to be prepared for 18.01.2024

Exercise 42. Again the situation of Exercise 37 prove that the number of unlucky primes is small, i.e., the cardinality of the set of all monic irreducibles $p \in \mathbb{F}[y]$ with a fixed degree $d + 1 + \deg b$ that divide $\mathrm{res}_x(f/h, g/h)$ is at most $2 \cdot \deg_x f$.

Exercise 43. Let $<$ be a monomial order on $\mathbb{N}^n$, and $R = \mathbb{F}[x_1, \dots, x_n]$. For $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in R \setminus 0$, the **multidegree** of f is the greatest n-tuple of exponents occuring in f with a nonzero coefficient

$$\mathrm{mdeg}(f) = \max\{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}.$$

Prove the following statements for $f, g \in R \setminus 0$:

1. $\mathrm{mdeg}(f\, g) = \mathrm{mdeg}(f) + \mathrm{mdeg}(g)$;
2. If $f + g \neq 0$ then $\mathrm{mdeg}(f + g) \leq \max\{\mathrm{mdeg}(f), \mathrm{mdeg}(g)\}$;
3. If $f+g \neq 0$ and $\mathrm{mdeg}(f) \neq \mathrm{mdeg}(g)$ then $\mathrm{mdeg}(f+g) = \max\{\mathrm{mdeg}(f), \mathrm{mdeg}(g)\}$.

Exercise 44. Let $<$ be a monomial order on $\mathbb{N}^n$, $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ an ideal and $G \subseteq I$. Show that

$$\langle \mathrm{LT}(G) \rangle = \langle \mathrm{LT}(I) \rangle \iff \forall_{p \in I}\, \exists_{g \in G}\, \mathrm{lt}(g) | \mathrm{lt}(p).$$

Exercise 45. Given a monomial order $<$ on $\mathbb{N}^n$. A **Gröbner basis** for an ideal $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ is a finite subset $G \subseteq I$ with the property $\langle \mathrm{LT}(G) \rangle = \langle \mathrm{LT}(I) \rangle$.

Let G be a Gröbner basis for $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ and $f \in \mathbb{F}[x_1, \dots, x_n]$. Prove that there exists a unique $r \in \mathbb{F}[x_1, \dots, x_n]$ such that

1. $r \equiv f \mod I$;
2. no term of r is divisible by any monomial in $\mathrm{LT}(G)$.

Exercise 46. Show that the result of applying the Euclidean Algorithm in $K[x]$ to any pair of polynomials f, g is a Gröbner basis for $\langle f, g \rangle$.

Exercise 47. Consider linear polynomials in $K[x_1, \dots, x_n]$

$$f_i = a_{i1} x_1 + \cdots + a_{in} x_n \qquad 1 \leq i \leq m$$

and let $A = (a_{ij})$ be the $m \times n$ matrix of their coefficients. Let B be the reduced row echelon matrix determined by A and let $g_1, \dots, g_r$ be the linear polynomials coming from the nonzero rows of B. Use lex order with $x_1 > \cdots > x_n$ and show that $\{g_1, \dots, g_r\}$ is a Groebner basis of $\langle f_1, \dots, f_m \rangle$.

Exercise 48. Compute a Gröbner basis w.r.t. the lexicographic ordering with $x < y$ for the ideal generated by

$$\begin{aligned} f_1 &= xy^2 + x^2 + x \\ f_2 &= x^2 y + x \end{aligned}$$

in $\mathbb{Z}_3[x, y]$ by using your favorite CA system.