WS 2022

Computer Algebra (selected slides)

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Lecture 2: October 13, 2022

Algorithm (Normalized) extended Euclidean algorithm
Input: $f, g \in R$ with R Euclidean domain.
Output: $\rho_i, r_i, s_i, t_i \in R$ for $0 \le i \le l+1$ and q_i for $0 \le i \le l$

1. $\rho_0 = \mathsf{lu}(f),\ r_0 = \mathsf{normal}(f)(= f/\rho_0),\ s_0 = \rho_0^{-1},\ t_0 = 0$
   $\rho_1 = \mathsf{lu}(g),\ r_1 = \mathsf{normal}(g)(= g/\rho_1),\ s_1 = 0,\ t_1 = \rho_1^{-1}$
2. i=1
   while $r_i \neq 0$ do
   $q_i = r_{i-1}\,\mathsf{quot}\,r_i$
   $r_{i+1} = r_{i-1}\mathsf{rem} r_i (= r_{i-1} - q_i\, r_i))$
   $\rho_{i+1} = \mathsf{lu}(r_{i+1})$
   $r_{i+1} = \mathsf{normal}(r_{i+1})(= r_{i+1}/\rho_{i+1})$
   $s_{i+1} = (s_{i-1} - q_i\, s_i)/\rho_{i+1}$
   $t_{i+1} = (t_{i-1} - q_i\, t_i)/\rho_{i+1}$
   $i = i + 1$
   od
3. $l = i - 1$
   return ρ_i, r_i, s_i, t_i for $0 \le i \le l+1$, q_i for $0 \le i \le l$

Define

$$R_0 = \begin{pmatrix} s_0 & t_0 \\ s_1 & t_1 \end{pmatrix}$$

$$Q_i = \begin{pmatrix} 0 & 1 \\ \rho_{i+1}^{-1} & -q_i \rho_{i+1}^{-1} \end{pmatrix}, \qquad R_i = Q_i \dots Q_1 R_0 \qquad 0 \le i \le l$$

EEA-Lemma. For $0 \le i \le l$:

(i) $R_i \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix}$

(ii) $R_i = \begin{pmatrix} s_i & t_i \\ s_{i+1} & t_{i+1} \end{pmatrix}$

(iii) $\gcd(f, g) = \gcd(r_i, r_{i+1}) = r_l$

(iv) $s_i\, f + t_i\, g = r_i$ (also $i = l + 1$)

(v) $s_i\, t_{i+1} - t_i\, s_{i+1} = (-1)^i (\rho_0 \dots \rho_{i+1})^{-1}$ and $\gcd(s_i, t_i) = 1$

(vi) $\gcd(r_i, t_i) = \gcd(f, t_i)$

(vii) $f = (-1)^i \rho_0 \dots \rho_{i+1} (t_{i+1}\, r_i - t_i\, r_{i+1})$ and
$g = (-1)^{i+1} \rho_0 \dots \rho_{i+1} (s_{i+1}\, r_i - s_i\, r_{i+1})$

(viii) If $R = \mathbb{F}[x]$ then $\deg(t_i) + \deg(r_{i-1}) = \deg(f)$,
$\deg(s_i) + \deg(r_{i-1}) = \deg(g)$.

Lecture 4: October 27, 2022

Algorithm CRA (Chinese Remainder Algorithm)
Input: $m_0, \dots, m_{r-1} \in R^* \setminus R^+$ pairwise coprime,
$v_0, \dots, v_{r-1} \in R$ with R ED.
Output: $f \in R$ with $d(f) < d(m_0) + \dots d(m_{r-1})$ such that for $0 \leq i < r$:

$$m_i \mid f - v_i \quad \Leftrightarrow \quad f \equiv v_i \mod m_i$$

1. $m = m_0 \dots m_{r-1} \in R$
2. for $0 \leq i < r$ do
3. $\quad f_i = m/m_i \in R$
4. $\quad$ call the EEA and compute $s_i, t_i \in R$ such that $s_i\, f_i + t_i\, m_i = 1$
5. $\quad c_i = v_i\, s_i \operatorname{rem} m_i \in R \quad$ [note:$d(c_i) < d(m_i)$]
6. od
7. return $f = \sum_{i=0}^{r-1} c_i\, f_i$

Remark 1: $l_i = s_i\, f_i$ and $c_i\, f_i = v_i\, l_i \operatorname{rem} m$.

