

Computer Algebra (selected slides)

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Lecture 2: October 13, 2022

Algorithm (Normalized) extended Euclidean algorithm **Input:** $f, g \in R$ with R Euclidean domain. **Output:** $\rho_i, r_i, s_i, t_i \in R$ for $0 \le i \le l+1$ and q_i for $0 \le i \le l$ 1. $\rho_0 = lu(f), r_0 = normal(f)(= f/\rho_0), s_0 = \rho_0^{-1}, t_0 = 0$ $\rho_1 = |\mathsf{u}(q), r_1 = \mathsf{normal}(q) (= q/\rho_1), s_1 = 0, t_1 = \rho_1^{-1}$ 2 i=1while $r_i \neq 0$ do $q_i = r_{i-1} \operatorname{quot} r_i$ $r_{i+1} = r_{i-1} \operatorname{rem} r_i (= r_{i-1} - q_i r_i))$ $\rho_{i+1} = \ln(r_{i+1})$ $r_{i+1} = \text{normal}(r_{i+1}) (= r_{i+1}/\rho_{i+1})$ $s_{i+1} = (s_{i-1} - q_i s_i) / \rho_{i+1}$ $t_{i+1} = (t_{i-1} - q_i t_i) / \rho_{i+1}$ i = i + 1od 3 l = i - 1

return ρ_i, r_i, s_i, t_i for $0 \le i \le l+1$, q_i for $0 \le i \le l$

Define

$$R_0 = \begin{pmatrix} s_0 & t_0 \\ s_1 & t_1 \end{pmatrix}$$

 $Q_{i} = \begin{pmatrix} 0 & 1\\ \rho_{i+1}^{-1} & -q_{i}\rho_{i+1}^{-1} \end{pmatrix}, \qquad R_{i} = Q_{i}\dots Q_{1}R_{0} \qquad 0 \le i \le l$

EEA-Lemma. For
$$0 \le i \le l$$
:
(i) $R_i \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix}$
(ii) $R_i = \begin{pmatrix} s_i & t_i \\ s_{i+1} & t_{i+1} \end{pmatrix}$
(iii) r_i, r_{i+1} and f, g have gcds, in particular: $gcd(f, g) = gcd(r_i, r_{i+1}) = r_l$
(iv) $s_i f + t_i g = r_i$ (also $i = l + 1$)
(v) $s_i t_{i+1} - t_i s_{i+1} = (-1)^i (\rho_0 \dots \rho_{i+1})^{-1}$ and $gcd(s_i, t_i) = 1$
(vi) $gcd(r_i, t_i) = gcd(f, t_i)$
(vii) $f = (-1)^i \rho_0 \dots \rho_{i+1}(t_{i+1} r_i - t_i r_{i+1})$ and
 $g = (-1)^{i+1} \rho_0 \dots \rho_{i+1}(s_{i+1} r_i - s_i r_{i+1})$
viii) If $R = \mathbb{F}[x]$ then $deg(t_i) + deg(r_{i-1}) = deg(f)$ for $i \ge 1$,
 $deg(s_i) + deg(r_{i-1}) = deg(g)$ for $i > 1$.

Lecture 4: November 9, 2023

Algorithm CRA (Chinese Remainder Algorithm) **Input:** $m_0, \ldots, m_{r-1} \in \mathbb{R}^* \setminus \mathbb{R}^+$ pairwise coprime, $v_0,\ldots,v_{r-1}\in R$ with R ED. **Output:** $f \in R$ with $d(f) < d(m_0) + \ldots d(m_{r-1})$ such that for $0 \le i < r$: $m_i \mid f - v_i \Leftrightarrow f \equiv v_i \mod m_i$ 1. $m = m_0 \dots m_{r-1} \in R$ 2. for $0 \leq i < r$ do 3. $f_i = m/m_i \in R$ 4. call the EEA and compute $s_i, t_i \in R$ such that $s_i f_i + t_i m_i = 1$ 5. $c_i = v_i s_i \operatorname{rem} m_i \in R$ [note: $d(c_i) < d(m_i)$] 6. od 7. return $f = \sum_{i=0}^{r-1} c_i f_i$

Remark 1: $l_i = s_i f_i$ and $c_i f_i = v_i l_i \operatorname{rem} m$.

Remark 2: If $d(v_i) < d(m_i)$ then $f \operatorname{rem} m_i = v_i$.

Remark 3: The computation of t_i in the EEA can be skipped.

Lecture 5: November 16, 2023

Theorem (Rational function reconstruction) Let $h, f \in \mathbb{F}[x]$ with $\deg(f) < \deg(h) =: n$ and $k \in \{1, \ldots, n\}$. Let $\{(r_j, s_j, t_j)\}$ be the ERS of h and f and let $j \in \mathbb{N}$ be minimal such that $\deg(r_j) < k$. Then:

1. There exist $r, t \in \mathbb{Z}$ with

$$\label{eq:relation} \begin{split} r &\equiv t\,f \mod m \quad \text{where} \ \deg(r_j) < k \text{ and } \deg(t_j) \leq n-k, \\ \text{namely} \ (r,t) &= (r_j,t_j). \\ \text{If } \gcd(r_j,t_j) &= 1 \text{ then } \gcd(t_j,h) = 1. \end{split}$$

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 $r \equiv t f \mod m$ where $\deg(r_i) < k$ and $\deg(t_i) \le n - k$, namely $(r, t) = (r_i, t_i)$. If $gcd(r_i, t_i) = 1$ then $gcd(t_i, h) = 1$. 2. If $\frac{r}{t} \in \mathbb{F}(x)$ is a canonical form solution to $r \equiv t f \mod h \iff r t^{-1} \equiv f \mod h$ with $\deg(t) \le n - k$, $\deg(r) \le k$ and $\gcd(t, h) = 1$, then $(r,t) = \frac{1}{\operatorname{lc}(t_i)} \Big(r_j, t_j \Big).$

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3. There is a solution as in 2 iff $gcd(r_j, t_j) = 1$.

