to be prepared for 16.11.2023

Exercise 16. Let $\mathbb{F}$ denote a field, and $a, b \in \mathbb{F}$ polynomials with $\deg a < (\deg b) \cdot e$, where $e \in \mathbb{N} \setminus 0$. Develop an algorithm that computes $a_1, \ldots, a_e \in \mathbb{F}[x]$ with $\deg a_i < \deg b$ $(1 \le i \le e)$ s.t.

$$\frac{a}{b^e} = \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots + \frac{a_e}{b^e}.$$

Hint: Consider polynomial division in a Horner scheme style.

Exercise 17. Develop a recursive algorithm that computes a solution of a Chinese remainder problem in a Euclidean domain.

Exercise 18. Let $f = \sum_{k=0}^{n} f_k x^k \in \mathbb{C}[x]$ with $\deg f = n \ge 0$. Considering f as an element of $\mathbb{C}^{n+1}$ we may assign to f the usual norms

$$||f||_p = \left(\sum_{k=0}^{n} |f_k| \right)^{1/p} \quad \text{for } p \ge 1 \text{ and } ||f||_\infty = \max_k |f_k|.$$

Check the following relations:

1. $||f||_\infty \le ||f||_2 \le \sqrt{n+1}||f||_\infty$;
2. $||f||_2 \le ||f||_1 \le (n+1)||f||_\infty$.

Exercise 19. Let $f \in \mathbb{C}[x]$ and $z \in \mathbb{C}$. Prove that

$$||(x-z)f||_2 = ||(\overline{z}x - 1)f||_2.$$

You may use $|w|^2 = w\overline{w}$ for $w \in \mathbb{C}$ and that conjugation $w \mapsto \overline{w}$ is an automorphism of the field $\mathbb{C}$.

Exercise 20. For a complex polynomial $f \in \mathbb{C}[x]$ written as product of linear factors $f = f_n(x - z_1)\cdots(x - z_n)$, where $z_k \in \mathbb{C}$, the real number $M(f)$ is

$$M(f) = |f_n| \cdot \prod_{k=1}^{n} \max(1, |z_k|).$$

Prove the following statements.

1. $f, g \in \mathbb{C}[x] \Rightarrow M(fg) = M(f) \cdot M(g)$ and $M(f) \ge |\mathrm{lc}(f)|$;
2. $f \in \mathbb{C}[x] \Rightarrow M(f) \le ||f||_2$.

Hint to point 2:
Sort the zeros $z_1, \ldots, z_n$ of f so that $|z_j| \ge |z_{j+1}|$ $(j = 1, \ldots, n-1)$ and let $k = \max\{j : |z_j| > 1\}$. Then show that $M(f)$ equals the absolute value of the leading coefficient of the polynomial

$$g = \mathrm{lc}(f) \cdot \prod_{j=1}^{k} (\overline{z}x - 1) \cdot \prod_{j=k+1}^{n} (x - z_j)$$

so that $M(f)^2 \le ||g||_2^2$. Then by repeatedly applying the identity in Ex. 19 prove that $||g||_2^2 = ||f||_2^2$.