Remark 2: If $d(v_i) < d(m_i)$ then $f \operatorname{rem} m_i = v_i$.

Remark 3: The computation of t_i in the EEA can be skipped.

Lecture 5: November 03, 2022

Theorem (Rational function reconstruction) Let $h, f \in \mathbb{F}[x]$ with $\deg(f) < \deg(h) =: n$ and $k \in \{1, \dots, n\}$. Let $\{(r_j, s_j, t_j)\}$ be the ERS of h and f and let $j \in \mathbb{N}$ be minimal such that $\deg(r_j) < k$. Then:

1. There exist $r, t \in \mathbb{Z}$ with

$$r \equiv t\,f \mod m \quad \text{where } \deg(r_j) < k \text{ and } \deg(t_j) \le n-k,$$

   namely $(r,t) = (r_j, t_j)$.
   If $\gcd(r_j, t_j) = 1$ then $\gcd(t_j, h) = 1$.

2. If $\frac{r}{t} \in \mathbb{F}(x)$ is a canonical form solution to

$$r \equiv t\,f \mod h \quad \Leftrightarrow \quad r\,t^{-1} \equiv f \mod h$$

   with $\deg(t) \le n-k$, $\deg(r) < k$ and $\gcd(t,h) = 1$, then

$$(r,t) = \frac{1}{\operatorname{lc}(t_j)}\Big(r_j, t_j\Big).$$

3. There is a solution as in 2 iff $\gcd(r_j, t_j) = 1$.

Lecture 6: November 10, 2022

Theorem (Rational number reconstruction) Let $m, f \in \mathbb{N}$ with $f < m$ and $k \in \{1, \ldots, m\}$. Let $\{(r_j, s_j, t_j)\}$ be the ERS of m and f and let $j \in \mathbb{N}$ be minimal such that $r_j < k$.

1. There exist $r, t \in \mathbb{Z}$ with
$$r \equiv t\, f \mod m \quad \text{where } |m| < k \text{ and } 0 \leq t \leq \frac{m}{k},$$
namely $(r, t) = \mathsf{sgn}(t_j)(r_j, t_j)$. If $\gcd(r_j, t_j) = 1$ then $\gcd(t_j, m) = 1$.
2. If $\frac{r}{t} \in \mathbb{Z}$ is a canonical form solution to
$$r \equiv t\, f \mod m \quad \Leftrightarrow \quad r\, t^{-1} \equiv f \mod m$$
with $\deg(t) \leq \frac{m}{k}$, $\deg(r) < k$ and $\gcd(t, m) = 1$, then
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Define $q \in \mathbb{N}^*$ with

$$r_{j-1} - q\, r_j < k \leq r_{j-1} - (q-1) r_j \quad [q = 0 \text{ if } j = l+1]$$

Set
$$r_j^* = r_{j-1} - q\, r_j, \quad t_j^* = t_{j-1} - q\, t_j.$$

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$$(r, t) = \mathrm{sgn}(t_j)\Big(r_j, t_j\Big) \quad \text{or} \quad (r, t) = \mathrm{sgn}(t_j^*)\Big(r_j^*, t_j^*\Big).$$
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3. There is a solution as in 2 iff $\gcd(r_j, t_j) = 1$
or $(\gcd(r_j^*, t_j^*) = 1$ and $|t_j^*| \leq \frac{m}{k})$
4. There is at most one solution as in 2 with $|r| < \frac{k}{2}$.

Corollary UFD-GCD R UFD. Let $f, g \in R[x]$ and define $h = \gcd_{R[x]}$. Then:

1. We can split gcd-calculation problem by

$$h = \gcd_R(\mathsf{cont}(f), \mathsf{cont}(g)) \cdot \gcd_{R[x]}(\mathsf{pp}(f), \mathsf{pp}(g))$$

   In particular, h is primitive if f or g are primitive.

2. We have

$$\frac{h}{\mathrm{lc}(h)} = \gcd_{\mathbb{K}[x]}(f, g)$$

   in the quotient field $\mathbb{K} = Q(R)$.