1. There exist $r, t \in \mathbb{Z}$ with $r \equiv t f \mod m$ where |m| < k and $0 \le t \le \frac{m}{k}$, namely $(r, t) = \operatorname{sgn}(t_j)(r_j, t_j)$. If $\operatorname{gcd}(r_j, t_j) = 1$ then $\operatorname{gcd}(t_j, m) = 1$. 2. If $\frac{r}{t} \in \mathbb{Z}$ is a canonical form solution to $r \equiv t f \mod m \iff r t^{-1} \equiv f \mod m$ with $\operatorname{deg}(t) \le \frac{m}{k}$, $\operatorname{deg}(r) < k$ and $\operatorname{gcd}(t, m) = 1$, then $(r, t) = \operatorname{sgn}(t_j)(r_j, t_j)$ 3. There is a solution as in 2 iff $\operatorname{gcd}(r_j, t_j) = 1$

$$r_{j-1} - q r_j < k \le r_{j-1} - (q-1)r_j \quad [q=0 \text{ if } j=l+1]$$

Set

$$r_j^* = r_{j-1} - q r_j, \quad t_j^* = t_{j-1} - q t_j.$$

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with $\deg(t) \leq \frac{m}{k}$, $\deg(r) < k$ and $\gcd(t, m) = 1$, then $(r, t) = \operatorname{sgn}(t_j) (r_j, t_j)$ There is a solution as in 2 iff $\operatorname{gad}(r_j, t_j) = 1$

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 $(r,t) = \operatorname{sgn}(t_j) \left(r_j, t_j \right) \quad \text{or} \quad (r,t) = \operatorname{sgn}(t_j^*) \left(r_j^*, t_j^* \right).$ 3. There is a solution as in 2 iff $\operatorname{gcd}(r_j, t_j) = 1$

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or $(\gcd(r_j^*,t_j^*)=1 \text{ and } |t_j^*| \leq \frac{m}{k})$

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3. There is a solution as in 2 iff $gcd(r_j, t_j) = 1$ or $(gcd(r_j^*, t_j^*) = 1$ and $|t_j^*| \le \frac{m}{k})$

4. There is at most one solution as in 2 with $|r| < \frac{k}{2}$.

Lecture 6: November 23, 2022

Corollary PP-Cont Let R be an UFD and let $f, g \in R[x]$. Then:

- 1. $\operatorname{pp}(f g) = \operatorname{pp}(f) \operatorname{pp}(g)$
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Corollary UFD-GCD Let R be an UFD. Let $f, g \in R[x]$ and define $h = \text{gcd}_{R[x]}(f,g)$. Then:

 $1. \ \mbox{We can split gcd-calculation problem by}$

$$h = \operatorname{gcd}_R(\operatorname{cont}(f), \operatorname{cont}(g)) \cdot \operatorname{gcd}_{R[x]}(\operatorname{pp}(f), \operatorname{pp}(g))$$

In particular, h is primitive if f or g are primitive.

2. We have

$$\frac{h}{\operatorname{lc}(h)} = \operatorname{gcd}_{\mathbb{K}[x]}(f,g)$$

in the quotient field $\mathbb{K} = Q(R)$.

Algorithm GCD for R[x]Input: $f, g \in R[x]^*$ with UFD R and its quotient field $\mathbb{K} = Q(R)$ where one can compute gcds in R and \mathbb{K} is computable. Output: $gcd(f,g) \in R[x]$

1.
$$\tilde{f} = pp(f)$$
, $\tilde{c} = cont(f)$
 $\tilde{g} = pp(g)$, $\tilde{d} = cont(g)$

2. Compute the following gcds in R:

$$\begin{split} a &= \gcd_R(\tilde{c}, \tilde{d}) \in R \\ b &= \gcd_R(\operatorname{lc}(\tilde{f}), \operatorname{lc}(\tilde{g})) \in R \end{split}$$

- Call the Euclidean algorithm in K[x] to get the monic polynomial
 v = gcd_{K[x]}(f̃, g̃) ∈ K[x]
 return a pp(b v)
- Remark: In step 1 (and most probably in step 3) we also utilize gcd computations in R.

As a consequence one obtains the following general statement.

Corollary Let $\mathbb{E} = G[x_1, \ldots, x_n]$ be a polynomial ring over a UFD G. Suppose that one can compute gcds in G and that the quotient field Q(G) is computable. Then one can compute gcds in \mathbb{E} and can carry out the field operations in $Q(\mathbb{E})$. As a consequence one obtains the following general statement.

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Proof. By induction on the number n of variables.

- If n = 0, the corollary holds.
- Suppose that one can compute gcds in the UFD $R = G[x_1, \ldots, x_{n-1}]$ and that the field operations in Q(R) can be executed. Then one can execute the algorithm above to compute gcds in $\mathbb{E} = R[x_n]$. In addition, one can carry out the field operations in $Q(R[x_n]) = Q(\mathbb{E})$; note that one can even calculate reduced representations in $Q(\mathbb{E})$, i.e., the numerators and denominators in $G[x_1, \ldots, x_{n-1}, x_n]$ are co-prime.

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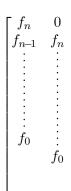
Remark: If G is a field, one obtains much more efficient algorithms; soon we will consider, e.g., R = G[x, y] for a field G.

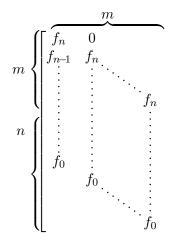
Lecture 7: November 30, 2023

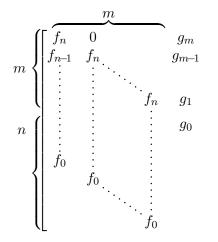
Thm. Let $f, g \in \mathbb{F}[x]^*$ with $n = \deg(f)$, $m = \deg(g)$. Then: 1. $\gcd(f,g) = 1$ iff ϕ is an isomorphism. 2. If $\gcd(f,g) = 1$, then the output s_l and t_l

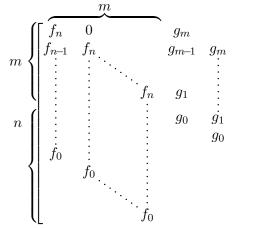
of the EEA is the unique solution in $\mathbb{F}[x]_{\leq m} \times \mathbb{F}[x]_{\leq n}$ of $\phi(s_l, t_l) = 1$.

