More on Functions 2

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Overview

- Sequences and Series
- Special Functions
- Asymptotic Bounds

Sequences and Series

Sequence Quantor

Definition: For every variable i and term T, the phrase

$$[T]_i$$

is a term with bound variable i whose value is the sequence

$$[T]_i : \mathbb{N} \to \mathbb{R}$$
$$[T]_i(i) := T.$$

Example: $[a^2 + c]_a = [0 + c, 1 + c, 4 + c, 9 + c, \ldots].$

Monotonicity

Definition: Let f be an infinite sequence over $\mathbb{R}$. f is monotonically increasing if every element of f is less than or equal the next element:

$$\begin{aligned}&f \text{ is monotonically increasing } :\Leftrightarrow \\ &\quad f : \mathbb{N} \to \mathbb{R} \land \forall i \in \mathbb{N} : f_i \leq f_{i+1}.\end{aligned}$$

f is strictly monotonically increasing if every element of f is less than the next element:

$$\begin{aligned}&f \text{ is strictly monotonically increasing } :\Leftrightarrow \\ &\quad f : \mathbb{N} \to \mathbb{R} \land \forall i \in \mathbb{N} : f_i < f_{i+1}.\end{aligned}$$

Monontonicity (Continued)

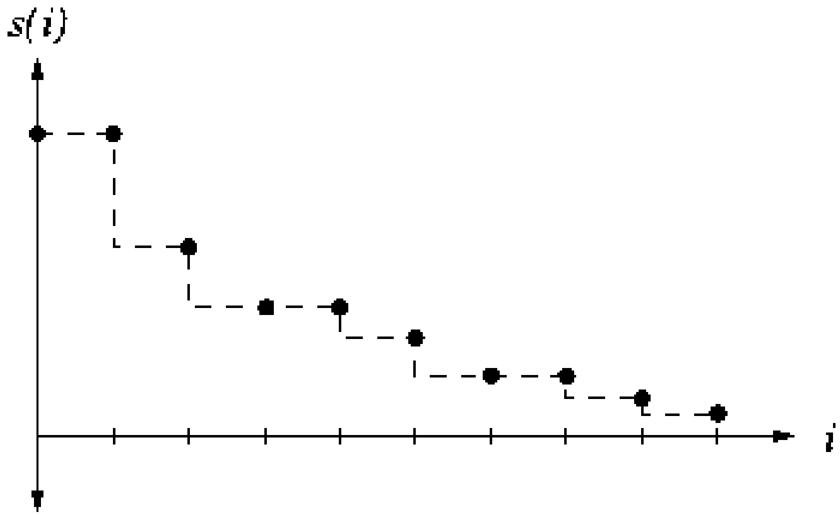
Definition: f is monotonically decreasing if every element of f is greater than or equal the next element:

$$
\begin{aligned}
&f \text{ is monotonically decreasing } :\Leftrightarrow \\
&\quad f : \mathbb{N} \to \mathbb{R} \land \forall i \in \mathbb{N} : f_i \geq f_{i+1}.
\end{aligned}
$$

f is strictly monotonically decreasing if every element of f is greater than the next element:

$$
\begin{aligned}
&f \text{ is strictly monotonically decreasing } :\Leftrightarrow \\
&\quad f : \mathbb{N} \to \mathbb{R} \land \forall i \in \mathbb{N} : f_i > f_{i+1}.
\end{aligned}
$$

Illustration



Sequence is (not strictly) monotonically decreasing.

Example

- $[\frac{1}{i+1}]_i$ is strictly monotonically decreasing:

$$[\frac{1}{i+1}]_i = [1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots]$$

- $[i \text{ div } 2]_i$ is monotonically increasing:

$$[i \text{ div } 2]_i = [0, 0, 1, 1, 2, 2, \ldots]$$

- $[(-1)^i]_i$ is neither monotonically increasing nor decreasing:

$$[(-1)^i]_i = [1, -1, 1, -1, \ldots]$$

Bounds

Definition: Let f be an infinite sequence over $\mathbb{R}$.

f has upper bound U, if every element of f is less than or equal U:

$$\begin{array}{l} U \text{ is upper bound of } f :\Leftrightarrow \\ \quad f : \mathbb{N} \to \mathbb{R} \land \forall i \in \mathbb{N} : f_i \leq U. \end{array}$$

f has lower bound L, if every elem. of f is greater than or equal L:

$$\begin{array}{l} L \text{ is lower bound of } f :\Leftrightarrow \\ \quad f : \mathbb{N} \to \mathbb{R} \land \forall i \in \mathbb{N} : f_i \geq L. \end{array}$$