Algorithm GCD for $R[x]$
Input: $f, g \in R[x]^*$ with UFD R and its quotient field $\mathbb{K} = Q(R)$ where one can compute gcds in R and $\mathbb{K}$ is computable.
Output: $\gcd(f, g) \in R[x]$

1. $\tilde{f} = \mathsf{pp}(f)$, $\tilde{c} = \mathsf{cont}(f)$
   $\tilde{g} = \mathsf{pp}(g)$, $\tilde{d} = \mathsf{cont}(g)$
2. Compute the following gcds in R:
   $a = \gcd_R(\tilde{c}, \tilde{d}) \in R$
   $b = \gcd_R(\mathrm{lc}(\tilde{f}), \mathrm{lc}(\tilde{g})) \in R$
3. Call the Euclidean algorithm in $\mathbb{K}[x]$ to get the monic polynomial
   $v = \gcd_{\mathbb{K}[x]}(\tilde{f}, \tilde{g}) \in \mathbb{K}[x]$
4. return $a\,\mathsf{pp}(b\,v)$

Remark: In step 1 (and most probably in step 3) we also utilize gcd computations in R.

As a consequence one obtains the following general statement.

Corollary Let $\mathbb{E} = G[x_1, \ldots, x_n]$ be a polynomial ring over a UFD G. Suppose that one can compute gcds in G and that the quotient field $Q(G)$ is computable. Then one can compute gcds in $\mathbb{E}$ and can carry out the field operations in $Q(\mathbb{E})$.

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Proof. By induction on the number n of variables.

- If $n = 0$, the corollary holds.
- Suppose that one can compute gcds in the UFD $R = G[x_1, \ldots, x_{n-1}]$ and that the field operations in $Q(R)$ can be executed.
  Then one can execute the algorithm above to compute gcds in $\mathbb{E} = R[x_n]$. In addition, one can carry out the field operations in $Q(R[x_n]) = Q(\mathbb{E})$; note that one can even calculate reduced representations in $Q(\mathbb{E})$, i.e., the numerators and denominators in $G[x_1, \ldots, x_{n-1}, x_n]$ are co-prime.

As a consequence one obtains the following general statement.

Corollary Let $\mathbb{E} = G[x_1, \ldots, x_n]$ be a polynomial ring over a UFD G. Suppose that one can compute gcds in G and that the quotient field $Q(G)$ is computable. Then one can compute gcds in $\mathbb{E}$ and can carry out the field operations in $Q(\mathbb{E})$.

Proof. By induction on the number n of variables.

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Remark: If G is a field, one obtains much more efficient algorithms; soon we will consider, e.g., $R = G[x, y]$ for a field G.

Lecture 7: November 16, 2022

$$\begin{bmatrix} f_n & & & & & & & & \\ f_{n-1} & & & & & & & & \\ \vdots & & & & & & & & \\ \vdots & & & & & & & & \\ \vdots & & & & & & & & \\ f_0 & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}$$

$$
\begin{bmatrix}
f_n & 0 & & & & & & \\
f_{n-1} & f_n & & & & & & \\
\vdots & \vdots & & & & & & \\
f_0 & \vdots & & & & & & \\
 & f_0 & & & & & & \\
 & & & & & & &
\end{bmatrix}
$$

$$
\begin{array}{cc}
 & \overbrace{\hphantom{f_{n-1}\quad f_n\quad \ddots\quad f_n}}^{m} \\
\begin{array}{r} m\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} &
\begin{bmatrix}
f_n & 0 & & & & & & \\
f_{n-1} & f_n & \ddots & & & & & \\
\vdots & \vdots & & f_n & & & & \\
\vdots & \vdots & & \vdots & & & & \\
f_0 & \vdots & & \vdots & & & & \\
 & f_0 & \ddots & \vdots & & & & \\
 & & & f_0 & & & &
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{l}\overbrace{\hphantom{f_{n-1} \quad f_n \quad \ddots \quad f_n}}^{m}\end{array} \\
\begin{array}{r} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \end{array} &
\left[\begin{array}{cccccccc}
f_n & 0 & & & g_m & & & \\
f_{n-1} & f_n & & & g_{m-1} & & & \\
\vdots & \vdots & \ddots & & & & & \\
 & & & f_n & g_1 & & & \\
 & & & \vdots & g_0 & & & \\
f_0 & & & & & & & \\
 & f_0 & & & & & & \\
 & & \ddots & & & & & \\
 & & & f_0 & & & &
\end{array}\right]
\end{array}
$$