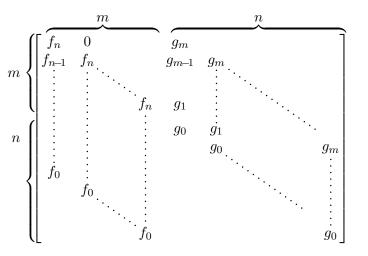
$\begin{bmatrix} f_n \\ f_{n-1} \\ \vdots \\ \vdots \\ \vdots \\ f_0 \end{bmatrix}$

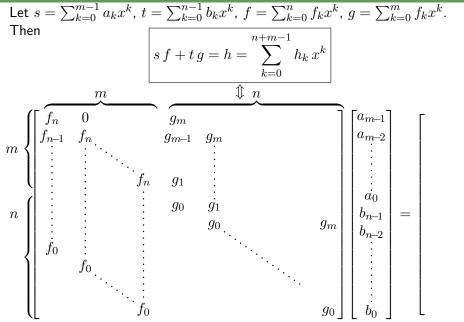


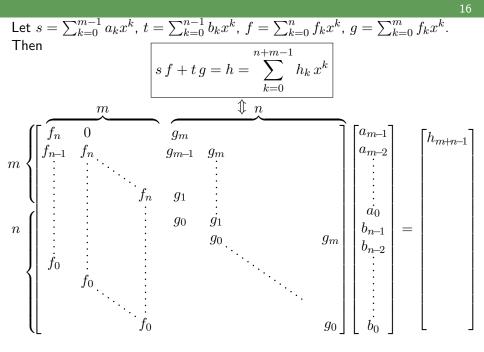


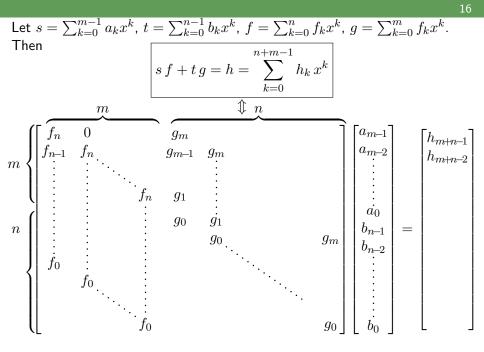


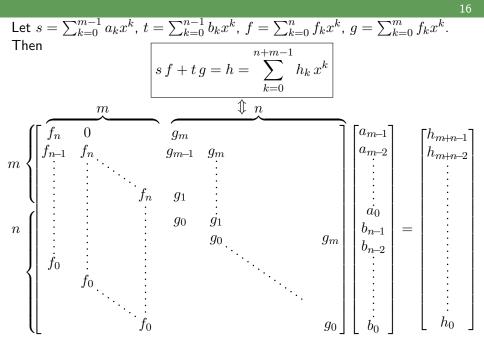


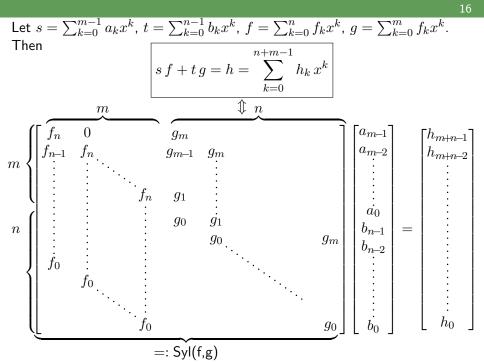












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Corollary $\mathbb{F}[x]$ -res. Let $f, g \in \mathbb{F}[x]^*$ with $n = \deg(f)$, $m = \deg(g)$. Then: 1. $\gcd(f,g) = 1$ iff $\det(S) \neq 0$.

↥

2. If gcd(f,g) = 1 and the a_i, b_i are a solution of the system above, then

$$s_l = \sum_{k=0}^{m-1} a_k x^k \qquad \text{ and } \qquad t_l = \sum_{k=0}^{n-1} b_k x^k$$

are the Bezout coefficients in $s_l f + t_l g = 1$ also produced by the EEA.

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are the Bezout coefficients in $s_l f + t_l g = 1$ also produced by the EEA.

Corollary UFD-res. Let R be an UFD, $f, g \in R[x]$, not both zero. Then: $gcd(f,g) \in R[x] \setminus R \quad \Leftrightarrow \quad res(f,g) = 0 \text{ in } R.$

 t_l

1. Coefficients bounds in $\mathbb{F}[y]$

Theorem. Let $f, g \in \mathbb{F}[x, y]$ with $n = \deg_x(f)$ and $m = \deg_x(g)$ and $\deg_y(f), \deg_y(g) \le d$. Then

 $\deg_y \operatorname{res}_x(f,g) \le (n+m)d.$

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2. Coefficients bounds in $\ensuremath{\mathbb{Z}}$

For
$$f = \sum_{n=0}^{d} f_n x^n \in \mathbb{C}[x]$$
 define the 2-norm
 $\|f\|_2 = \Big(\sum_{n=0}^{d} |f_n|^2\Big)^{1/2}, \quad |a| = (a \cdot \bar{a})^{1/2} \in \mathbb{R}$
and the max-norm

$$||f||_{\infty} = \max\{|f_n|: 0 \le n \le d\}.$$

Note:

$$||f||_{\infty} \le ||f||_2 \le (n+1)^{1/2} ||f||_{\infty}$$

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$$||f||_{\infty} = \max\{|f_n|: 0 \le n \le d\}.$$

Note:

$$\begin{split} \|f\|_{\infty} &\leq \|f\|_{2} \leq (n+1)^{1/2} \|f\|_{\infty} \\ \text{Theorem. Let } f,g \in \mathbb{Z}[x] \text{ with } n = \deg(f) \text{ and } m = \deg(g). \text{ Then} \\ |\operatorname{res}_{x}(f,g)| &\leq \|f\|_{2}^{m} \|g\|_{2}^{n} \leq (n+1)^{m/2} (m+1)^{n/2} \|f\|_{\infty}^{m} \|g\|_{\infty}^{n}. \end{split}$$

Lemma. Let $f, g \in R[x]^*$ and I be an ideal in R with $I \neq R$. Suppose that $\overline{\operatorname{lc}(f)} \in R/I$ is not a zero-divisor. Then:

1.
$$\overline{\operatorname{res}(f,g)} = \overline{0} \quad \Leftrightarrow \quad \operatorname{res}(\overline{f},\overline{g}) = \overline{0}.$$

2. If R/I is a UFD then

$$\overline{\operatorname{\mathsf{res}}(f,g)} = \bar{0} \quad \Leftrightarrow \quad \gcd(\bar{f},\bar{g}) \notin R/I.$$

Proof. (1) will be settled as a homework.

