 \{ R_1 \} \quad R_1 \Rightarrow R_2 \quad \{ R_2 \} c_2 \{ Q \} \)

- **Interpretation**
  - To show that, if \( P \) holds before the execution of \( c_1; c_2 \), then \( Q \) holds afterwards, it suffices to show for some \( R_1 \) and \( R_2 \) with \( R_1 \Rightarrow R_2 \) that
    - if \( P \) holds before \( c_1 \), that \( R_1 \) holds afterwards, and that
    - if \( R_2 \) holds before \( c_2 \), then \( Q \) holds afterwards.
  - **Problem:** find suitable \( R_1 \) and \( R_2 \)
    - Easy in many cases (see later).

\[
\begin{align*}
\{ x + y - 1 > 0 \} & \quad y := y - 1 \quad \{ x + y > 0 \} \quad \{ x + y > 0 \} \quad x := x + y \quad \{ x > 0 \} \\
\{ x + y - 1 > 0 \} & \quad y := y - 1 \quad \{ x + y > 0 \} \quad x := x + y \quad \{ x > 0 \}
\end{align*}
\]
Conditionals

\[
\begin{align*}
\{P \land b\} & \ c_1 \ \{Q\} \ \{P \land \neg b\} \ c_2 \ \{Q\} \\
\{P\} & \ \text{if } b \ \text{then } c_1 \ \text{else } c_2 \ \{Q\}
\end{align*}
\]

- Interpretation
  - To show that, if \(P\) holds before the execution of the conditional, then \(Q\) holds afterwards,
  - it suffices to show that the same is true for each conditional branch, under the additional assumption that this branch is executed.

\[
\begin{align*}
\{x \neq 0 \land x \geq 0\} & \ y := x & \{y > 0\} \ \{x \neq 0 \land x \geq 0\} & \ y := -x & \{y > 0\} \\
\{x \neq 0\} & \ \text{if } x \geq 0 \ \text{then } y := x \ \text{else } y := -x & \{y > 0\}
\end{align*}
\]

Backward Reasoning

Implication of rule for command sequences and rule for assignments:

\[
\begin{align*}
\{P\} & \ c \ \{Q[e/x]\} \\
\{P\} & \ c; x := e \ \{Q\}
\end{align*}
\]

- Interpretation
  - If the last command of a sequence is an assignment, we can remove the assignment from the proof obligation.
  - By multiple application, assignment sequences can be removed from the back to the front.

\[
\begin{align*}
\{P\} & \ x := x+1; \ x := x+1; \ x := x+1; \ {x + 1 = 5} \\
y := 2 \times x; \ y := 2 \times x; \ {x + 2 \times x = 15} & \ \Rightarrow \ {x + 3 \times x = 15} & \ \Rightarrow \ {x = 5}
\end{align*}
\]

Weakest Preconditions

A calculus for “backward reasoning”.

- Predicate transformer \(wp\)
  - Function “\(wp\)” that takes a command \(c\) and a postcondition \(Q\) and returns a precondition.
  - Read \(wp(c, Q)\) as “the weakest precondition of \(c\) w.r.t. \(Q\)”.
  - \(wp(c, Q)\) is a precondition for \(c\) that ensures \(Q\) as a postcondition.
  - Must satisfy \(\{wp(c, Q)\} \ c \ \{Q\}\).
  - \(wp(c, Q)\) is the weakest such precondition.
  - Take any \(P\) such that \(\{P\} \ c \ \{Q\}\).
  - Then \(P \Rightarrow wp(P, Q)\).
  - Consequence: \(\{P\} \ c \ \{Q\} \ \text{iff } (P \Rightarrow wp(c, Q))\)
    - We want to prove \(\{P\} \ c \ \{Q\}\).
    - We may prove \(P \Rightarrow wp(c, Q)\) instead.