Supremum and Infimum

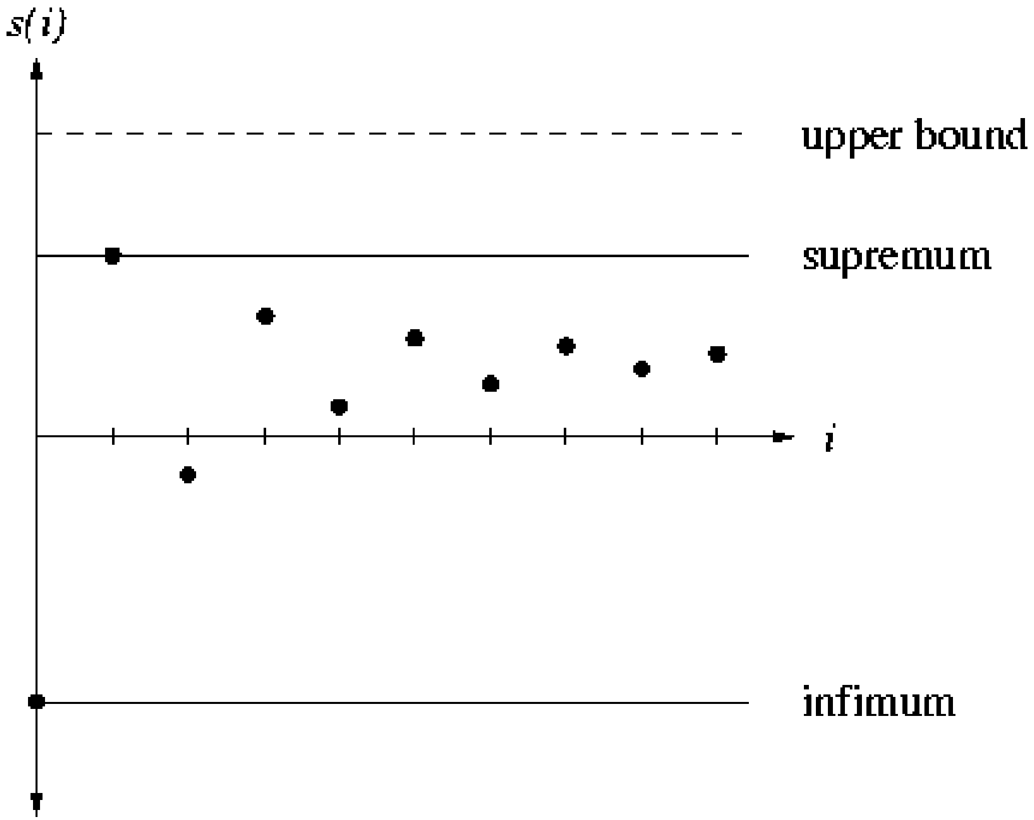
Definition: The supremum of f is the smallest upper bound of f:

$$
\begin{aligned}
&\sup(f) := \textbf{such } S : \\
&\quad S \text{ is upper bound of } f \land \\
&\quad (\forall S' : S' \text{ is upper bound of } f \Rightarrow S \leq S').
\end{aligned}
$$

The infimum of f is the greatest lower bound of f:

$$
\begin{aligned}
&\inf(f) := \textbf{such } I : \\
&\quad I \text{ is lower bound of } f \land \\
&\quad (\forall I' : I' \text{ is lower bound of } f \Rightarrow I \geq I').
\end{aligned}
$$

Illustration



Unicity of Supremum

Proposition:

$$\forall f, S_0, S_1 : \\ S_0 \text{ is supremum of } f \wedge S_1 \text{ is supremum of } f \Rightarrow S_0 = S_1.$$

where

$$S \text{ is supremum of } f :\Leftrightarrow \\ S \text{ is upper bound of } f \wedge \\ (\forall S' : S' \text{ is upper bound of } f \Rightarrow S \leq S')$$

i.e., $\sup(f) =$ **such** $S : S$ is supremum of f.