¹Note that $\deg(f) = \deg(\overline{f})$ and thus $\overline{\operatorname{lc}(f)} = \operatorname{lc}(\overline{f})$ does not hold in general.

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$$\overline{\operatorname{res}(f,g)} = \bar{0} \qquad \stackrel{(1)}{\Longleftrightarrow} \qquad \operatorname{res}(\bar{f},\bar{g}) = \bar{0}$$

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$$\overline{\operatorname{res}(f,g)} = \overline{0} \qquad \stackrel{(1)}{\longleftrightarrow} \qquad \operatorname{res}(\bar{f},\bar{g}) = \overline{0}$$

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¹Note that $\deg(f) = \deg(\overline{f})$ and thus $\overline{\operatorname{lc}(f)} = \operatorname{lc}(\overline{f})$ does not hold in general.

GCD-Theorem. Let R be an ED, $f, g \in R[x]^*$ and $p \in R$ be prime with $p \nmid \operatorname{gcd}_R(\operatorname{lc}(f), \operatorname{lc}(g))$; let $\mathbb{F} = R/\langle p \rangle$ be its quotient field. Then:

(i) $\operatorname{lc}(\operatorname{gcd}_{R[x]}(f,g)) | \operatorname{gcd}_R(\operatorname{lc}(f),\operatorname{lc}(g)).$

GCD-Theorem. Let R be an ED, $f, g \in R[x]^*$ and $p \in R$ be prime with $p \nmid \operatorname{gcd}_R(\operatorname{lc}(f), \operatorname{lc}(g))$; let $\mathbb{F} = R/\langle p \rangle$ be its quotient field. Then:

- (i) $\operatorname{lc}(\operatorname{gcd}_{R[x]}(f,g)) | \operatorname{gcd}_R(\operatorname{lc}(f),\operatorname{lc}(g)).$
- (ii) $\deg(\gcd_{\mathbb{F}[x]}(\bar{f},\bar{g})) \ge \deg(\gcd_{R[x]}(f,g)).$

GCD-Theorem. Let R be an ED, $f, g \in R[x]^*$ and $p \in R$ be prime with $p \nmid \operatorname{gcd}_R(\operatorname{lc}(f), \operatorname{lc}(g))$; let $\mathbb{F} = R/\langle p \rangle$ be its quotient field. Then:

Lecture 9: November 14, 2023

Algorithm modGCD for $\mathbb{F}[x, y]$ (big prime version) Input: primitive $f, g \in \mathbb{F}[x, y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \ge \deg_x(g) \ge 1$ and $\deg_y(f), \deg_y(g) \le d$ for some $d \in \mathbb{N}$. Output: $h = \gcd(f, g) \in \mathbb{F}[x, y]$

- 1. Compute $b := \gcd_{\mathbb{F}[y]}(\operatorname{lc}_x(f), \operatorname{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$
- 2. repeat
- 3. choose a random monic irreducible $p \in \mathbb{F}[y]$ with $\deg_y(p) = \ell$
- 4. call the EEA for $\bar{f}, \bar{g} \in \mathbb{E}[x]$ over the field $\mathbb{E} = \mathbb{F}[y]/\langle p \rangle$ to get the monic $v \in R[x]$ with $\deg_y(v) < \ell$ such that $\bar{v} = \gcd(\bar{f}, \bar{g}) \in \mathbb{E}[x]$.
- 5. Compute $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ where \bar{z}

$$\bar{w} = \overline{b}\,\overline{v}, \quad \bar{f^*} = \frac{f}{\bar{v}}, \quad \bar{g^*} = \frac{\bar{g}}{\bar{v}}$$

6. until $\deg_y(f^*w) = \deg_y(bf)$ and $\deg_y(g^*w) = \deg_y(bg)$

Theorem. Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \operatorname{gcd}_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$. Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

$$\begin{array}{ll} 1. \ \deg(r) \leq 2nd \ \text{where} \ n = \deg_x(f) \geq \deg_x(g) \geq 1 \ \text{and} \\ d \geq \deg_y(f), \deg_y(g). \end{array}$$

2. $p \nmid_R r$ if and only if the halting condition holds.

3. If
$$p \nmid_R r$$
 then $h = \mathsf{pp}_x(w)$.

For $f = \sum_{n=0}^{d} f_n x^n \in \mathbb{C}[x]$ define the q-norm $(q \in \mathbb{N}^*)$ $\|f\|_q = \left(\sum_{n=0}^{d} |f_n|^q\right)^{1/q}, \quad |a| = (a \cdot \bar{a})^{1/2} \in \mathbb{R}$ and the max-norm

 $||f||_{\infty} = \max\{|f_n|: 0 \le n \le d\}.$

Note (Ex. 18):

$$\begin{aligned} \|f\|_{\infty} &\leq \|f\|_{2} \leq (n+1)^{1/2} \, \|f\|_{\infty} \\ \|f\|_{2} &\leq \|f\|_{1} \leq (n+1) \, \|f\|_{\infty} \end{aligned}$$

$$||f||_{\infty} \leq ||f||_{2} \leq (n+1)^{1/2} ||f||_{\infty}$$

$$||f||_{2} \leq ||f||_{1} \leq (n+1) ||f||_{\infty}$$

Theorem Mignotte. Let $f, g, h \in \mathbb{Z}[x]$ with $\deg(f) = n$, $\deg(g) = m$ and $\deg(h) = k$. Suppose that $g h \mid f$. Then

 $\|g\|_{\infty} \, \|h\|_{\infty} {\leq} \|g\|_2 \, \|h\|_2$

$$||f||_{\infty} \leq ||f||_{2} \leq (n+1)^{1/2} ||f||_{\infty}$$

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$$||f||_{\infty} \leq ||f||_{2} \leq (n+1)^{1/2} ||f||_{\infty}$$

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Theorem Mignotte. Let $f, g, h \in \mathbb{Z}[x]$ with $\deg(f) = n$, $\deg(g) = m$ and $\deg(h) = k$. Suppose that $gh \mid f$. Then

