to be prepared for 12.01.2023

Exercise 44. Let R be a commutative ring with 1. Prove the equivalence of the following statements.

1. Every ideal in R has a finite basis.
2. Each non empty set of ideals has an element that is maximal with respect to inclusion.
3. Each increasing chain of ideals terminates, i.e., if $M_1 \subseteq M_2 \subseteq \cdots$ is an ascending chain of ideals then there is an $N \in \mathbb{N}$ s.t. $\forall n \geq N \; M_n = M_N$.

Note that this is true for submodules of a module over an arbitrary ring.

Exercise 45. Let $\leq$ be a linear order on $\mathbb{N}^n$ satisfying the 2 axioms

1. $\forall_{\alpha \in \mathbb{N}^n} \; 0 \leq \alpha$;
2. $\forall_{\alpha,\beta,\gamma \in \mathbb{N}^n} \big(\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma\big)$.

Show that $\leq$ is a well-order, so that it is indeed a monomial order.

Hint: You may consider the relation $\leq$ as a linear order of monomials in a polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ and apply the Hilbert Basis Theorem.

Exercise 46. Show that the following relations on $\mathbb{N}^n$ are monomial orders:

1. $\alpha <_{\text{lex}} \beta \iff$ the leftmost nonzero entry in $\alpha - \beta$ is negative;
2. $\alpha <_{\text{grlex}} \beta \iff \deg \alpha < \deg \beta$ or $\Big(\deg \alpha = \deg \beta$ and $\alpha <_{\text{lex}} \beta\Big)$.

Exercise 47. Let $<$ be a monomial order on $\mathbb{N}^n$, and $R = \mathbb{F}[x_1, \ldots, x_n]$. For $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in R \setminus 0$, the **multidegree** of f is the greatest n-tuple of exponents occuring in f with a nonzero coefficient

$$\operatorname{mdeg}(f) = \max\{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}.$$

Prove the following statements for $f, g \in R \setminus 0$:

1. $\operatorname{mdeg}(f\,g) = \operatorname{mdeg}(f) + \operatorname{mdeg}(g)$;
2. If $f + g \neq 0$ then $\operatorname{mdeg}(f + g) \leq \max\{\operatorname{mdeg}(f), \operatorname{mdeg}(g)\}$;
3. If $f+g \neq 0$ and $\operatorname{mdeg}(f) \neq \operatorname{mdeg}(g)$ then $\operatorname{mdeg}(f+g) = \max\{\operatorname{mdeg}(f), \operatorname{mdeg}(g)\}$.

Exercise 48. Given a monomial order on $\mathbb{N}^n$, the **leading monomial** $\operatorname{lm}(f)$ of a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n] \setminus 0$ is the monomial $x^{\operatorname{mdeg}(f)}$.

Thus, if $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$, then $\operatorname{lm}(f)$ is the power product x^α with $\alpha = \max\{\beta \in \mathbb{N}^n : c_\beta \neq 0\}$. The **leading coefficient** of f is then $\operatorname{lc}(f) = c_{\operatorname{lm}(f)}$.

Consider the polynomial $f = 2xy^2 - xy + x^3$ in the ring $\mathbb{F}[x, y]$. Find monomial orderings $<_1$ and $<_2$ so that $\operatorname{lm}_{<_1}(f) \neq \operatorname{lm}_{<_2}(f)$.