Term Algebras

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Term Algebra

Take signature $\Sigma = (S, \Omega)$.

- Term algebra $T(\Sigma)$:
  - Σ-algebra whose carriers are Σ-*terms*.
  - $T(\Sigma)(s) = T_{\Sigma,s}$, for every $s \in S$.
  - $T(\Sigma)(\omega) = n$
    - for every $\omega = (n :\to s) \in \Omega$.
  - $T(\Sigma)(\omega)(t_1, \ldots, t_k) = n(t_1, \ldots, t_k)$
    - for every $\omega = (n : s_1 \times \ldots \times s_k \to s) \in \Omega$, $t_i \in T(\Sigma)(s_i)$.

$T(\Sigma)$ is the algebra of (well-typed) ground terms of Σ.

Term Algebras

- Example: $\mathrm{NAT} = (\{nat\}, \{0 :\rightarrow nat, Succ : nat \rightarrow nat\})$.
  - $T(\mathrm{NAT})(nat) = \{0, Succ(0), Succ(Succ(0)), \ldots\}$.
  - $T(\mathrm{NAT})(0) = 0$.
  - $T(\mathrm{NAT})(Succ)(t) = Succ(t)$, for every $t \in T(\mathrm{NAT})(nat)$.
- Term value $T(\Sigma)(t) = t$, for every ground term $t \in T(\Sigma)$.
  - A ground term denotes itself.
- $T(\Sigma)$ is freely generated.
  - Generated: every carrier is denoted by itself.
  - Free: two different ground terms denote two different carriers.

In a term algebra, a ground term and its interpretation coincide.

Initiality

Take signature Σ, class $\mathcal{C} \subseteq Alg(\Sigma)$ of Σ-algebras, and Σ-algebra $A \in \mathcal{C}$.

- A is initial in $\mathcal{C}$ if
  - for every $B \in \mathcal{C}$, there exists exactly one homomorphism $h : A \rightarrow B$.
  - A distinguishes most among all algebras of $\mathcal{C}$.
- Initial algebras are unique up to isomorphism:
  - If A is initial in $\mathcal{C}$, then B is initial in $\mathcal{C}$ iff $A \simeq B$.
- Theorem: $T(\Sigma)$ is initial in $Alg(\Sigma)$.
  - For every $A \in Alg(\Sigma)$, there exists the unique evaluation homomorphism:

$$h : T(\Sigma) \rightarrow A$$
$$h(t) := A(t), \text{ for every ground term } t \in T_\Sigma.$$

The term algebra $T(\Sigma)$ distinguish most among all Σ-algebras.

Congruence Relation

Take signature $\Sigma = (S, \Omega)$, Σ-algebra A.

- Congruence relation $Q = (Q_s)_{s \in S}$ on A:
  - Q_s is an equivalence relation on $A(s)$ for every $s \in S$.
  - $(a_1, a'_1) \in Q_{s_1} \wedge \ldots \wedge (a_k, a'_k) \in Q_{s_k} \Rightarrow$
    $(A(\omega)(a_1, \ldots, a_k), A(\omega)(a'_1, \ldots, a'_k)) \in Q_s$
    - for every $w = (n : s_1 \times \ldots \times s_k \to s) \in \Omega$, and
    - for every $a_1, a'_1 \in A(s_1), \ldots, a_k, a'_k \in A(s_k)$.
  - Equivalent arguments yield equivalent results.

A congruence relation preserves equivalence across function applications.

Example

- BOOL-algebra D:

$$D(bool) = \mathbb{N}$$

$$D(\neg)(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{otherwise} \end{cases}$$

$$D(\wedge)(n,m) = n * m$$

- Q is a congruence relation on D.

$$(m,n) \in Q_{bool} :\Leftrightarrow m+n \text{ is even.}$$

  - Take $\omega = \neg : bool \rightarrow bool$:
    - Take $n, n' \in D(bool)$ with $(n, n') \in Q_{bool}$.
    - We have to show $(D(\neg)(n), D(\neg)(n')) \in Q_{bool}$.
    - $n + n'$ is even. Thus n and n' are either both even or both odd.
    - Case 1: we have to show $(n+1, n'+1) \in Q_{bool}$, i.e., $(n+1)+(n'+1) = (n+n')+2$ is even. ...
    - Case 2: we have to show $(n-1, n'-1) \in Q_{bool}$, i.e., $(n-1)+(n-1) = (n+n')-2$ is even. ...
  - Take $\omega = \wedge : bool \times bool \rightarrow bool$:
    - ...

Quotient Algebra

Take signature $\Sigma = (S, \Omega)$, Σ-algebra A, congruence relation Q on A.

- Quotient (algebra) $A/_Q$ of A by Q:
  - Σ-algebra whose carriers are congruence classes.
    - $[a]_Q = \{a' : (a, a') \in Q\}$.
    - Class of a with respect to congruence relation Q.
  - $A/_Q(s) = \{[a]_{Q_s} | a \in A(s)\}$
    - for every $s \in S$.
  - $A/_Q(\omega) = [A(\omega)]_{Q_s}$
    - for every $\omega = (n :\to s) \in \Omega$.
  - $A/_Q(\omega)([a_1]_{Q_{s_1}}, \ldots, [a_k]_{Q_{s_k}}) = [A(\omega)(a_1, \ldots, a_k)]_{Q_s}$
    - for every $\omega = (n : s_1 \times \ldots \times s_k \to s) \in \Omega$.