Verification is reduced to the calculation of weakest preconditions.
**Weakest Preconditions**

The weakest precondition of each program construct.

\[
\begin{align*}
wp(\text{skip}, Q) & \Leftrightarrow Q \\
wp(\text{abort}, Q) & \Leftrightarrow \text{true} \\
wp(x := e, Q) & \Leftrightarrow Q[e/x] \\
wp(c_1; c_2, Q) & \Leftrightarrow \wp(c_1, \wp(c_2, Q)) \\
wp(\text{if } b \text{ then } c_1 \text{ else } c_2, Q) & \Leftrightarrow (\Leftrightarrow b \Rightarrow \wp(c_1, Q)) \land (\Leftrightarrow \neg b \Rightarrow \wp(c_2, Q))
\end{align*}
\]

Alternative formulation of a program calculus.

**Strongest Postconditions**

A calculus for forward reasoning.

- **Predicate transformer** \( sp \)
  - Function "sp" that takes a precondition \( P \) and a command \( c \) and returns a postcondition.
  - Read \( sp(P, c) \) as "the strongest postcondition of \( c \) w.r.t. \( P \)."
  - \( sp(P, c) \) is a postcondition for \( c \) that is ensured by precondition \( P \).
  - Must satisfy \( \{ P \} \ c \ { sp(P, c) } \).
  - \( sp(P, c) \) is the strongest such postcondition.
  - Take any \( P, Q \) such that \( \{ P \} \ c \ { Q } \).
  - Then \( sp(P, c) \Rightarrow Q \).
  - **Consequence:** \( \{ P \} \ c \ { Q } \) iff \( sp(P, c) \Rightarrow Q \).
  - We want to prove \( \{ P \} \ c \ { Q } \).
  - We may prove \( sp(P, c) \Rightarrow Q \) instead.

Verification is reduced to the calculation of strongest postconditions.

**Forward Reasoning**

Sometimes, we want to derive a postcondition from a given precondition.

\[
\{ P \} \ x := e \begin{cases} \exists x_0 : P[x_0/x] \land x = e[x_0/x] \end{cases}
\]

**Forward Reasoning**

- **What is the maximum we know about the post-state of an assignment \( x := e \), if the pre-state satisfies \( P \)?**
- We know that \( P \) holds for some value \( x_0 \) (the value of \( x \) in the pre-state) and that \( x \) equals \( e[x_0/x] \).

\[
\begin{align*}
\{ x \geq 0 \land y = a \} \\
x := x + 1 \\
\{ \exists x_0 : x_0 \geq 0 \land y = a \land x = x_0 + 1 \} \\
(\Leftrightarrow (\exists x_0 : x_0 \geq 0 \land x = x_0 + 1) \land y = a) \\
(\Leftrightarrow x > 0 \land y = a)
\end{align*}
\]

The use of predicate transformers is an alternative/supplement to the Hoare calculus; this view is due to Dijkstra.
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The Hoare Calculus and Loops

\[ \{\text{true}\} \text{ loop } \{\text{false}\} \]

\[ P \Rightarrow I \quad \{I \land b\} \ c \{I\} \quad \{I \land \lnot b\} \Rightarrow Q \]

\[ \{P\} \text{ while } b \text{ do } c \{Q\} \]

- **Interpretation:**
  - The loop command does not terminate and thus trivially satisfies partial correctness.
  - Axiom implies \( \{P\} \text{ loop } \{Q\} \) for arbitrary \( P, Q \).
  - To show that, if before the execution of a while-loop the property \( P \) holds, after its termination the property \( Q \) holds, it suffices to show for some property \( I \) (the loop invariant) that
    - \( I \) holds before the loop is executed (i.e. that \( P \) implies \( I \)),
    - if \( I \) holds when the loop body is entered (i.e. if also \( b \) holds), that after the execution of the loop body \( I \) still holds,
    - when the loop terminates (i.e. if \( b \) does not hold), \( I \) implies \( Q \).

- **Problem:** find appropriate loop invariant \( I \).
  - Strongest relationship between all variables modified in loop body.