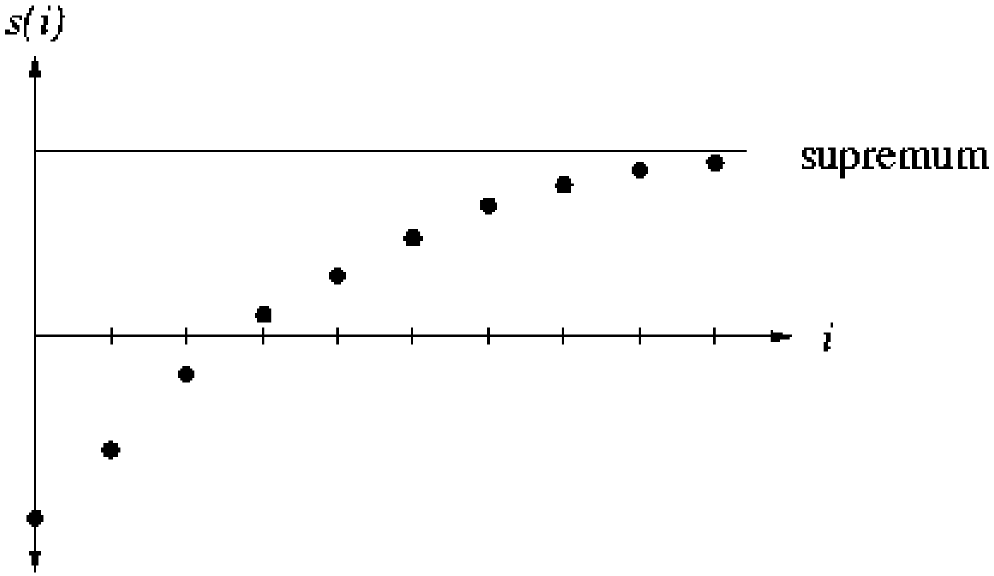
Proof: Take arbitrary f and suprema S_0 and S_1 of f. Since S_0 is a supremum and S_1 is an upper bound of f, we have $S_0 \leq S_1$. Conversely, since S_1 is a supremum and S_0 is an upper bound of f, we have $S_1 \leq S_0$. Since $S_0 \leq S_1$ and $S_1 \leq S_0$, we have $S_0 = S_1$.

Example

- $[i]_i$ has infimum 0 but no upper bound.
- $[1 - i]_i$ has supremum 1 but no lower bound.
- $[\frac{1}{i+1}]_i$ has supremum 1 and infimum 0.
- $[i^3 * (-1)^i]_i$ has no upper bound and no lower bound.

Infimum and supremum need not exist.

Illustration



Infimum and supremum need not be among sequence elements.

Convergence and Limit

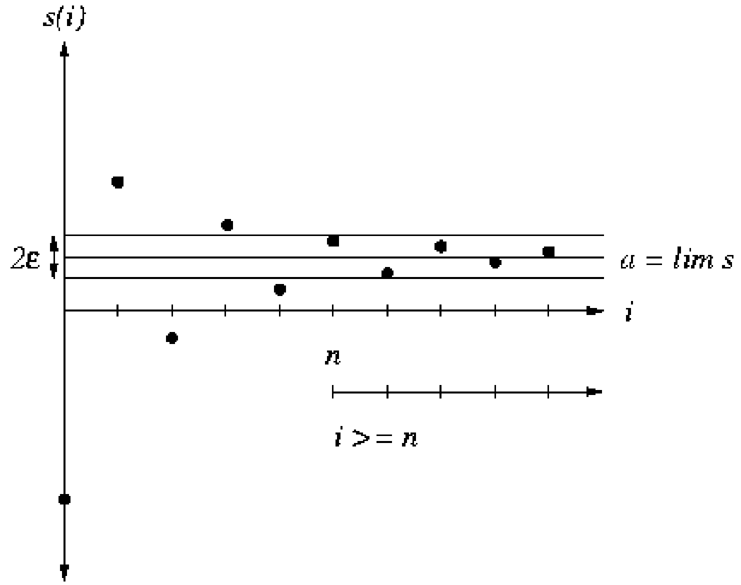
Definition: An infinite sequence s over $\mathbb{R}$ converges to limit a, if its members approach a arbitrarily close:

$$
\begin{aligned}
&s \text{ converges to } a :\Leftrightarrow \\
&\quad \forall \epsilon > 0 : \exists n \in \mathbb{N} : \forall i \geq n : |s_i - a| < \epsilon; \\
&\lim(s) := \\
&\quad \textbf{such } a : s \text{ converges to } a.
\end{aligned}
$$

A non-convergent series is called divergent (divergent):

$$s \text{ is divergent} :\Leftrightarrow \neg\exists a : s \text{ converges to } a.$$

Illustration



For every $\epsilon > 0$, all members of a convergent sequence are eventually in an "ϵ-tunnel" around limit a, i.e., in the interval $]a - \epsilon, a + \epsilon[$.

Example

Let $s := [(-1)^i * \frac{1}{i}]_i$. We show that s converges to 0.
Take arbitrary $\epsilon > 0$. We have to find some $n \in \mathbb{N}$ such that $\forall i \geq n :$ $|(-1)^i * \frac{1}{i} - 0| < \epsilon$ which can be simplified to

$$\forall i \geq n : \frac{1}{i} < \epsilon.$$

Take $n :=$ **such** $n \in \mathbb{N} : \frac{1}{\epsilon} < n$. Because $\epsilon > 0$, we know $1/\epsilon \in \mathbb{R}_{>0}$. Because $\mathbb{N}$ is unbounded (i.e., $\forall r \in \mathbb{R} : \exists n \in \mathbb{N} : r < n$), we know $\frac{1}{\epsilon} < n$ and thus $n > 0$. Take arbitrary $i \geq n$. We know $i > 0$ and

