to be prepared for 23.11.2023

Exercise 21. Consider the polynomials $f, g \in \mathbb{Q}[x]$

$$h := x^8 + x^6 - 3x^4 - 3x^3 + 8x^2 + 2x - 5$$
$$f := 3x^6 + 5x^4 - 4x^2 - 9x + 2.$$

Compute polynomials $r, t \in \mathbb{Q}[x]$ such that

$$r \equiv tf \mod h \text{ with } \deg r < 4 \text{ and } \deg t \leq 4.$$

Hint: Write down the extended remainder sequence produced when executed the extended Euclidean algorithm on the input (h, f). Decide which of the triples (r_i, s_i, t_i) determine rational function approximations, i.e., a pair (r, t) of polynomials in $\mathbb{Q}[x]$ such that $\deg r < n$, $\deg t \leq n - \deg r$ and so that

$$\mathrm{GCD}(t, h) = 1 \text{ and } rt^{-1} \equiv f \mod h.$$

Exercise 22. Let F be a field, $(u_0, v_0), \ldots, (u_{n-1}, v_{n-1})$ n points in the plane $F \times F$, where the u_i are pairwise distinct, and $f \in F[x]$ an approximating polynomial, i.e., $\forall_i\, f(u_i) = v_i$ and $\deg f < n$. Apply the extended Euclidean algorithm to the polynomials $h := (x - u_0) \cdots (x - u_{n-1})$ and f, thereby producing the remainder sequence (r_j, s_j, t_j). In this situation prove the following:

If $k \in \{0, \ldots, n\}$ and, in the course of the algorithm, j is the first index with $\deg r_j < k$, then

$$\deg t_j \leq n - k \text{ and } \forall_{0 \leq i < n}\, r_j(u_i) = t_j(u_i) v_i.$$

If in addition r_j and t_j are coprime, then $t_j(u_i) \neq 0$ and $\frac{r_j(u_i)}{t_j(u_i)} = v_i$ for all $0 \leq i < n$.

Exercise 23. Compute a $(3, 4)$-Pade approximant for the sine function, that is, compute polynomials $r, t \in \mathbb{Q}[x]$ with $\deg r < 3$, $\deg t \leq 4$ such that t has a nonzero constant coefficient and $\frac{r}{t} \equiv T_7 \mod x^{3+4}$ where T_7 is the degree 7 Taylor polynomial of $\sin x$ with center 0.

Exercise 24. Consider polyomials $f, g \in k[x]$ of positive degrees m, n respectively. Let I denote the ideal in $k[x]$ generated by f, and let μ denote the multiplication map

$$\mu\colon k[x]/I \longrightarrow k[x]/I, \quad h + I \mapsto gh + I.$$

Demonstrate that $\mathrm{res}_x(f, g) = LC(f)^{\deg(g)} \det(\mu)$.