$$
\begin{array}{r l}
 & \quad\overbrace{\hphantom{f_{n-1} \quad f_n \quad \ddots \quad f_n}}^{m} \\
\begin{array}{r} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right. \end{array} &
\begin{bmatrix}
f_n & 0 & & & g_m & & & \\
f_{n-1} & f_n & & & g_{m-1} & g_m & & \\
\vdots & \vdots & \ddots & & & \vdots & & \\
\vdots & \vdots & & f_n & g_1 & \vdots & & \\
\vdots & \vdots & & \vdots & g_0 & g_1 & & \\
\vdots & \vdots & & \vdots & & g_0 & & \\
f_0 & \vdots & & \vdots & & & & \\
 & f_0 & & \vdots & & & & \\
 & & \ddots & \vdots & & & & \\
 & & & f_0 & & & &
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\hphantom{f_{n-1}\quad f_n\quad \ddots\quad f_n}}^{m} & \overbrace{\hphantom{g_{m-1}\quad g_m\quad \ddots\quad \ddots\quad g_m}}^{n} \end{array} \\
\begin{array}{c} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \end{array} &
\begin{bmatrix}
f_n & 0 & & g_m & & & \\
f_{n-1} & f_n & & g_{m-1} & g_m & & \\
\vdots & \vdots & \ddots & & \vdots & \ddots & \\
 & & f_n & g_1 & & & \\
 & & \vdots & g_0 & g_1 & & \\
 & & & & g_0 & & g_m \\
f_0 & & & & & \ddots & \vdots \\
 & f_0 & \ddots & & & & \\
 & & f_0 & & & & g_0
\end{bmatrix}
\end{array}
$$

Let $s = \sum_{k=0}^{m-1} a_k x^k$, $t = \sum_{k=0}^{n-1} b_k x^k$, $f = \sum_{k=0}^{n} f_k x^k$, $g = \sum_{k=0}^{m} f_k x^k$. Then

$$\boxed{s\,f + t\,g = h = \sum_{k=0}^{n+m-1} h_k\,x^k}$$

$$\Updownarrow$$

$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\hphantom{f_{n-1}\quad f_n \quad\quad f_n}}^{m} & \overbrace{\hphantom{g_{m-1}\quad g_m \quad\quad\quad\quad g_m}}^{n} \end{array} \\
\begin{array}{c} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right. \end{array} &
\begin{bmatrix}
f_n & 0 & & g_m & & & \\
f_{n-1} & f_n & & g_{m-1} & g_m & & \\
\vdots & \vdots & \ddots & & \vdots & & \\
\vdots & \vdots & f_n & g_1 & \vdots & & \\
\vdots & \vdots & \vdots & g_0 & g_1 & & \\
\vdots & \vdots & \vdots & & g_0 & & g_m \\
f_0 & \vdots & \vdots & & & \ddots & \\
 & f_0 & \vdots & & & & \\
 & & \ddots & & & & \\
 & & f_0 & & & & g_0
\end{bmatrix}
\end{array}
\begin{bmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_0 \\ b_n \\ b_{n-1} \\ \vdots \\ b_0 \end{bmatrix}
=
\begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}
$$

Let $s = \sum_{k=0}^{m-1} a_k x^k$, $t = \sum_{k=0}^{n-1} b_k x^k$, $f = \sum_{k=0}^{n} f_k x^k$, $g = \sum_{k=0}^{m} f_k x^k$.
Then

$$\boxed{s\,f + t\,g = h = \sum_{k=0}^{n+m-1} h_k\,x^k}$$

$$\Updownarrow$$

$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\hspace{7em}}^{m} & \overbrace{\hspace{9em}}^{n} \end{array} \\
\begin{array}{c} m\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right. \\ n\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right. \end{array} &
\begin{bmatrix}
f_n & 0 & & g_m & & \\
f_{n-1} & f_n & & g_{m-1} & g_m & \\
\vdots & \vdots & \ddots & & \vdots & \\
\vdots & \vdots & f_n & g_1 & \vdots & \\
\vdots & \vdots & \vdots & g_0 & g_1 & \\
\vdots & \vdots & \vdots & & g_0 & g_m \\
f_0 & \vdots & \vdots & & & \\
 & f_0 & \vdots & & \ddots & \\
 & & f_0 & & & g_0
\end{bmatrix}
\end{array}
\begin{bmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_0 \\ b_n \\ b_{n-1} \\ \vdots \\ b_0 \end{bmatrix}
=
\begin{bmatrix} h_{m+n-1} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}
$$

