to be prepared for 9.11.2023

Exercise 11. Let R be an integral domain and $\sim$ the equivalence relation resulting from divisibility

$$a \sim b \iff a|b \wedge b|a.$$

Then $R/\sim$ is partially ordered: $[a] \leq [b] \iff a|b$.

1. Check that the equivalence class of $a \in R$ is the complex product $R^\times a$ (where $R^\times$ is the group of multiplicative units).
2. Demonstrate that the complex product resulting from $\mathcal{P}(R)$ turns $R/\sim$ into a commutative monoid and that the projection $\pi\colon R \to R/\sim$ is a homomorphism of monoids.
3. Let $\operatorname{Sec}(\pi)$ denote the set of all (set-theoretic) cross sections

$$\operatorname{Sec}(\pi) = \{s\colon R/\sim \longrightarrow R \mid \pi \circ s = 1_{R/\sim}\}.$$

   Establish a bijective correspondence of normalization maps $R \longrightarrow R$ with $\operatorname{Sec}(\pi)$ and show that, under this bijection, homomorphisms correspond to homomorphisms.

Exercise 12. Prove the following statements listed in the **EEA-Lemma**, where the input $f, g \in R$ (R a Euklidean domain), and the output $\rho_i, r_i, s_i, t_i \in R$ for $0 \leq i \leq l+1$ and q_i for $0 \leq i \leq l$.

4. $s_i f + t_i g = r_i$ (also $i = l+1$)
5. $s_i t_{i+1} - t_i s_{i+1} = (-1)^i(\rho_0 \cdots \rho_{i+1})^{-1}$ and $\gcd(s_i, t_1) = 1$
6. $\gcd(r_i, t_i) = \gcd(f, t_i)$
7. $f = (-1)^i \rho_0 \cdots \rho_{i+1}(t_{i+1} r_i - t_i r_{i+1})$ and
   $g = (-1)^{i+1} \rho_0 \cdots \rho_{i+1}(s_{i+1} r_i - s_i r_{i+1})$
8. If $R = \mathbb{F}[x]$ then $\deg(t_i) + \deg(r_{i-1}) = \deg(f)$ for $i \geq 1$,
   $\deg(s_i) + \deg(r_{i-1}) = \deg(g)$ for $i > 1$.

Exercise 13. Let I be a unique factorization domain. Every $f \in I[x]$ can be decomposed into "content $\times$ primitive part"

$$f = \operatorname{cont}(f)\operatorname{pp}(f)$$

where $\operatorname{cont}(f) \in I$ and $\operatorname{pp}(f) \in I[x]$ is primitive, i.e., the GCD of all coefficients is 1. This decomposition is unique up to multiplication by units.

Given $f, g \in I[x]$, we write $f \sim g$ if there is a unit ε with $g = \varepsilon f$. Then prove the following:

1. $\operatorname{cont}(fg) \sim \operatorname{cont}(f) \cdot \operatorname{cont}(g)$

2. $\mathrm{pp}(fg) \sim \mathrm{pp}(f) \cdot \mathrm{pp}(g)$.

Exercise 14. Let R be a Euclidean domain and $m_1, \ldots, m_n \in R \setminus 0$ pairwise coprime. Prove the following statements.

1. The rings $R/\langle \prod_{i=1}^{n} m_i \rangle$ and $\prod_{i=1}^{n} R/\langle m_i \rangle$ are isomorphic.

2. $\prod_{i=1}^{n-1} m_i$ and m_n are relatively prime.

3. If $m_1, m_2 \in R \setminus 0$ are coprime and $r, r' \in R$ arbitrary, then $r \equiv r' \bmod m_1$ and $r \equiv r' \bmod m_2$ if and only if $r \equiv r' \bmod m_1 m_2$.

Exercise 15. Solve the Chinese remainder problem

$$\begin{aligned} r &\equiv 62 \mod 79 \\ r &\equiv 66 \mod 83 \\ r &\equiv 72 \mod 89 \end{aligned}$$

over the integers.