$$\frac{1}{i} \leq \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

Example

We show that $s := [(-1)^i]_i$ is divergent. Assume that s is convergent, i.e., s converges to some limit $a \in \mathbb{R}$. We show a contradiction.
Let $\epsilon := \frac{1}{2}$. There exists (by definition of convergence) some $n \in \mathbb{N}$ such that $\forall i \geq n : |(-1)^i - a| < \frac{1}{2}$. We thus have

$$|1 - a| < \tfrac{1}{2} \wedge |-1 - a| < \tfrac{1}{2}$$

We then have (using the absolute value laws)

$$\begin{aligned} & 1 = \tfrac{1}{2} + \tfrac{1}{2} > |1 - a| + |-1 - a| \\ & = |1 - a| + |1 + a| \geq |(1 - a) + (1 + a)| = 2 \end{aligned}$$

which represents a contradiction.

Limit Quantor

Definition: For every variable i and term T, the following phrase represents a term with bound variable i:

$$\lim_{i \to \infty} T$$

The value of this term is that of the term

$$\lim([T]_i).$$

Further Results: see lecture notes.

Series

Definition: Let a be an infinite sequence over $\mathbb{R}$. The series corresponding to a is the sequence where every element s_n is the sum of the first $n+1$ elements of a:

$$\begin{aligned} &\mathrm{series} : (\mathbb{N} \to \mathbb{R}) \to (\mathbb{N} \to \mathbb{R}) \\ &\mathrm{series}(a)_n := \textstyle\sum_{0 \le i \le n} a_i. \end{aligned}$$

Consequence: If $a = [T]_i$, then $\mathrm{series}(a) = [\sum_{0 \le i \le n} T]_n$.

Sequence of (partial) sequence sums.

Example

Let $a = [i^2]_i$:

$$a = [0, 1, 4, 9, 16, 25, \ldots].$$

Then $\mathrm{series}(a) = [\sum_{0 \leq i \leq n} i^2]_n$:

$$\mathrm{series}(a) = [0, 1, 5, 14, 30, 55, \ldots].$$

Arithmetic Series

For every $c \in \mathbb{R}$, the sequence $[i * c]_i$ is called an arithmetic sequence. Correspondingly, $\text{series}([i * c]_i)$ is called an arithmetic series.

We have, for every $n \in \mathbb{N}$,

$$\text{series}([i * c]_i)_n = \sum_{0 \leq i \leq n} i * c = c * \frac{n(n+1)}{2}.$$

(which can be proved by induction on n).

For instance, for $c = 1$, we have

$$\begin{aligned} [i]_i &= [0, 1, 2, 3, 4, 5, \ldots], \\ \text{series}([i]_i) &= [0, 1, 3, 6, 10, 15, \ldots]. \end{aligned}$$

Geometric Series

For every $q \in \mathbb{R}$, the sequence $[q^i]_i$ is called a geometric sequence. Correspondingly, $\text{series}([q^i]_i)$ is called a geometric series.

We have, for every $n \in \mathbb{N}$,

$$\text{series}([q^i]_i)_n = \sum_{0 \leq i \leq n} q^i = \frac{q^n - 1}{q - 1}.$$

(which can be proved by induction on n).

For instance, for $q = 2$, we have

$$\begin{aligned} [2^i]_i &= [1, 2, 4, 8, 16, \ldots], \\ \text{series}([2^i]_i) &= [1, 3, 7, 15, 31]. \end{aligned}$$

Series Limit Quantor

Definition: For every variable i and term T, the following phrase denotes a term with bound variable i:

$$\sum_{i=0}^{\infty} T$$

The value of this term is

$$\lim_{n\to\infty} \sum_{0\leq i\leq n} T.$$

$\sum_{i=0}^{\infty} T$ is only well defined if $[\sum_{0\leq i\leq n} T]_n$ is convergent.

Limit of Geometric Series

Proposition: For $q \in \mathbb{R}$, $|q| < 1$, $[\sum_{0 \leq i \leq n} q^i]_n$ converges to $\frac{1}{1-q}$:

$$\forall q \in \mathbb{R} : |q| < 1 \Rightarrow \sum_{i=0}^{\infty} q^i = \frac{1}{1-q}.$$

Proof: Take arbitrary $q \in \mathbb{R}$ with $|q| < 1$. We then have

$$\begin{aligned}
\sum_{i=0}^{\infty} q^i &= \\
\lim_{n\to\infty} \sum_{0 \leq i \leq n} T &= \\
\lim_{n\to\infty} \frac{q^n-1}{q-1} &= \\
\frac{\lim_{n\to\infty}\ (q^n-1)}{\lim_{n\to\infty}\ (q-1)} &= \\
\frac{(\lim_{n\to\infty}\ q^n)-(\lim_{n\to\infty}\ 1)}{q-1} &= (*) \\
\frac{0-1}{q-1} &= \\
\frac{1}{1-q}.
\end{aligned}$$

(*) The fact $\lim_{n\to\infty}\ q^n = 0$ has to be shown in a separate proof.