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$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\hphantom{f_n \quad 0 \quad \cdots \quad f_n}}^{m} & \overbrace{\hphantom{g_m \quad g_m \quad \cdots \quad \cdots \quad g_m}}^{n} \end{array} \\
\begin{array}{c} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \end{array} &
\begin{bmatrix}
f_n & 0 & & g_m & & & \\
f_{n-1} & f_n & & g_{m-1} & g_m & & \\
\vdots & \vdots & \ddots & & \vdots & & \\
\vdots & \vdots & f_n & g_1 & \vdots & & \\
\vdots & \vdots & \vdots & g_0 & g_1 & & \\
\vdots & \vdots & \vdots & & g_0 & & g_m \\
f_0 & \vdots & \vdots & & & \ddots & \\
 & f_0 & \vdots & & & & \\
 & & f_0 & & & & g_0
\end{bmatrix}
\end{array}
\begin{bmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_0 \\ b_n \\ b_{n-1} \\ \vdots \\ \vdots \\ b_0 \end{bmatrix}
=
\begin{bmatrix} h_{m+n-1} \\ h_{m+n-2} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}
$$

Let $s = \sum_{k=0}^{m-1} a_k x^k$, $t = \sum_{k=0}^{n-1} b_k x^k$, $f = \sum_{k=0}^{n} f_k x^k$, $g = \sum_{k=0}^{m} f_k x^k$.
Then

$$\boxed{s\, f + t\, g = h = \sum_{k=0}^{n+m-1} h_k\, x^k}$$

$$\Updownarrow$$

$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\hphantom{f_n \quad 0 \quad \cdots \quad f_n}}^{m} & \overbrace{\hphantom{g_m \quad g_m \quad \cdots \quad \cdots \quad g_m}}^{n} \end{array} \\
\begin{array}{c} m \left\{ \vphantom{\begin{array}{c}1\\1\\1\\1\end{array}} \right. \\ n \left\{ \vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}} \right. \end{array} &
\begin{bmatrix}
f_n & 0 & & & g_m & & & \\
f_{n-1} & f_n & & & g_{m-1} & g_m & & \\
\vdots & \vdots & \ddots & & & \vdots & & \\
\vdots & \vdots & & f_n & g_1 & \vdots & & \\
\vdots & \vdots & & \vdots & g_0 & g_1 & & \\
\vdots & \vdots & & \vdots & & g_0 & & g_m \\
f_0 & \vdots & & \vdots & & & \ddots & \\
 & f_0 & & \vdots & & & & \\
 & & \ddots & \vdots & & & & \\
 & & & f_0 & & & & g_0
\end{bmatrix}
\end{array}
\begin{bmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_0 \\ b_n \\ b_{n-1} \\ \vdots \\ b_0 \end{bmatrix}
=
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$$

Let $s = \sum_{k=0}^{m-1} a_k x^k$, $t = \sum_{k=0}^{n-1} b_k x^k$, $f = \sum_{k=0}^{n} f_k x^k$, $g = \sum_{k=0}^{m} f_k x^k$.
Then

