Analysis Exercises **Wintersemester 2022**
https://www3.risc.jku.at/education/courses/ws2022/mathematik1/

Sheet 11

Discussion on **Jan. 26, 2022**

Exercise 1 Prove the following extension of Theorem 7.21 (l'Hospital's Rule): Given two functions $f, g : (0, \infty) \to \mathbb{R}$ which are both differentiable on $(0, \infty)$,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Hint: Consider substituting $1/x$ *in for* x*, and taking* $\lim_{x\to 0^+}$.

Exercise 2 Using l'Hospital's Rule, compute

$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$

Does this result depend on the fact that the power of x in the numerator is 2? What if it was 5, or 10, or 1000? What does this tell you about how fast e^x grows, compared to polynomials in x?

Exercise 3 Using l'Hospital's Rule, and remembering that $x^{1/n} = \sqrt[n]{x}$ compute

$$\lim_{x\to\infty} \frac{\sqrt[7]{x}}{\ln(x)}.$$

Does this result depend on the fact that the root of x in the numerator is 7? What if it was 5, or 10, or 1000? What does this tell you about how fast $\ln(x)$ grows, compared to roots of x?

Exercise 4 Compute the following antiderivatives.

a) $\int (4x^3 - 6x^2 + 3)\, dx$

b) $\int (x^5 + 9x^4 - 7x^3 + x^2 - x - 8)\, dx$

c) $\int \sin(x)^2 dx$ *Hint: Trigonometric identities involving* $\sin(x)^2$

d) $\int \sec(x)^2 dx$

Exercise 5 Compute the following antiderivatives.

a) $\int x \ln(x) dx$
b) $\int (-7x^{-1} + 13x^{-2})\, dx$
c) $\int (5x^{-5} - 8x^{-4} + 2x^{-1})\, dx$

Exercise 6 Imagine that you throw a ball through the air. The ball will accelerate downward due to gravity at a constant rate $g = 9.81\ m/s^2$. So our equation for the ball's acceleration as a function of time is a constant function:

$$a(t) = g.$$

a) The acceleration of the ball is the derivative of its velocity with respect to time. Can we find the equation for velocity? Is there missing information?
b) The velocity of the ball is the derivative of its height with respect to time. Can we find the equation for the height of the ball? Is there missing information?
c) What kind of a path do we expect a ball to trace through the air as we throw it? Should it look like a straight line, a parabola, some other sort of curve?

Exercise 7 Consider the function

$$f(x) = x^3 + x^2 - 6x + 3.$$

a) Using Theorem 7.25, determine the critical points of $f(x)$.
b) Without looking at a graph of f, how can we determine the local minima (valleys) and local maxima (hills) of f?
c) Find the local minima and maxima.

Exercise 8 This problem is to show how Theorem 7.25 can be used to estimate the size of an atom. The energy of an electron in a hydrogen atom can be estimated by

$$E = -\underbrace{ke^2}_{\text{a}} \cdot \frac{1}{x} + \underbrace{\hbar^2/2m}_{\text{b}} \cdot \frac{1}{x^2}.$$

The term $-\frac{ke^2}{x}$ gives the classical energy of the electron to the nucleus, while the term $\frac{\hbar^2}{2mx^2}$ is a quantum mechanical effect. Here, e is the charge of the electron, m is the mass of the electron, k and $\hbar$ are certain physical constants, and x is the distance of the electron from the nucleus. Note: $e, m, k, \hbar$ are all *positive* constants.

a) Compute the first derivative $E'(x)$. *Hint: do not be intimidated by the fancy symbols. Think of the equation E as $E(x) = -a \cdot x^{-1} + b \cdot x^{-2}$, with $a, b > 0$ as positive constants.*
b) Set $E'(x) = 0$ and solve for x.
c) Denote x_0 as the solution to (b) above. Compute $E''(x_0)$. What can you say about the energy at $x = x_0$?

d) In physics, particles tend to occupy the minimal energy state. With this in mind, what does $x = x_0$ represent?

e) The constants $\hbar, k, m, e$ can be estimated by the following:

$$\begin{aligned}\hbar &= 1.0546 \times 10^{-34} \mathrm{J} \cdot \mathrm{s},\\ k &= 8.988 \times 10^{9} \frac{\mathrm{kg} \cdot \mathrm{m}^3}{\mathrm{s}^2 \mathrm{C}^2},\\ m &= 9.109 \times 10^{-31} \mathrm{kg},\\ e &= 1.602 \times 10^{-19} \mathrm{C}.\end{aligned}$$

Substitute these numbers for the solution x_0 in (b) in order to give a numerical estimation of x_0. *Hint: This is a straightforward computation. Don't worry about the units on the right of each number, e.g.,* $\mathrm{J} \cdot \mathrm{s}$ *or* $\frac{\mathrm{kg} \cdot \mathrm{m}^3}{\mathrm{s}^2 \mathrm{C}^2}$. *These are there for reasons connected with the underlying physics.*