to be prepared for 25.01.2024

Exercise 49. Let W be a set, ordered linearly by some relation $<$ and let $P_{\text{fin}}(W)$ denote the set of finite subsets of W. For $A, B \in P_{\text{fin}}(W)$ define

$$A < B \iff \max(A \triangle B) \in B \tag{1}$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.

Show that:

1. (1) is a linear order on $P_{\text{fin}}(W)$ that extends both, the partial order of containment ($A \subset B$) and, via embedding $w \mapsto \{w\}$, the linear order $<$.
2. If $<$ is a well-order on W then (1) is a well-order on $P_{\text{fin}}(W)$.

In the presence of a monomial order we use this concept to compare the sets of monomials of two polynomials.

Exercise 50. Write $[X]$ for the set of all monomials in $\mathbb{F}[x_1, \ldots, x_n]$ and let $\mathrm{M}(f)$ denote the set of monomials that occur in a polynomial f with a nonzero coefficient. Moreover, given polynomials $f, g, h \in \mathbb{F}[x_1, \ldots, x_n]$, we set

$$f \longrightarrow_g h \iff \exists_{c \in \mathbb{F}} \exists_{\mu \in [X]}\, h = f - c\mu g \text{ and } \mathrm{M}(h) < \mathrm{M}(f). \tag{2}$$

Consider a Gröbner basis for $I \trianglelefteq \mathbb{F}[x_1, \ldots, x_n]$ and let $g, h \in G$ with $g \neq h$. Prove the following statements.

1. If $\mathrm{lt}(g) | \mathrm{lt}(h)$ then $G \setminus \{h\}$ is a Gröbner basis for I.
2. If $h \longrightarrow_g h'$ then $(G \setminus \{h\}) \cup \{h'\}$ is a Gröbner basis for I.

Exercise 51. A set $G \subseteq \mathbb{F}[x_1, \ldots, x_n] \setminus 0$ is called a **reduced Gröbner basis** (w.r.t. some monomial order) provided that

- G is a Gröbner basis;
- $\forall_{g \in G}\, \mathrm{lc}(g) = 1$;
- $\forall_{g \in G}\, \mathrm{M}(g) \cap \langle \mathrm{lt}(G \setminus \{g\}) \rangle = \emptyset$.

Let G be a Gröbner basis for the ideal $I \trianglelefteq \mathbb{F}[x_1, \ldots, x_n]$. Describe an algorithm which, starting from G, produces a reduced Gröbner basis for I.

Exercise 52. Let $<$ be a monomial order and $I \trianglelefteq \mathbb{F}[x_1, \ldots, x_n]$. Prove that I has a unique reduced Gröbner basis.

Hint: Given two reduced Gröbner bases G and G' for I, use the definition of Gröbner basis to prove that $\mathrm{LT}(G) = \mathrm{LT}(G')$. Then use property $(\star)$ in **Proposition Red0**: $f \in I \iff f \text{ rem } G = 0$.

Exercise 53. Let $<$ be a monomial order and $G \subseteq \mathbb{F}[x_1, \ldots, x_n]$ an arbitrary set. Write $f \overset{\star}{\longrightarrow} h$ if there is a finite chain of reductions (2)

$$f \longrightarrow_{g_1} h_1 \longrightarrow_{g_2} h_2 \longrightarrow_{g_3} \cdots \longrightarrow_{g_r} h \text{ where all } g_i \text{ are in } G.$$

A polynomial f is called irreducible (w.r.t. the given reduction defined by G) or a **normal form**, if there is no h s.t. $f \longrightarrow h$. Let N denote the set of all normal forms and $Z = \{f \mid f \overset{\star}{\longrightarrow} 0\}$ the set of all polynomials that reduce to 0.

Prove the following statements.

1. The reduction terminates always (each f has a normal form).
2. Let $I = \langle G \rangle$ be the ideal generated by G. Then
   (a) $N + I = \mathbb{F}[x_1, \ldots, x_n]$;
   (b) $N \cap I = 0 \iff I = Z$;
   (c) G is a Gröbner basis for $I \iff N \oplus Z = \mathbb{F}[x_1, \ldots, x_n]$.