$$\boxed{s\,f + t\,g = h = \sum_{k=0}^{n+m-1} h_k\, x^k}$$

$$\Updownarrow$$

$$\begin{array}{cc}
 & \overbrace{\hphantom{f_{n-1}\quad f_n\quad \ddots\quad f_n}}^{m}\quad \overbrace{\hphantom{g_{m-1}\quad g_m\quad \ddots\quad g_m}}^{n} \\
\begin{array}{c} m\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right. \\ n\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right. \end{array} &
\underbrace{\begin{bmatrix}
f_n & 0 & & & g_m & & & \\
f_{n-1} & f_n & \ddots & & g_{m-1} & g_m & & \\
\vdots & \vdots & & & & \vdots & & \\
\vdots & \vdots & & f_n & g_1 & \vdots & & \\
\vdots & \vdots & & \vdots & g_0 & g_1 & & \\
\vdots & \vdots & & \vdots & & g_0 & \ddots & g_m \\
f_0 & \vdots & & \vdots & & & \ddots & \\
 & f_0 & \ddots & \vdots & & & & \\
 & & & f_0 & & & & g_0
\end{bmatrix}}_{=:\ \mathrm{Syl(f,g)}}
\begin{bmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_0 \\ b_n \\ b_{n-1} \\ \vdots \\ b_0 \end{bmatrix}
=
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\end{array}$$

Corollary UFD-res. Let R be a UFD and let $f, g \in R[x]$, not both zero. Then

$$\gcd(f,g) \in R[x] \setminus R \quad \Leftrightarrow \quad \mathsf{res}(f,g) = 0 \text{ in } R.$$

Proof. See Exercise 30.

1. Coefficients bounds in $\mathbb{F}[y]$

Theorem. Let $f, g \in \mathbb{F}[x, y]$ with $n = \deg_x(f)$ and $m = \deg_x(g)$ and $\deg_y(f), \deg_y(g) \leq d$. Then

$$\deg_y \mathsf{res}_x(f, g) \leq (n + m)d.$$

1. Coefficients bounds in $\mathbb{F}[y]$

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2. Coefficients bounds in $\mathbb{Z}$

For $f = \sum_{n=0}^{d} f_n\, x^n \in \mathbb{C}[x]$ define the 2-norm

$$\|f\|_2 = \Big(\sum_{n=0}^{d} |f_n|^2\Big)^{1/2}, \quad |a| = (a \cdot \bar{a})^{1/2} \in \mathbb{R}$$

and the max-norm

$$\|f\|_\infty = \max\{|f_n| : 0 \le n \le d\}.$$

Note:

$$\|f\|_\infty \le \|f\|_2 \le (n+1)^{1/2}\, \|f\|_\infty$$

1. Coefficients bounds in $\mathbb{F}[y]$

Theorem. Let $f, g \in \mathbb{F}[x, y]$ with $n = \deg_x(f)$ and $m = \deg_x(g)$ and $\deg_y(f), \deg_y(g) \le d$. Then

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2. Coefficients bounds in $\mathbb{Z}$

For $f = \sum_{n=0}^{d} f_n \, x^n \in \mathbb{C}[x]$ define the 2-norm

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$$\|f\|_\infty = \max\{|f_n| : 0 \le n \le d\}.$$

Note:

$$\|f\|_\infty \le \|f\|_2 \le (n+1)^{1/2} \, \|f\|_\infty$$

Theorem. Let $f, g \in \mathbb{Z}[x]$ with $n = \deg(f)$ and $m = \deg(g)$. Then

$$|\mathsf{res}_x(f, g)| \le \|f\|_2^m \, \|g\|_2^n \le (n+1)^{m/2}(m+1)^{n/2} \, \|f\|_\infty^m \, \|g\|_\infty^n .$$

Lecture 8: November 24, 2022

Lemma. Let $f, g \in R[x]^*$ and I be an ideal in R with $I \neq R$. Suppose that[1] $\overline{\mathrm{lc}(f)} \in R/I$ is not a zero-divisor. Then:

1. $\overline{\mathsf{res}(f,g)} = \bar{0} \quad \Leftrightarrow \quad \mathsf{res}(\bar{f}, \bar{g}) = \bar{0}$.
2. If R/I is a UFD then

$$\overline{\mathsf{res}(f,g)} = \bar{0} \quad \Leftrightarrow \quad \gcd(\bar{f}, \bar{g}) \notin R/I.$$

Proof. (1) is settled by Exercise 32.

[1] Note that $\deg(f) = \deg(\bar{f})$ and thus $\overline{\mathrm{lc}(f)} = \mathrm{lc}(\bar{f})$ does not hold in general.