Special Functions

Pointwise Function Definition

Definition: Let $\text{RealFun} = \mathbb{R} \rightarrow \mathbb{R}$.

$$
\begin{aligned}
&. : \mathbb{R} \rightarrow \text{RealFun}\\
&c(x) := c;\\
\\
&+ : (\text{RealFun} \times \text{RealFun}) \rightarrow \text{RealFun}\\
&(f+g)(x) := f(x) + g(x);\\
&(f-g)(x) := f(x) - g(x);\\
&(f*g)(x) := f(x) * g(x);\\
&(\tfrac{f}{g})(x) := \tfrac{f(x)}{g(x)};\\
\\
&.^{\cdot} : (\text{RealFun} \times \mathbb{Z}) \rightarrow (\mathbb{R} \xrightarrow{\text{partial}} \mathbb{R})\\
&(f^n)(x) := f(x)^n;\\
&\sqrt{\ } : (\mathbb{Z} \times \text{RealFun}) \rightarrow (\mathbb{R} \xrightarrow{\text{partial}} \mathbb{R})\\
&(\sqrt[n]{f})(x) := \sqrt[n]{f(x)};
\end{aligned}
$$

Example

- Let $f(x) := 2x^2$ and $g(y) := y + 4$. Then, for every $x \in \mathbb{R}$,
$$(\sqrt{f+g})(x) = \sqrt{2x^2 + x + 4}.$$
- The function $f(x) := 3x^2 + x\sqrt{x}$ equals
$$3 * g + h$$
where $g(x) := x^2$ and $h(x) = x\sqrt{x}$.
- The function $\frac{1_{\mathbb{R}}{}^2 + \sqrt{\ }}{1_{\mathbb{R}} + 1}$ equals $f(x) := \frac{x^2+\sqrt{x}}{x+1}$.

Composition of functions at each argument value.

Floor and Ceiling

Definition: The floor of a real number x is the largest integer less than or equal x.

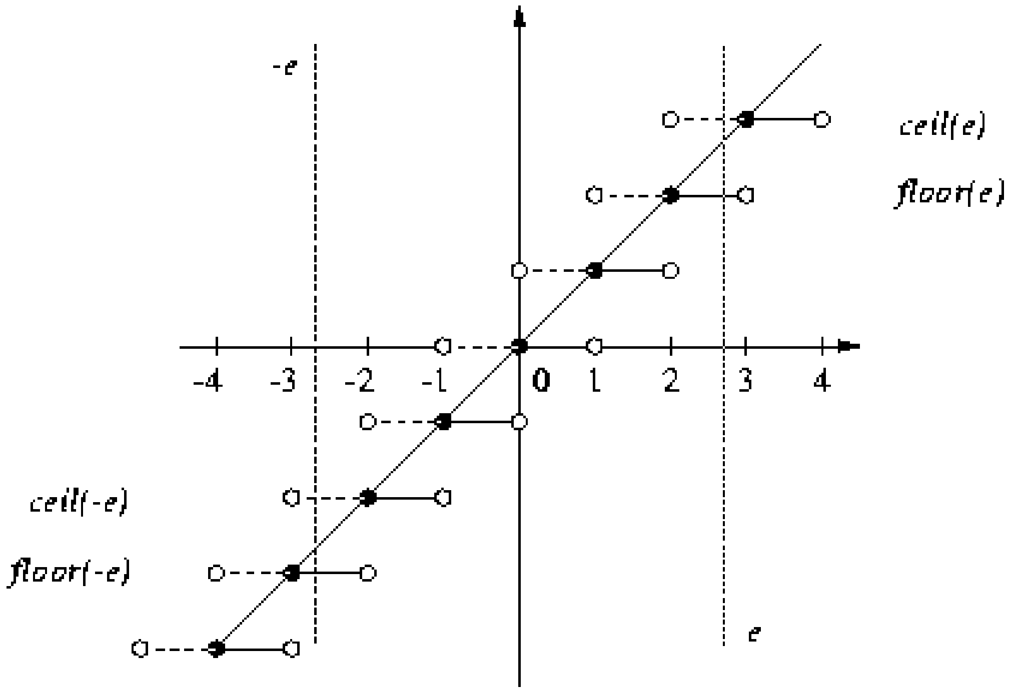
$$\lfloor x \rfloor := \max\{y \in \mathbb{Z} : y \leq x\}.$$

Analogously, the ceiling of x is the smallest integer greater than or equal x:

$$\lceil x \rceil := \min\{y \in \mathbb{Z} : y \geq x\}.$$

Functions mapping real numbers to integers.