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$$||f||_{\infty} \le ||f||_2 \le (n+1)^{1/2} ||f||_{\infty}$$

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Theorem Mignotte. Let $f, g, h \in \mathbb{Z}[x]$ with $\deg(f) = n$, $\deg(g) = m$ and $\deg(h) = k$. Suppose that $gh \mid f$. Then

$$\begin{split} \|g\|_{\infty} \|h\|_{\infty} \leq \|g\|_{2} \|h\|_{2} &\leq \|g\|_{1} \|h\|_{1} \overset{\text{Ex.}}{\leq} 2^{m+k} \|f\|_{2} &\leq (n+1)^{1/2} 2^{m+k} \|f\|_{\infty} \\ \text{Special case } (g=1): \end{split}$$

 $||h||_{\infty} \le ||h||_2 \le ||h||_1 \le 2^k ||f||_2 \le (n+1)^{1/2} 2^k ||f||_{\infty}.$

Corollary. Let $f, g \in \mathbb{Z}[x]$ with $n = \deg(f) \ge \deg(g) \ge 1$ and $\|f\|_{\infty}, \|g\|_{\infty} \le A$. Then

$$\|\gcd(f,g)\|_{\infty} \le (n+1)^{1/2} 2^n A.$$

Lemma. Let $f, g \in \mathbb{Z}[x]$ with $||f||_{\infty}, ||g||_{\infty} < \frac{p}{2}$. Then

$$\bar{f} = \bar{g} \quad \Leftrightarrow \quad f = g.$$

Recall: Algorithm modGCD for $\mathbb{F}[x, y]$ (big prime version) Input: primitive $f, g \in \mathbb{F}[x, y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \ge \deg_x(g) \ge 1$ and $\deg_y(f), \deg_y(g) \le d$ for some $d \in \mathbb{N}$. Output: $h = \gcd(f, g) \in \mathbb{F}[x, y]$

- 1. Compute $b := \gcd_{\mathbb{F}[y]}(\operatorname{lc}_x(f), \operatorname{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$
- 2. repeat
- 3. choose a random monic irreducible $p \in \mathbb{F}[y]$ with $\deg_y(p) = \ell$
- 4. call the EEA for $\bar{f}, \bar{g} \in \mathbb{E}[x]$ over the field $\mathbb{E} = \mathbb{F}[y]/\langle p \rangle$ to get the monic $v \in R[x]$ with $\deg_y(v) < \ell$ such that $\bar{v} = \gcd(\bar{f}, \bar{g}) \in \mathbb{E}[x]$.
- 5. Compute $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ where \bar{z}

$$\bar{w} = \overline{b}\,\overline{v}, \quad \bar{f^*} = \frac{f}{\bar{v}}, \quad \bar{g^*} = \frac{\bar{g}}{\bar{v}}$$

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Algorithm modGCD for $\mathbb{Z}[x]$ (big prime version) Input: primitive $f, g \in \mathbb{Z}[x]$ with $n = \deg(f) \ge \deg(g) \ge 1$ and $\|f\|_{\infty}, \|g\|_{\infty} \le A$ for some $A \in \mathbb{N}$. Output: $h = \gcd(f, g) \in \mathbb{Z}[x]$

1. Compute $b := \operatorname{gcd}_{\mathbb{Z}}(\operatorname{lc}(f), \operatorname{lc}(g))$ and set $B = (n+1)^{1/2} 2^n A b$

2. repeat

- 3. choose a random prime p with 2B < p
- 4. call the EEA for $\overline{f}, \overline{g} \in \mathbb{Z}_p[x]$ over the finite field \mathbb{Z}_p to get the monic $v \in R[x]$ with $||v||_{\infty} < p/2$ such that $\overline{v} = \gcd(\overline{f}, \overline{g}) \in \mathbb{Z}_p[x]$.
- 5. Compute $w, f^*, g^* \in \mathbb{Z}[x]$ with $\|w\|_{\infty}, \|f^*\|_{\infty}, \|g^*\|_{\infty} < p/2$ where

$$\bar{w} = \overline{b \, v}, \quad \bar{f^*} = \frac{\bar{f}}{\bar{v}}, \quad \bar{g^*} = \frac{\bar{g}}{\bar{v}}$$

- 6. until $||f^*||_1 ||w||_1 \le B$ and $||g^*||_1 ||w||_1 \le B$
- 7. return pp(w)

Recall: **Theorem.** Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$.

Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

1.
$$\deg(r) \le 2nd$$
 where $n = \deg_x(f) \ge \deg_x(g) \ge 1$ and $d \ge \deg_y(f), \deg_y(g)$.

2. $p \nmid_R r$ if and only if the halting condition holds.

3. If
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Theorem. Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \operatorname{gcd}_{\mathbb{Z}[x]}(f, g)$ and $r = \operatorname{res}(f/h, g/h) \in \mathbb{Z}$; note that $\operatorname{lc}(h) > 0$. Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop. Then:

²There is the improved version $|r| \leq (n+1)^n A^{2n}$.

Recall: Theorem. Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \operatorname{gcd}_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$.

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- 1. $\deg(r) \le 2nd$ where $n = \deg_x(f) \ge \deg_x(g) \ge 1$ and $d \ge \deg_y(f), \deg_y(g)$.
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Recall: **Theorem.** Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$.

Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

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Theorem. Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \operatorname{gcd}_{\mathbb{Z}[x]}(f, g)$ and $r = \operatorname{res}(f/h, g/h) \in \mathbb{Z}$; note that $\operatorname{lc}(h) > 0$.

Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop. Then:

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- 2. $p \nmid_{\mathbb{Z}} r$ if and only if the halting condition holds.
- 3. If $p \nmid_{\mathbb{Z}} r$ then h = pp(w).

²There is the improved version $|r| \leq (n+1)^n A^{2n}$.