Lemma. Let $f, g \in R[x]^*$ and I be an ideal in R with $I \neq R$. Suppose that[1] $\overline{\mathrm{lc}(f)} \in R/I$ is not a zero-divisor. Then:

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Proof. (1) is settled by Exercise 32. (2) follows by

$$\overline{\mathsf{res}(f,g)} = \bar{0} \qquad \stackrel{(1)}{\Longleftrightarrow} \qquad \mathsf{res}(\bar{f}, \bar{g}) = \bar{0}$$

[1] Note that $\deg(f) = \deg(\bar{f})$ and thus $\overline{\mathrm{lc}(f)} = \mathrm{lc}(\bar{f})$ does not hold in general.

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Proof. (1) is settled by Exercise 32. (2) follows by

$$\begin{aligned} \overline{\mathsf{res}(f,g)} = \bar{0} \quad & \overset{(1)}{\Longleftrightarrow} \quad \mathsf{res}(\bar{f}, \bar{g}) = \bar{0} \\ & \overset{\text{Cor. UFD-res}}{\Longleftrightarrow} \quad \gcd(\bar{f}, \bar{g}) \notin R/I. \end{aligned}$$

[1] Note that $\deg(f) = \deg(\bar{f})$ and thus $\overline{\mathrm{lc}(f)} = \mathrm{lc}(\bar{f})$ does not hold in general.

GCD-Theorem. Let R be an ED, $f, g \in R[x]^*$ and $p \in R$ be prime with $p \nmid \gcd_R(\mathrm{lc}(f), \mathrm{lc}(g))$; let $\mathbb{F} = R/\langle p \rangle$ be its quotient field. Then:

(i) $\mathrm{lc}(\gcd_{R[x]}(f, g)) \mid \gcd_R(\mathrm{lc}(f), \mathrm{lc}(g))$.

(ii) $\deg(\gcd_{\mathbb{F}[x]}(\bar{f}, \bar{g})) \geq \deg(\gcd_{R[x]}(f, g))$.

(iii)

$$\deg(\gcd_{\mathbb{F}[x]}(\bar{f}, \bar{g})) = \deg(\gcd_{R[x]}(f, g))$$

$$\Updownarrow_1$$

$$\overline{\mathrm{lc}(\gcd_{R[x]}(f, g))} \cdot \gcd_{\mathbb{F}[x]}(\bar{f}, \bar{g}) = \overline{\gcd_{R[x]}(f, g)}$$

$$\Updownarrow_2$$

$$p \nmid_R \mathrm{res}\left(\frac{f}{\gcd_{R[x]}(f, g)}, \frac{f}{\gcd_{R[x]}(f, g)}\right)$$

Algorithm modGCD for $\mathbb{F}[x,y]$ (big prime version)
Input: primitive $f, g \in \mathbb{F}[x,y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $\deg_y(f), \deg_y(g) \leq d$ for some $d \in \mathbb{N}$.
Output: $h = \gcd(f,g) \in \mathbb{F}[x,y]$

1. Compute $b := \gcd_{\mathbb{F}[y]}(\mathrm{lc}_x(f), \mathrm{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$
2. repeat
3. choose a random monic irreducible $p \in \mathbb{F}[y]$ with $\deg_y(p) = \ell$
4. call the EEA for $\bar{f}, \bar{g} \in \mathbb{E}[x]$ over the field $\mathbb{E} = \mathbb{F}[y]/\langle p \rangle$ to get the monic $v \in R[x]$ with $\deg_y(v) < \ell$ such that $\bar{v} = \gcd(\bar{f}, \bar{g}) \in \mathbb{E}[x]$.
5. Compute $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ where
$$\bar{w} = \overline{b\,v}, \quad \bar{f^*} = \frac{\bar{f}}{\bar{v}}, \quad \bar{g^*} = \frac{\bar{g}}{\bar{v}}$$
6. until $\deg_y(f^* w) = \deg_y(bf)$ and $\deg_y(g^* w) = \deg_y(bg)$
7. return $\mathrm{pp}_x(w)$

Theorem. Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

1. $\deg(r) \leq 2nd$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $d \geq \deg_y(f), \deg_y(g)$.
2. $p \nmid_R r$ if and only if the halting condition holds.
3. If $p \nmid_R r$ then $h = \mathsf{pp}_x(w)$.

Recall: **Theorem.** Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
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2. $p \nmid_R r$ if and only if the halting condition holds.
3. If $p \nmid_R r$ then $h = \mathsf{pp}_x(w)$.