Illustration



Floor and Ceiling Rules

Proposition: For every $x \in \mathbb{R}$, we have:

$$
\begin{aligned}
&x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1, \\
&\lfloor x \rfloor = x \Leftrightarrow x \in \mathbb{Z} \Leftrightarrow \lceil x \rceil = x, \\
&\lfloor -x \rfloor = -\lceil x \rceil, \lceil -x \rceil = -\lfloor x \rfloor.
\end{aligned}
$$

1. Floor lies on or below the first diagonal; ceiling lies above diagonal; if we shift diagonal down one unit, it lies completely below the floor; if we shift it up one unit, it lies completely above the ceiling.
2. Floor and ceiling intersect each other at the diagonal.
3. Floor and ceiling are reflections of each other about both axes.

Floor and Ceiling Rules

Proposition: For every $x \in \mathbb{R}$ and $i \in \mathbb{Z}$, we have:

$$\lfloor x + i \rfloor = \lfloor x \rfloor + i,$$
$$\lceil x + i \rceil = \lceil x \rceil + i.$$

$$x < i \Leftrightarrow \lfloor x \rfloor < i,$$
$$i < x \Leftrightarrow i < \lceil x \rceil,$$
$$x \leq i \Leftrightarrow \lceil x \rceil \leq i,$$
$$i \leq x \Leftrightarrow i \leq \lfloor x \rfloor.$$

May shift integer terms out of a floor or ceiling; may get rid of floor and ceiling under some assumptions.

Truncation

Definition: The truncated part of a real number is the number without its fractional part:

$$\mathrm{trunc}(x) := \textbf{if } x < 0 \textbf{ then } \lceil x \rceil \textbf{ else } \lfloor x \rfloor.$$

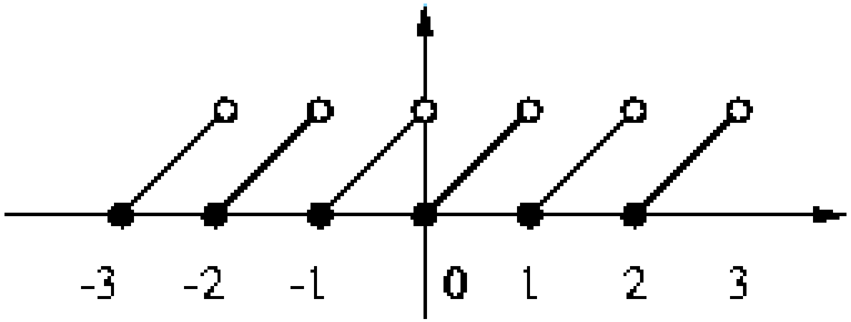
Proposition: For every $x \in \mathbb{R}$, the truncated part of the negation of x is the negation of the truncated part of x:

$$\forall x \in \mathbb{R} : \mathrm{trunc}(-x) = -\mathrm{trunc}(x).$$

Not many nice other properties.

Example

The graph of $f(x) := x - \lfloor x \rfloor$ is depicted by



Many interesting functions can be defined by floor and ceiling.

Periodic Functions

Definition: A function has period a, if the function values are repeated in intervals of width a:

$$\begin{aligned}&f \text{ has period } a :\Leftrightarrow \\ &\quad f : \mathbb{R} \to \mathbb{R} \wedge a \in \mathbb{R} \wedge \forall x \in \mathbb{R} : f(x+a) = f(x).\end{aligned}$$

A function is periodic if it has some period:

$$f \text{ is periodic } :\Leftrightarrow \exists a \in \mathbb{R} : f \text{ has period } a.$$

Example: f in previous example has period 1.

Real Remainder

Definition:

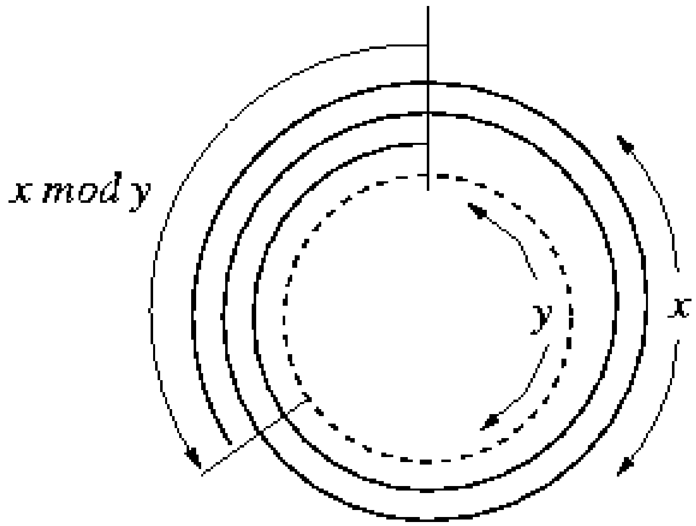
$$\begin{aligned}&\mathrm{mod} : (\mathbb{R} \times \mathbb{R}) \to \mathbb{R}\\&x \bmod y := x - y * \lfloor x/y \rfloor.\end{aligned}$$

Proposition: We have, for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $y \neq 0$,

$$x = y * \lfloor x/y \rfloor + x \bmod y.$$

Division and remainder on $\mathbb{R}$.