Congruent elements of A are combined to a single element of $A/_Q$.

Example

- BOOL-algebra D and congruence relation Q on D (as before).

  $(m, n) \in Q_{bool} :\Leftrightarrow m + n \text{ is even.}$
- Quotient algebra $D/_Q$:

  $[0] = \{n \in \mathbb{N} \mid 0 + n \text{ is even}\} = \{n \in \mathbb{N} \mid n \text{ is even}\}$

  $[1] = \{n \in \mathbb{N} \mid 1 + n \text{ is even}\} = \{n \in \mathbb{N} \mid n \text{ is odd}\}$
  - $(D/_Q)(\text{bool}) = \{[0], [1]\}$.
  - $(D/_Q)(\neg)(n) = \begin{cases} [1] & \text{if } n = [0] \\ [0] & \text{if } n = [1] \end{cases}$
  - $(D/_Q)(\wedge)(n, m) = \begin{cases} [1] & \text{if } n = m = [1] \\ [0] & \text{else} \end{cases}$
- $(D/_Q) \simeq C$

  $C(bool) = \{0, 1\}$

  $C(True) = 1$

  $C(False) = 0$

  $C(\neg)(n) = 1 - n$

  $C(\wedge)(n, m) = n * m$

Quotient Term Algebra

Take signature $\Sigma = (S, \Omega)$ and class of algebras $\mathcal{C} \subseteq Alg(\Sigma)$.

- Congruence relation $\equiv_\mathcal{C}$ of $\mathcal{C}$:
  - $\equiv_\mathcal{C} := (\equiv_{\mathcal{C},s})_{s \in S}$.
  - $\equiv_{\mathcal{C},s} := \{(t, u) \in T_{\Sigma,s} \times T_{\Sigma,s} \mid \forall A \in \mathcal{C} : A(t) = A(u)\}$.
  - All ground terms are congruent that have the same value in all algebras of $\mathcal{C}$.
- Quotient Term Algebra $T(\Sigma, \mathcal{C})$ of $\mathcal{C}$:
  - $T(\Sigma, \mathcal{C}) := T(\Sigma)/_{\equiv_\mathcal{C}}$.
  - Σ-algebra whose carrier are congruence classes of ground terms of Σ.
- Theorem: If $T(\Sigma, \mathcal{C}) \in \mathcal{C}$, then $T(\Sigma, \mathcal{C})$ is initial in $\mathcal{C}$.
  - For every $A \in \mathcal{C}$, there exists the unique evaluation homomorphism:

$$h : T(\Sigma, \mathcal{C}) \to A$$
$h([t]) := A(t)$, for every ground term $t \in T_\Sigma$.

$T(\Sigma, \mathcal{C})$ relates similarly to $\mathcal{C}$ as $T(\Sigma)$ relates to $Alg(\Sigma)$.

Examples

- $T(\Sigma, Alg(\Sigma)) \simeq T(\Sigma)$.
  - Carriers of $T(\Sigma, Alg(\Sigma))$ are singletons $[t] = \{t\}$ for every ground term $t \in T_\Sigma$.
- $T(\Sigma, \{A\}) \simeq A$, for every Σ-algebra A.
  - Carriers of $T(\Sigma, \{A\})$ are classes of all those terms that denote the same carrier in A.
- Let B be the "classical" $\mathrm{NATBOOL}$-algebra.
  - Terms *True* and $\neg$*False* belong to the same carrier of $T(\Sigma, \{B\})$.
  - Terms 0 and $0+0$ belong to the same carrier of $T(\Sigma, \{B\})$.

Quotient Term Algebra of a Set of Formulas

Take logic L, signature Σ, set of formulas $\Phi \subseteq L(\Sigma)$.

- Quotient term algebra $T(\Sigma, \Phi)$ of Φ:
  - $T(\Sigma, \Phi) := T(\Sigma, Mod_\Sigma(\Phi))$ $(= T(\Sigma)/_{\equiv_{Mod_\Sigma(\Phi)}})$.
    - $Mod_\Sigma(\Phi) = \{A \in Alg(\Sigma) \mid A \text{ is a model of } \Phi\}$.
    - $\equiv_{Mod_\Sigma(\Phi),s} = \{(t, u) \in T_{\Sigma,s} \times T_{\Sigma,s} \mid \forall A \in Mod_\Sigma(\Phi) : A(t) = A(u)\}$.
  - Σ-algebra whose carriers are classes of those terms that have the same value in all models of Φ.
- Theorem: If $T(\Sigma, \Phi)$ is model of Φ, $T(\Sigma, \Phi)$ is initial in $Mod_\Sigma(\Phi)$.
  - For every model A of Φ, there exists the unique evaluation homomorphism:

$$h : T(\Sigma, \Phi) \to A$$
$$h([t]) := A(t), \text{ for every ground term } t \in T_\Sigma.$$

Basis of initial specification semantics.