to be prepared for 17.11.2022

Exercise 24. Let R be a unique factorization domain with a normalization function, i.e., a map $N\colon R \to R$ s.t.

1. $\forall_{a,b} \big(a \sim b \Rightarrow N(a) = N(b)\big)$ and $\forall_a N(a) \sim a$;
2. $\forall_{a,b} N(ab) = N(a)N(b)$.

We may then define the **leading unit** $\mathrm{lu}(a)$ as

$$\mathrm{lu}(a) = \begin{cases} \text{the unique } u \in R^+ \text{ with } a = u \cdot N(a) & \dots\ a \neq 0 \\ 1 & \dots\ a = 0. \end{cases}$$

Thus, $\forall_{a,b\in R}\, a = \mathrm{lu}(a) \cdot N(a)$, and the leading unit provides a map $\mathrm{lu}\colon R \to R^+$. Moreover we say that $a \in R$ is **normalized**[1], provided that $\mathrm{lu}(a) = 1$.

Prove the following statements:

1. $N(1) = 1$;
2. $\forall_{a,b\in R\setminus 0}\, \mathrm{lu}(ab) = \mathrm{lu}(a)\,\mathrm{lu}(b)$;
3. $\forall_{a,b\in R} \big(a \text{ and } b \text{ normalized } \Rightarrow ab \text{ normalized}\big)$;
4. $\forall_{a,b\in R\setminus 0} \big(a \text{ and } b \text{ normalized and } \exists_c\, a = bc \Rightarrow c \text{ normalized}\big)$.

Exercise 25. Let R be a UFD with normalization, i.e., with a map $N\colon R \to R$ as it is described in the previous exercise. For a polynomial $f \in R[x]$ set

$$\mathrm{LU}(f) := \begin{cases} \mathrm{lu}\big(\mathrm{lc}(f)\big) & \dots\ f \neq 0 \\ 1 & \dots\ f = 0 \end{cases}$$

where $\mathrm{lc}(f)$ is the leading coefficient of $f \in R[x] \setminus 0$. Show that $N(f) := \frac{f}{\mathrm{LU}(f)}$ defines a normalization on $R[x]$ with corresponding leading unit LU.

Moreover show that the primitive part of a nonzero normalized polynomial $f \in R[x]$ is normalized.

Exercise 26. Let R be a UFD, $f, g \in R[x]$. Prove that[2]

1. $\mathrm{pp}(fg) = \mathrm{pp}(f) \cdot \mathrm{pp}(g)$;
2. $\mathrm{cont}(fg) = \mathrm{cont}(f) \cdot \mathrm{cont}(g)$.

Exercise 27. Let R be a UFD, $f, g \in R[x]$ and $h = \mathrm{GCD}_{R[x]}(f, g)$. Prove that

1. $\mathrm{cont}(h) = \mathrm{gcd}_R\big(\mathrm{cont}(f), \mathrm{cont}(g)\big)$ and $\mathrm{pp}(h) = \mathrm{GCD}_{R[x]}\big(\mathrm{pp}(f), \mathrm{pp}(g)\big)$;
2. $h = \mathrm{gcd}_R\big(\mathrm{cont}(f), \mathrm{cont}(g)\big) \cdot \mathrm{GCD}_{R[x]}\big(\mathrm{pp}(f), \mathrm{pp}(g)\big)$;
3. If f or g is primitive then so is h.

[1] See the lecture notes for various instances of these concepts.
[2] Compare with ex. 10 on exercise sheet 2.