Example



Unwind a line of length x around a circle of circumference y.

Remainder Laws

Proposition: For every $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we have

$$y > 0 \Rightarrow (0 \leq x \bmod y < y),$$
$$y < 0 \Rightarrow (0 \geq x \bmod y > y).$$

- $\lfloor 5.44/4 \rfloor = 1$; $5.44 \bmod 4 = 5.44 - 4 * 1 = 1.44$.
- $\lfloor -5.44/4 \rfloor = -2$; $-5.44 \bmod 4 = -5.44 - 4 * (-2) = 2.56$.
- $\lfloor 5.44/-4 \rfloor = -2$; $5.44 \bmod -4 = 5.44 - (-4) * (-2) = -2.56$.
- $\lfloor -5.44/-4 \rfloor = 1$; $-5.44 \bmod -4 = -5.44 - (-4) * 1 = -1.44$.

More Real Functions

- Polynomial functions

$$p(x) := \sum_{0 \le i \le n} a_i x^i$$

- Rational functions

$$r(x) := \frac{p(x)}{q(x)}$$

- Exponentiation and natural logarithm

$$\exp(x) := \textstyle\sum_{i=0}^{\infty} \frac{x^i}{i!};\ \ln(x) := \exp^{-1}(x).$$

- Trigonometric functions

$$\sin, \cos, \tan, \cot.$$

See lecture notes.

Asymptotic Bounds

Motivation

- Detailed execution time of an algorithm for input size n.

$$7n^2 + 3n + 19$$

- "Essential" growth of the execution time with increasing n.

$$n^2$$

- Constant factors and smaller sums ignored.

Capture the "essential" growth of a function.

Big O Quantor

Definition: For every variable n and terms S and T, the phrase

$$S = \mathrm{O}_n(T)$$

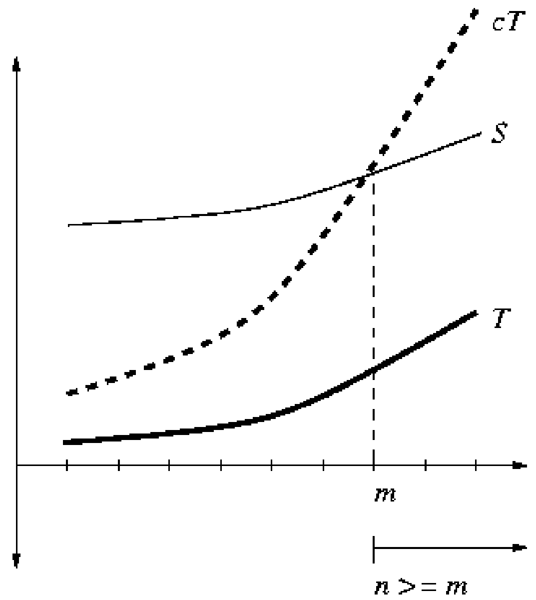
is a proposition with bound variable n which is read as "S is big O of T" or "T asymptotically dominates S".

Its meaning is equivalent to the proposition

$$\exists c \in \mathbb{R}, m \in \mathbb{N} : \forall n \geq m : |S| \leq c * |T|.$$

Usually the subscript n is dropped and the bound variable has to be deduced from the context.

Illustration



From a certain point m on and scaled by some factor c, the absolute value of T is at least as large as the absolute value of S.

Usage

- Instead of saying that the execution time T of some algorithm A in dependence on input size n is $T_A(n) = 7n^2 + 3n + 19$, we may say
$$T_A(n) = \mathrm{O}(n^2),$$
- The proposition
$$S_0 = S_1 + \mathrm{O}(T)$$
is to be understood as $S_0 - S_1 = \mathrm{O}(T)$.

  We say by
$$f(n) = 5n^3 + 2n^2 + \mathrm{O}(n)$$
that $f(n)$ differs from $5n^3 + 2n^2$ not more than by a linear factor.

Example

We have $10n + 100 = \mathrm{O}(n)$.

Proof: Let $c := 110$ and $m := 1$ and take arbitrary $n \geq m$. We show

$$|10n + 100| \leq c * |n|.$$

We know

$$|10 * n + 100| = 10 * n + 100 \leq 110n = 110|n| = c|n|.$$

Example

We have not $n^2 = \mathrm{O}(n)$.