Algorithm modGCD for $\mathbb{F}[x, y]$ (small prime version) Input: primitive $f, g \in \mathbb{F}[x, y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \ge \deg_x(g) \ge 1$ and $\deg_y(f), \deg_y(g) \le d$ for some $d \in \mathbb{N}$. Output: $h = \gcd(f, g) \in \mathbb{F}[x, y]$

1. Compute $b := \gcd_{\mathbb{F}[y]}(\operatorname{lc}_x(f), \operatorname{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$

2. repeat

- 3. choose a set $S \subseteq \mathbb{F}$ of ℓ evaluation points u with $b(u) \neq 0$.
- 4. for each $u \in S$ call the EEA to get $v_u = \operatorname{gcd}_{\mathbb{F}[x]}(f(x, u), g(x, u))$

7. Compute by interpolation each coefficient in $\mathbb{F}[y]$ of the polynomials $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ such that for each $u \in S$ we have

$$w(x,u) = b(u)v_u, \quad f^*(x,u) = \frac{f(x,u)}{v_u}, \quad g^*(x,u) = \frac{g(x,u)}{v_u}$$

8. until $\deg_y(f^*w) = \deg_y(bf)$ and $\deg_y(g^*w) = \deg_y(bg)$

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1. Compute
$$b := \gcd_{\mathbb{F}[y]}(\operatorname{lc}_x(f), \operatorname{lc}_x(g)) \in \mathbb{F}[y]$$
 and set $\ell = d + 1 + \deg_y(b)$

2. repeat

- 3. choose a set $S \subseteq \mathbb{F}$ of 2ℓ evaluation points u with $b(u) \neq 0$.
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1. Compute $b := \gcd_{\mathbb{F}[y]}(\operatorname{lc}_x(f), \operatorname{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$

2. repeat

- 3. choose a set $S \subseteq \mathbb{F}$ of 2ℓ evaluation points u with $b(u) \neq 0$.
- 4. for each $u \in S$ call the EEA to get $v_u = \operatorname{gcd}_{\mathbb{F}[x]}(f(x, u), g(x, u))$
- 5. $\lambda = \min\{\deg(v_u) \mid u \in S\}$ and refine $S := \{u \in S \mid \deg(v_u) = \lambda\}$
- 6. if $|S| \ge \ell$ then remove $|S| \ell$ points from S else goto 3.
- 7. Compute by interpolation each coefficient in $\mathbb{F}[y]$ of the polynomials $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ such that for each $u \in S$ we have

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Recall: **Theorem.** Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \operatorname{gcd}_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$. Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

$$\begin{array}{ll} 1. \ \deg(r) \leq 2nd \ \text{where} \ n = \deg_x(f) \geq \deg_x(g) \geq 1 \ \text{and} \\ d \geq \deg_y(f), \deg_y(g). \end{array}$$

2. $p \nmid_R r$ if and only if the halting condition holds.

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Theorem. Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$. Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop using the ℓ given points from S. Then: Recall: **Theorem.** Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$. Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

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Theorem. Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h, g/h) \in R$. Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop using the ℓ given points from S. Then:

1. $\deg(r) \le 2nd$ where $n = \deg_x(f) \ge \deg_x(g) \ge 1$ and $d \ge \deg_y(f), \deg_y(g)$.

Recall: **Theorem.** Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \operatorname{gcd}_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$. Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

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2. $p \nmid_R r$ if and only if the halting condition holds.

3. If
$$p \nmid_R r$$
 then $h = \mathsf{pp}_x(w)$.

Theorem. Let $f, g \in R[x]$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f,g)$ and $r = \operatorname{res}_x(f/h,g/h) \in R$. Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop using the ℓ given points from S. Then:

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$$\deg(r) \le 2nd$$
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2. $r(s) \neq 0$ for all $s \in S$ if and only if the halting condition holds.

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Algorithm modGCD for $\mathbb{Z}[x]$ (small prime version) Input: primitive $f, g \in \mathbb{Z}[x]$ with $n = \deg(f) \ge \deg(g) \ge 1$ and $||f||_{\infty}, ||g||_{\infty} \le A$ for some $A \in \mathbb{N}$. Output: $h = \gcd(f, g) \in \mathbb{Z}[x]$

- 1. Compute $b:=\gcd_{\mathbb{Z}}(\mathrm{lc}(f),\mathrm{lc}(g))$ and set $B=(n+1)^{1/2}2^nA\,b.$ Take $\ell=\log_2(2B+1)$
- 2. repeat
- 3. choose a set S of 2ℓ primes p with $p \nmid b$.
- 4. for each $p \in S$ call the EEA to get the monic $v_p \in \mathbb{Z}[x]$ where the coefficients are from $\{0, \ldots, p-1\}$ with $\bar{v}_p = \gcd_{\mathbb{Z}_p[x]}(\bar{f}, \bar{g})$
- 5. $\lambda = \min\{\deg(v_p) \mid p \in S\}$ and refine $S := \{p \in S \mid \deg(v_p) = \lambda\}$
- 6. if $|S| \ge \ell$ then remove $|S| \ell$ points from S else goto 3.
- 7. Compute by CRA the coefficients of the polynomials $w, f^*, g^* \in \mathbb{Z}[x]$ with $||w||_{\infty}, ||f^*||_{\infty}, ||g^*||_{\infty} \leq (\prod_{p \in S} p)/2$ s.t. for each $p \in S$ we have

$$\bar{w} = \overline{b v_p}, \quad \bar{f^*} = \frac{J}{\overline{v_p}}, \quad \bar{g^*} = \frac{g}{\overline{v_p}} \pmod{p}$$
 (reduction mod p)

- 8. until $||f^*||_1 ||w||_1 \le B$ and $||g^*||_1 ||w||_1 \le B$
- 9. return pp(w)

Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop. Then:

- 1. $|r|\leq (n+1)^nA^{2n}$ where $n=\deg(f)\geq \deg(g)\geq 1$ and $A\geq \|f\|_\infty,\|g\|_\infty.$
- 2. $p \nmid_{\mathbb{Z}} r$ if and only if the halting condition holds.
- 3. If $p \nmid_{\mathbb{Z}} r$ then $h = \mathsf{pp}(w)$.

Theorem. Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \operatorname{gcd}_{\mathbb{Z}[x]}(f, g)$ and $r = \operatorname{res}(f/h, g/h) \in \mathbb{Z}$; note that $\operatorname{lc}(h) > 0$. Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop using the ℓ primes given in S. Then:

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Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop. Then:

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$$|r| \leq (n+1)^n A^{2n}$$
 where $n = \deg(f) \geq \deg(g) \geq 1$ and $A \geq ||f||_{\infty}, ||g||_{\infty}.$

2. $p \nmid_{\mathbb{Z}} r$ for all $p \in S$ if and only if the halting condition holds.

