to be prepared for 19.10.2023

Exercise 7. Give an example of a commutative ring with unit, that is not a Noetherian ring.

Exercise 8. Prove the following statement for univariate polynomials:

Let $f, g \in \mathbb{F}[x]$ where $\mathbb{F}$ denotes a field. Then

$$\gcd(f, g) \neq 1 \iff \exists_{s,t \in \mathbb{F}[x] \setminus 0} \big(sf + tg = 0 \wedge \deg s < \deg g \wedge \deg t < \deg f \big).$$

Exercise 9. Let $(S, \cdot)$ be a semigroup. For $X, Y \subseteq S$ we define the 'complex product'

$$X \cdot Y = \{xy \mid x \in X \wedge y \in Y\}.$$

This gives the power set $\mathcal{P}(S)$ of S the structure of a semigroup; if S is commutative (a monoid) then so is $\mathcal{P}(S)$.

Now assume that R is a commutative ring with 1.

1. If $I \trianglelefteq R$ is an ideal, show that the congruence class $\operatorname{mod} I$ of an element $a \in R$ is the complex sum (the complex product w.r.t addition) of $\{a\}$ and I.
2. Show that the sum in the ring R/I is the complex sum inherited from $\mathcal{P}(R)$.
3. Describe the relation between the product in R/I and the complex product.

Exercise 10. Let R be an integral domain and $\sim$ the equivalence relation resulting from divisibility

$$a \sim b \iff a|b \wedge b|a.$$

Then $R/\sim$ is partially ordered: $[a] \leq [b] \iff a|b$.

1. Check that the equivalence class of $a \in R$ is the complex product $R^\times a$ (where $R^\times$ is the group of multiplicative units).
2. Demonstrate that the complex product resulting from $\mathcal{P}(R)$ turns $R/\sim$ into a commutative monoid and that the projection $\pi \colon R \to R/\sim$ is a homomorphism of monoids.
3. Let $\operatorname{Sec}(\pi)$ denote the set of all (set-theoretic) cross sections

$$\operatorname{Sec}(\pi) = \big\{ s \colon R/\sim \longrightarrow R \mid \pi \circ s = 1_{R/\sim} \big\}.$$

   Establish a bijective correspondence of normalization maps $R \longrightarrow R$ with $\operatorname{Sec}(\pi)$ and show that, under this bijection, homomorphisms correspond to homomorphisms.

Exercise 11. Prove the following statements listed in the **EEA-Lemma**, where the input $f, g \in R$ (R a Euklidean domain), and the output $\rho_i, r_i, s_i, t_i \in R$ for $0 \leq i \leq l+1$ and q_i for $0 \leq i \leq l$.

4. $s_i f + t_i g = r_i$ (also $i = l+1$)
5. $s_i t_{i+1} - t_i s_{i+1} = (-1)^i (\rho_0 \cdots \rho_{i+1})^{-1}$ and $\gcd(s_i, t_1) = 1$
6. $\gcd(r_i, t_i) = \gcd(f, t_i)$
7. $f = (-1)^i \rho_0 \cdots \rho_{i+1} (t_{i+1} r_i - t_i r_{i+1})$ and
   $g = (-1)^{i+1} \rho_0 \cdots \rho_{i+1} (s_{i+1} r_i - s_i r_{i+1})$
8. If $R = \mathbb{F}[x]$ then $\deg(t_i) + \deg(r_{i-1}) = \deg(f)$ for $i \geq 1$,
   $\deg(s_i) + \deg(r_{i-1}) = \deg(g)$ for $i > 1$.