Proof: We suppose $n^2 = \mathrm{O}(n)$ and show a contradiction. By the assumption, we have some $c \in \mathbb{R}$ and $m \in \mathbb{N}$ with

$$\forall n \geq m : |n^2| \leq c|n|.$$

If $c < 0$, let $k := \max(1, m)$; then we have $k \geq m$ but

$$|k^2| > 0 > c|k|$$

If $c \geq 0$, let $k := \max(m, \lceil c+2 \rceil)$. Then we have $k \geq m$ but

$$\begin{aligned} |k^2| &= k^2 \geq \lceil c+2 \rceil^2 \geq (c+2)^2 = c^2 + 4c + 4 \\ &> c^2 + 3c = c(c+3) \geq c\lceil c+2 \rceil \geq ck = c|k| \end{aligned}$$

Example

We have $n^2 = \mathrm{O}(2^n)$:

Proof: We assume not $n^2 = \mathrm{O}(2^n)$ and show a contradiction. By assumption, we have $\neg\exists c \in \mathbb{R}, m \in \mathbb{N} : \forall n \geq m : |n^2| \leq c|2^n|$, i.e.,

$$\forall c \in \mathbb{R}, m \in \mathbb{N} : \neg\forall n \geq m : |n^2| \leq c|2^n|.$$

We prove

$$\forall n \geq 0 : n^2 \leq 2 * 2^n$$

by induction on n and thus have a contradiction.

See lecture notes.

O Manipulation

See lecture notes.

Asymptotic of Polynomial Sequences

Proposition: A polynomial sequence of degree n is dominated by $[x^n]_x$:

$$\forall n \in \mathbb{N}, a : \mathbb{N}_n \to \mathbb{R} :$$
$$\textstyle\sum_{0 \leq i \leq n} a x^i = \mathrm{O}_n(x^n).$$

Only the highest exponent matters.

Asymptotic Classes

Proposition: For every variable n and terms S and T, the phrase

$$S \prec_n T$$

is a proposition with bound variable n which is read as "S is strictly dominated by T" and that is equivalent to

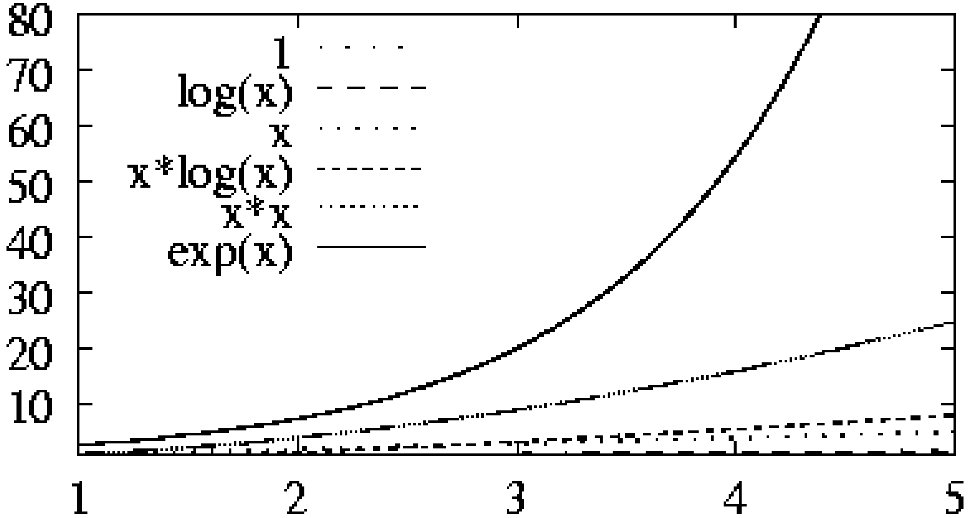
$$S = \mathrm{O}_n(T) \wedge T \neq O_n(S).$$

Means to compare the growth of classes of functions.

Asymptotic Classes

Proposition: For every $k \in \mathbb{N}$, $c \in \mathbb{R}$, and $a \in \mathbb{R}_{>0}$, we have:

$$1 \prec_n \log_a(n) \prec_n n \prec_n n\log_a(n) \prec_n n^k \prec_n c^n \prec_n n!$$



Comparison of Algorithms

Largest problem that can be solved in fixed time.

Execution Steps	1 second	1 minute	1 hour
$\log_2(n)$	2^{10^6}	2^{6*10^7}	2^{36*10^8}
n	10^6	$6*10^7$	$3.6*10^9$
$n\log_2(n)$	62746	$2.8*10^6$	$1.3*10^8$
n^2	1000	7746	60000
2^n	23	26	32
$n!$	9	11	12

Problems that can be solved only by exponential time algorithms are intractable.

Summary

- Sequences.
  - Monotonicity, bounds, infimum, supremum.
  - Convergence and limit.
  - Series from sequence.
  - Limit of series.
- Special functions.
  - Pointwise function definition.
  - Floor, ceiling, remainder.
  - ...
- Asymptotic function bounds.
  - Big O quantor.
  - Asymptotic classes.