3. If $p \nmid_{\mathbb{Z}} r$ for all $p \in S$ then h = pp(w).

Lecture 8: December 7, 2023

Definition. A monomial order on $\mathbb{F}[\mathbf{x}]$ is a relation < on \mathbb{N}^n such that

- 1. < is a total order;
- 2. For all $\alpha, \beta, \gamma \in \mathbb{N}^n$:

$$\alpha < \beta \qquad \Rightarrow \qquad \alpha + \gamma < \beta + \gamma$$

3. < is well-ordered, i.e.,

 $\forall S \subseteq \mathbb{N}^n \; \exists m \in S \; \forall s \in S : \; m \leq s$ $\Leftrightarrow \quad \forall s \in \mathbb{N}^n : \; s \ge \mathbf{0}.$

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Lemma Deg. Let < be a monomial order on $\mathbb{F}[\mathbf{x}]$ and $f,g \in \mathbb{F}[\mathbf{x}]^*$. Then:

1. $\deg(f g) = \deg(f) + \deg(g);$ 2. If $f + g \neq 0$ then

 $\deg(f+g) \le \max(\deg(f), \deg(g));$

equality holds if $\deg(f) \neq \deg(g)$.

Algorithm PolynomialReduce

Input: $f, g_1, \ldots, g_s \in \mathbb{F}[x_1, \ldots, x_n] =: R$ with a monomial order <. **Output:** $r, q_1, \ldots, q_s \in R$ with $f = r + q_1 g_1 + \cdots + q_s g_s$

1.
$$r = 0$$
, $p = f$, $q_i = 0$ for $1 \le i \le s$

2. while $p \neq 0$ do

3. if $lt(g_i) \mid lt(p)$ for some $1 \le i \le s$ then

4. choose such an
$$i$$
 and set $q_i = q_i + \frac{|\mathbf{t}(p_i)|}{|\mathbf{t}(g_i)|}$
$$p = p - \frac{|\mathbf{t}(p_i)|}{|\mathbf{t}(f_i)|}g_i$$

5. else

6.
$$r = r + \operatorname{lt}(p), \ p = p - \operatorname{lt}(p)$$

8. od

9. return $q_1, ..., q_s$, r

Remark: If s = n = 1 then $q_1 = quot(f, g_1)$ and $r = rem(f, g_1)$

Algorithm PolynomialReduce

Input: $f, g_1, \ldots, g_s \in \mathbb{F}[x_1, \ldots, x_n] =: R$ with a monomial order <. **Output:** $r, q_1, \ldots, q_s \in R$ with $f = r + q_1 g_1 + \cdots + q_s g_s$ where no monomial in r is divisible by $\mathsf{lt}(g_i)$ for all $1 \le i \le s$.

1.
$$r = 0$$
, $p = f$, $q_i = 0$ for $1 \le i \le s$

- 2. while $p \neq 0$ do
- 3. if $\mathsf{lt}(g_i) \mid \mathsf{lt}(p)$ for some $1 \le i \le s$ then

4. choose such an
$$i$$
 and set $q_i = q_i + \frac{|\mathbf{t}(p_i)|}{|\mathbf{t}(g_i)|}$
$$p = p - \frac{|\mathbf{t}(p_i)|}{|\mathbf{t}(f_i)|}g_i$$

5. else

6.
$$r = r + \operatorname{lt}(p), \ p = p - \operatorname{lt}(p)$$

- 7. fi
- 8. od
- 9. return $q_1, ..., q_s$, r

Remark: If s = n = 1 then $q_1 = \operatorname{quot}(f, g_1)$ and $r = \operatorname{rem}(f, g_1)$

Lemma Mon. Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ that is generated by a set M of monomials, and let h be a monomial. Then:

$$h \in I \quad \Leftrightarrow \quad \exists m \in M : m \mid h.$$

Lecture 11: January 11, 2024

Definition. Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$, $G \subseteq I$ finite, < monomial order.

 $\langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle \qquad \Leftrightarrow: \quad G \text{ is a GB of I}$

$$\begin{array}{l} \langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle & \text{(i.e., } G \text{ is a GB of I)} \\ \\ \updownarrow \\ \forall p \in I \ \exists g \in G : \ \mathsf{lt}(g) \mid \mathsf{lt}(p). \end{array}$$

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Lemma. Let G be a GB for $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ w.r.t. < and $f \in \mathbb{F}[\mathbf{x}]$. Then there is a unique $r \in \mathbb{F}[\mathbf{x}]$:

1. $f - r \in I;$

2. no term of r is divisible by any monomial in LT(G).

$$\begin{split} \langle \mathsf{LT}(I) \rangle &= \langle \mathsf{LT}(G) \rangle \qquad \text{(i.e., } G \text{ is a GB of I)} \\ & \updownarrow \\ \forall p \in I \ \exists g \in G : \ \mathsf{lt}(g) \mid \mathsf{lt}(p). \end{split}$$

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Notation: For $G \subseteq \mathbb{F}[\mathbf{x}]$ finite, $f \in \mathbb{F}[\mathbf{x}]$,

 $f \operatorname{rem} G = \operatorname{PolynomialReduce}(f, G) = r \in \mathbb{F}[\mathbf{x}].$

$$\begin{array}{l} \langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle & \text{(i.e., } G \text{ is a GB of I)} \\ & \\ & \\ \forall p \in I \ \exists g \in G : \ \mathsf{lt}(g) \mid \mathsf{lt}(p). \end{array}$$

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Lemma Red. Let G be a GB for $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ w.r.t. <. Then

 $\forall f \in \mathbb{F}[\mathbf{x}]: \quad f \in I \qquad \Longleftrightarrow \qquad f \operatorname{rem} G = 0. \qquad (*)$

$$\begin{array}{l} \langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle & \text{(i.e., } G \text{ is a GB of I)} \\ \\ \updownarrow \\ \forall p \in I \ \exists g \in G : \ \mathsf{lt}(g) \mid \mathsf{lt}(p). \end{array}$$

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Proposition Red0. Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$, $G \subseteq I$ finite, < monomial order. Then G is a GB of $I \Leftrightarrow$ property (*) holds.

$$f = \sum_{i=1}^{s} c_i \, \mathbf{x}^{\alpha_i} g_i \in \mathbb{F}[\mathbf{x}]$$

$$g_1, \dots, g_s \in \mathbb{F}[\mathbf{x}]$$

$$\alpha_1, \dots, \alpha_s \in \mathbb{N}^n$$

$$c_1, \dots, c_s \in \mathbb{F}^*$$

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$$\alpha_1, \dots, \alpha_s \in \mathbb{N}^n$$

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with the following extra properties.