Example

\[ I \iff (n \geq 0 \Rightarrow 1 \leq i \leq n + 1) \land s = \sum_{j=1}^{i-1} j \]

\[ \{i = 1 \land s = 0\} \Rightarrow I \]

\[ \{I \land i \leq 0\} \ s := s + i; i := i + 1 \ \{I\} \]

\[ \{I \land i \leq n\} \Rightarrow s = \sum_{j=1}^{i} j \]

\[ \{i = 1 \land s = 0\} \text{ while } i \leq n \text{ do } (s := s + i; i := i + 1) \ \{s = \sum_{j=1}^{i} j\} \]

The invariant captures the "essence" of a loop; only by giving its invariant, a true understanding of a loop is demonstrated.

Practical Aspects

We want to verify the following program:

\[ \{P\} \ c_1; \text{ while } b \text{ do } c; c_2 \ \{Q\} \]

- Assume \( c_1 \) and \( c_2 \) do not contain loop commands.
- It suffices to prove
  \[ \{\text{sp}(P, c_1)\} \text{ while } b \text{ do } c \ \{\text{wp}(c_2, Q)\} \]

Verification of loops is the core of most program verifications.
Weakest Liberal Preconditions for Loops

\[ wp(\text{loop}, Q) \Leftrightarrow \text{true} \]
\[ wp(\text{while } b \text{ do } c, Q) \Leftrightarrow \forall i \in \mathbb{N} : L_i(Q) \]

- **Interpretation**
  - Weakest precondition that ensures that loops stops in a state satisfying \( Q \), unless it aborts or runs forever.
- **Infinite sequence of predicates \( L_i(Q) \):**
  - Weakest precondition that ensures that loops runs after less than \( i \) iterations in a state satisfying \( Q \), unless it aborts or runs forever.
- **Alternative view:** \( L_i(Q) \Leftrightarrow wp(\text{if } i, Q) \)
  
  \[
  \text{if}_0 := \text{loop} \\
  \text{if}_{i+1} := \text{if } b \text{ then } (c; \text{if}_i) 
  \]

Weakest Liberal Preconditions for Loops

- **Sequence \( L_i(Q) \) is monotonically increasing in strength:**
  - \( \forall i \in \mathbb{N} : L_{i+1}(Q) \Rightarrow L_i(Q) \).
- The weakest precondition is the "lowest upper bound":
  - \( wp(\text{while } b \text{ do } c, Q) \Rightarrow \forall i \in \mathbb{N} : L_i(Q) \).
  - \( \forall P : (P \Rightarrow \forall i \in \mathbb{N} : L_i(Q)) \Rightarrow (P \Rightarrow wp(\text{while } b \text{ do } c, Q)) \).
- We can only compute weaker approximation \( L_i(Q) \).
  - \( wp(\text{while } b \text{ do } c, Q) \Rightarrow L_i(Q) \).
- We want to prove \( \{ P \} \text{ while } b \text{ do } c \{ Q \} \).
  - This is equivalent to proving \( P \Rightarrow wp(\text{while } b \text{ do } c, Q) \).
  - Thus \( P \Rightarrow L_i(Q) \) must hold as well.
- If we can prove \( \neg(P \Rightarrow L_i(Q)) \), . . .
  - \( \{ P \} \text{ while } b \text{ do } c \{ Q \} \) does not hold.
- If we fail, we may try the easier proof \( \neg(P \Rightarrow L_{i+1}(Q)) \).

Falsification is possible by use of approximation \( L_i \), but verification is not.