1. There is $\delta \in \mathbb{N}^n$ such that for all $1 \leq i \leq n$:

$$\alpha_i + \deg(g_i) = \delta$$
 i.e., $\deg(x^{\alpha_i}g_i) = \delta$

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 $\deg(f) < \delta.$

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2. we have

 $\deg(f) < \delta.$

Then for $\gamma_{i,j} \in \mathbb{N}^n$ with $\mathbf{x}^{\gamma_{i,j}} = \operatorname{lcm}(\operatorname{Im}(g_i), \operatorname{Im}(g_j))$ with $1 \leq i < j \leq s$: (a) $\delta - \gamma_{i,j} \in \mathbb{N}^n$, i.e., $\mathbf{x}^{\delta - \gamma_{i,j}} \in \mathbb{F}[\mathbf{x}]$

$$f = \sum_{i=1}^{s} c_i \mathbf{x}^{\alpha_i} g_i \in \mathbb{F}[\mathbf{x}]$$

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$$f = \sum_{i=1}^{s} c_i \mathbf{x}^{\alpha_i} g_i \in \mathbb{F}[\mathbf{x}]$$

$$g_1, \dots, g_s \in \mathbb{F}[\mathbf{x}]$$

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(b)
$$\deg(\mathbf{x}^{\delta-\gamma_{i,j}}S(g_i,g_j)) < \delta$$

(c) There exist $c_{i,j} \in \mathbb{F}$ such that

$$f = \sum_{1 \le i < j \le s} c_{i,j} \mathbf{x}^{\delta - \gamma_{i,j}} S(g_i, g_j)$$

Lecture 11: January 18, 2024

Algorithm GetGroebnerBasis (Buchberger's algorithm) Input: $f_1, \ldots, f_s \in \mathbb{F}[x_1, \ldots, x_n]$ with a monomial order <. Output: A Gröbner basis G of $\langle f_1, \ldots, f_s \rangle$ w.r.t. <.

- 1. Set $G = \{f_1, \dots, f_s\}$
- 2. repeat do

3.
$$S = \{\}$$

(*let $G = \{g_1, \dots, g_\sigma\}^*$)

4. for all
$$i, j$$
 with $1 \le i < j < \sigma$ do

5.
$$r = \mathsf{PolynomialReduce}(S(g_i, g_j), G) = S(g_i, g_j) \operatorname{rem} G$$

6. if
$$r \neq 0$$
 then $S = S \cup \{r\}$ fi

7. od

8. if $S = \{\}$ then return G fi

9.
$$G = G \cup S$$

10. od

11. return G

Definition. Let G be a GB of $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ w.r.t. <. G is called reduced iff

1.
$$lc(g) = 1$$
 for all $g \in G$.

2. for all $g \in G$ no monomial of g lies in $\langle \mathsf{LT}(G \setminus \{g\}) \rangle$.

Theorem ReducedGB.

There is an algorithm which computes for a given GB G of $I \leq \mathbb{F}[\mathbf{x}]$ w.r.t. < a reduced GB G' of I w.r.t. <.

Theorem UniqueGB.

Let G_1 and G_2 be two reduced GB of $I \leq \mathbb{F}[\mathbf{x}]$ w.r.t. <. Then $G_1 = G_2$.

Lecture 12: January 25, 2024

1. $\langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle$, i.e., G is a GB of I

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- 1. $\langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle$, i.e., G is a GB of I
- 2. $\forall p \in I \ \exists g \in G : \ \mathsf{lt}(g) \mid \mathsf{lt}(p).$
- 3. $\forall f \in \mathbb{F}[\mathbf{x}]$:

$$f\in I \iff f \operatorname{rem} G = 0.$$

- 1. $\langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle$, i.e., G is a GB of I
- $2. \ \forall p \in I \ \exists g \in G: \ \mathsf{lt}(g) \mid \mathsf{lt}(p).$

3. $\forall f \in \mathbb{F}[\mathbf{x}]$:

$$f\in I \iff f \operatorname{rem} G = 0.$$

4. PolynomialReduce implements a function,
i.e., for each input there is a unique output.
("don't care nondeterministic" → "don't know nondeterministic")

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- 4. PolynomialReduce implements a function,
 i.e., for each input there is a unique output.
 ("don't care nondeterministic" → "don't know nondeterministic")
- 5. $\forall 1 \leq i < j \leq s$: $S(g_i, g_j) \operatorname{rem} G = 0$

- 1. $\langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle$, i.e., G is a GB of I
- $2. \ \forall p \in I \ \exists g \in G: \ \mathsf{lt}(g) \mid \mathsf{lt}(p).$

3. $\forall f \in \mathbb{F}[\mathbf{x}]$:

$$f\in I \iff f \operatorname{rem} G = 0.$$

- 4. PolynomialReduce implements a function,
 i.e., for each input there is a unique output.
 ("don't care nondeterministic" → "don't know nondeterministic")
- 5. $\forall 1 \leq i < j \leq s$: $S(g_i, g_j) \operatorname{rem} G = 0$

6. $B = \{b + I \mid b \in \hat{B}\}$ forms a basis of the \mathbb{F} -vector space $\mathbb{F}[\mathbf{x}]/I$ with

$$\hat{B} = \{m \in [\mathbf{x}] \mid m \operatorname{rem} G = m\}.$$

Applications

- 1. Computation in the quotient ring $R=\mathbb{F}[\mathbf{x}]/I$
- 2. Ideal membership
- 3. Test ideal equality
- 4. Elimination property
- 5. Finding zeros

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- 6. Radical ideal membership
- 7. Ideal operations (and the corresponding operations of varieties)
 - (a) sum of ideals
 - (b) product of ideals
 - $(\ensuremath{\mathsf{c}})$ intersection of ideals