Example

\[ wp(\text{while } i < n \text{ do } i := i + 1, Q) \]

\[
L_0(Q) \Leftrightarrow \text{true} \\
L_1(Q) \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow wp(i := i + 1, Q)) \\
L_2(Q) \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow wp(i := i + 1, i < n \Rightarrow Q[i + 1/i])) \\
L_3(Q) \Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow wp(i := i + 1, i < n \Rightarrow Q[i + 1/i]) \land (i + 1 < n \Rightarrow Q[i + 2/i])))
\]

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Total Correctness of Loops

Hoare rules for loop and while are replaced as follows:

\[
P \Rightarrow \{ I \land b \Rightarrow t > 0 \} \land b \Rightarrow t > 0
\]

\[
\{ \text{false}\} \text{ loop } \{ \text{false}\}
\]

\[
\{ I \land t < N \} \land t < N
\]

\[
\{ P \} \text{ while } b \text{ do } c \{ Q \}
\]

- New interpretation of \( \{ P \} \) \( \Rightarrow \) \( \{ Q \} \).
  - If execution of \( c \) starts in a state where \( P \) holds, then execution terminates in a state where \( Q \) holds, unless it aborts.
  - Non-termination is ruled out, abortion not (yet).
  - The loop command thus does not satisfy total correctness.

- Termination term \( t \).
  - Denotes a natural number before and after every loop iteration.
  - If \( t = N \) before an iteration, then \( t < N \) after the iteration.
  - Consequently, if term denotes zero, loop must terminate.

Instead of the natural numbers, any well-founded ordering may be used for the domain of \( t \).

Weakest Preconditions for Loops

**Example**

\[
I \Rightarrow (n \geq 0 \Rightarrow 1 \leq i \leq n + 1) \land s = \sum_{j=1}^{i-1} j
\]

\[
(i = 1 \land s = 0) \Rightarrow I \land i \leq n \Rightarrow n - i + 1 > 0
\]

\[
\{ I \land i \leq 0 \land n - i + 1 = N \} s := s + i; i := i + 1 \{ I \land n - i + 1 < N \}
\]

\[
(1 \land i \leq n) \Rightarrow s = \sum_{j=1}^{n-1} j
\]

\[
\{ i = 1 \land s = 0 \} \text{ while } i \leq n \text{ do } (s := s + i; i := i + 1) \{ s = \sum_{j=1}^{n} j \}
\]

In practice, termination is easy to show (compared to partial correctness).

**Example**

\[
\text{wp}(\text{while } i < n \text{ do } i := i + 1, Q)
\]

\[
L_0(Q) : \Rightarrow \text{false}
\]

\[
L_1(Q) : \Rightarrow (i \not< n) \land (i < n \Rightarrow \text{wp}(i := i + 1, L_0(Q)))
\]

\[
\{ i \not< n \land Q \}
\]

\[
L_2(Q) : \Rightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, L_1(Q)))
\]

\[
\{ i \not< n \Rightarrow Q \}
\]

\[
L_3(Q) : \Rightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \text{wp}(i := i + 1, L_2(Q)))
\]

\[
\{ i \not< n \Rightarrow Q \} \land
\]

\[
(i < n \Rightarrow ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \land (i + 1 < n \Rightarrow (i + 2 \not< n \land Q[i + 2/i])))
\]

\[
\ldots
\]
Weakest Preconditions for Loops

- Sequence $L_i(Q)$ is now monotonically decreasing in strength:
  - $\forall i \in \mathbb{N}: L_i(Q) \Rightarrow L_{i+1}(Q)$.

- The weakest precondition is the “greatest lower bound”:
  - $(\forall i \in \mathbb{N}: Q) \Rightarrow \text{wp(while } b \text{ do } c, Q)$.
  - $\forall P: ((\forall i \in \mathbb{N}: L_i(Q)) \Rightarrow P) \Rightarrow (\text{wp(while } b \text{ do } c, Q) \Rightarrow P)$.

- We can only compute a stronger approximation $L_i(Q)$.
  - $L_i(Q) \Rightarrow \text{wp(while } b \text{ do } c, Q)$.

- We want to prove $\{P\} c \{Q\}$.
  - It suffices to prove $P \Rightarrow \text{wp(while } b \text{ do } c, Q)$.
  - It thus also suffices to prove $P \Rightarrow L_i(Q)$.
  - If proof fails, we may try the easier proof $P \Rightarrow L_{i+1}(Q)$

Verifications are typically not successful with finite approximation of weakest precondition.

Abortion

New rules to prevent abortion.

\[
\begin{align*}
\{\text{false}\} & \text{ abort } \{\text{true}\} \\
\{Q[e/x] \land D(e)\} & \text{ x := e } \{Q\} \\
\{Q[a[i \rightarrow e]/a] \land D(e) \land 0 \leq i < \text{length}(a)\} & a[i] := e \{Q\}
\end{align*}
\]

- New interpretation of $\{P\} c \{Q\}$.
  - If execution of $c$ starts in a state, in which property $P$ holds, then it does not abort and eventually terminates in a state in which $Q$ holds.

- Sources of abortion.
  - Division by zero.
  - Index out of bounds exception.

\[D(e)\] makes sure that every subexpression of $e$ is well defined.

Definedness of Expressions

- $D(0) :\leftrightarrow \text{true}$.
- $D(1) :\leftrightarrow \text{true}$.
- $D(x) :\leftrightarrow \text{true}$.
- $D(a[i]) :\leftrightarrow D(i) \land 0 \leq i < \text{length}(a)$.
- $D(e_1 + e_2) :\leftrightarrow D(e_1) \land D(e_2)$.
- $D(e_1 - e_2) :\leftrightarrow D(e_1) \land D(e_2) \land e_2 \neq 0$.
- $D(\text{true}) :\leftrightarrow \text{true}$.
- $D(\text{false}) :\leftrightarrow \text{true}$.
- $D(\neg b) :\leftrightarrow D(b)$.
- $D(b_1 \land b_2) :\leftrightarrow D(b_1) \land D(b_2)$.
- $D(b_1 \lor b_2) :\leftrightarrow D(b_1) \lor D(b_2)$.
- $D(e_1 < e_2) :\leftrightarrow D(e_1) \land D(e_2)$.
- $D(e_1 \leq e_2) :\leftrightarrow D(e_1) \land D(e_2)$.
- $D(e_1 > e_2) :\leftrightarrow D(e_1) \land D(e_2)$.
- $D(e_1 \geq e_2) :\leftrightarrow D(e_1) \land D(e_2)$.

Assumes that expressions have already been type-checked.
Abortion

Slight modification of existing rules.

\[
\begin{align*}
&\{ P \land b \land D(b) \} \ x_1 \ \{ Q \} \\
&\{ P \} \ \text{if} \ b \ \text{then} \ x_1 \ \text{else} \ x_2 \ \{ Q \}
\end{align*}
\]

\[
\begin{align*}
&\{ P \land b \land D(b) \} \ x_3 \ \{ Q \} \\
&\{ P \} \ \text{if} \ b \ \text{then} \ x_3 \ \text{else} \ x_2 \ \{ Q \}
\end{align*}
\]

\[
\begin{align*}
P &\Rightarrow I \ 
I &\Rightarrow (T \in \mathbb{N} \land D(b)) \\
\{ I \land b \land T = t \} \ x_1 \ \{ I \land T < t \} \ (I \land \neg b) &\Rightarrow Q \\
\{ P \} \ \text{while} \ b \ \text{do} \ x_1 \ \{ Q \}
\end{align*}
\]

Expressions must be defined in any context.

Similar modifications of weakest preconditions.

\[
\begin{align*}
\wp(\text{abort}, Q) &\Leftrightarrow \text{false} \\
\wp(x := e, Q) &\Leftrightarrow Q[e/x] \land D(e) \\
\wp(\text{if} \ b \ \text{then} \ c \ \text{else} \ c, Q) &\Leftrightarrow \\
&\text{D}(b) \land (b \Rightarrow \wp(c_1, Q)) \land (\neg b \Rightarrow \wp(c_2, Q))
\end{align*}
\]

\[
\begin{align*}
\wp(\text{if} \ b \ \text{then} \ c, Q) &\Leftrightarrow \\
&\text{D}(b) \land (b \Rightarrow \wp(c, Q)) \land (\neg b \Rightarrow Q)
\end{align*}
\]

\[
\begin{align*}
\wp(\text{while} \ b \ \text{do} \ c, Q) &\Leftrightarrow \\
&\exists i \in \mathbb{N} : L_i(Q)
\end{align*}
\]

wp(\(c, Q\)) now makes sure that the execution of \(c\) does not abort but eventually terminates in a state in which \(Q\) holds.

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Procedure Specifications

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Procedure Calls

First let us give an alternative (equivalent) version of the assignment rule.

- Original:
  \[
  \{D(e) \land Q[e/x]\} \\
  x := e \\
  \{Q\}
  \]

- Alternative:
  \[
  \{D(e) \land \forall x' : x' = e \Rightarrow Q[x'/x]\} \\
  x := e \\
  \{Q\}
  \]

The new value of \(x\) is given name \(x'\) in the precondition.

Corresponding Predicate Transformers

- \(wp(p(e, x, y), Q) \iff\)
  \[
  D(e) \land Pre[e/i, x/t] \land \\
  \forall x', y', f' : Post[e/i, x/t, y'/o, f/f_0, f'/f] \Rightarrow Q[x'/x, y'/y, f'/f]
  \]

- \(sp(P, p(e, x, y)) \iff\)
  \[
  \exists x_0, y_0, f_0 : P[x_0/x, y_0/y, f_0/f] \land \\
  Post[e[x_0/x, y_0/y, f_0/f]/i, x_0/t, x/t, y/o]
  \]

Explicit naming of old/new values required.

Procedure Calls

From this, we can derive a rule for the correctness of procedure calls.

\[
\{D(e) \land Pre[e/i, x/t] \land \\
\forall x', y', f' : Post[e/i, x/t, y'/o, f/f_0, f'/f] \Rightarrow Q[x'/x, y'/y, f'/f]\} \\
p[e, x, y] \\
\{Q\}
\]

- \(Pre[e/i, x/t]\) refers to the values of the actual arguments \(e\) and \(x\) (rather than to the formal parameters \(i\) and \(t\)).
- \(x', y', f'\) denote the values of the vars \(x\), \(y\), and \(f\) after the call.
- \(Post[\ldots]\) refers to the argument values before and after the call.
- \(Q[x'/x, y'/y, f'/f]\) refers to the argument values after the call.

Modular reasoning: rule only relies on the specification of \(p\), not on its implementation.

Procedure Calls Example

- Procedure specification:
  \[
  \text{global } f \\
  \text{requires } f \geq 0 \land i > 0 \\
  \text{ensures } f_0 = f \cdot i + o \land 0 \leq o < i \\
  \text{dividesF}(i, o)
  \]

- Procedure call:
  \[
  \{f \geq 0 \land f = N \land b \geq 0\} \\
  \text{dividesF}(b + 1, y) \\
  \{f \cdot (b + 1) \leq N < (f + 1) \cdot (b + 1)\}
  \]

- To be ultimately proved:
  \[
  f \geq 0 \land f = N \land b \geq 0 \Rightarrow \\
  D(b + 1) \land f \geq 0 \land b + 1 > 0 \land \\
  \forall y', f' : \\
  f = f' \cdot (b + 1) + y' \land 0 \leq y' < b + 1 \Rightarrow \\
  f' \cdot (b + 1) \leq N < (f' + 1) \cdot (b + 1)
  \]
Not Yet Covered

- Primitive data types.
  - int values are actually finite precision integers.
- More data and control structures.
  - switch, do-while (easy); continue, break, return (more complicated).
  - Records can be handled similar to arrays.
- Recursion.
  - Procedures may not terminate due to recursive calls.
- Exceptions and Exception Handling.
  - Short discussion in the context of ESC/Java2 later.
- Pointers and Objects.
  - Here reasoning gets complicated.
- ...

The more features are covered, the more complicated reasoning becomes.