Lemma. Let R be an ED and $f, g \in R[x]^*$, and p prime in R with $p \nmid \gcd_R(\mathrm{lc}(f), \mathrm{lc}(g))$; let $\mathbb{F} = R/\langle p \rangle$ be the quotient field.
If

$$\gcd_{\mathbb{F}[x]}(\bar{f}, \bar{g}) = 1$$

then

$$\gcd_{R[x]}(f, g) = \gcd_R(\mathsf{cont}(f), \mathsf{cont}(g)).$$

Lecture 9: December 1, 2022

For $f = \sum_{n=0}^{d} f_n\, x^n \in \mathbb{C}[x]$ define the q-norm ($q \in \mathbb{N}^*$)

$$\|f\|_q = \Big(\sum_{n=0}^{d} |f_n|^q\Big)^{1/q}, \quad |a| = (a \cdot \bar{a})^{1/2} \in \mathbb{R}$$

and the max-norm

$$\|f\|_\infty = \max\{|f_n| : 0 \leq n \leq d\}.$$

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and the max-norm

$$\|f\|_\infty = \max\{|f_n| : 0 \le n \le d\}.$$

Note (Ex. 34):

$$\begin{aligned} \|f\|_\infty \ &\le \|f\|_2 \le \ (n+1)^{1/2} \ \|f\|_\infty \\ \|f\|_2 \ &\le \|f\|_1 \le \ (n+1) \ \|f\|_\infty \end{aligned}$$

For $f = \sum_{n=0}^{d} f_n\, x^n \in \mathbb{C}[x]$ define the q-norm ($q \in \mathbb{N}^*$)

$$\|f\|_q = \Big(\sum_{n=0}^{d} |f_n|^q\Big)^{1/q}, \quad |a| = (a \cdot \bar{a})^{1/2} \in \mathbb{R}$$

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$$\|f\|_\infty \le \|f\|_2 \le (n+1)^{1/2}\, \|f\|_\infty$$
$$\|f\|_2 \le \|f\|_1 \le (n+1)\, \|f\|_\infty$$

Theorem Mignotte. Let $f, g, h \in \mathbb{Z}[x]$ with $\deg(f) = n$, $\deg(g) = m$ and $\deg(h) = k$. Suppose that $g\, h \mid f$. Then

$\|g\|_\infty\, \|h\|_\infty \le \|g\|_2\, \|h\|_2$

For $f = \sum_{n=0}^{d} f_n\, x^n \in \mathbb{C}[x]$ define the q-norm ($q \in \mathbb{N}^*$)

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$$\|g\|_\infty\, \|h\|_\infty \leq \|g\|_2\, \|h\|_2 \leq \|g\|_1\, \|h\|_1 \overset{\text{Ex. 38}}{\leq} 2^{m+k}\, \|f\|_2$$

For $f = \sum_{n=0}^{d} f_n\, x^n \in \mathbb{C}[x]$ define the q-norm ($q \in \mathbb{N}^*$)

$$\|f\|_q = \Big(\sum_{n=0}^{d} |f_n|^q \Big)^{1/q}, \quad |a| = (a \cdot \bar{a})^{1/2} \in \mathbb{R}$$

and the max-norm

$$\|f\|_\infty = \max\{|f_n| : 0 \le n \le d\}.$$

Note (Ex. 34):

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Theorem Mignotte. Let $f, g, h \in \mathbb{Z}[x]$ with $\deg(f) = n$, $\deg(g) = m$ and $\deg(h) = k$. Suppose that $g\, h \mid f$. Then

$$\|g\|_\infty \|h\|_\infty \le \|g\|_2 \|h\|_2 \le \|g\|_1 \|h\|_1 \overset{\text{Ex. 38}}{\le} 2^{m+k} \|f\|_2 \le (n+1)^{1/2} 2^{m+k} \|f\|_\infty$$

For $f = \sum_{n=0}^{d} f_n\, x^n \in \mathbb{C}[x]$ define the q-norm ($q \in \mathbb{N}^*$)

$$\|f\|_q = \Big(\sum_{n=0}^{d} |f_n|^q\Big)^{1/q}, \quad |a| = (a \cdot \bar{a})^{1/2} \in \mathbb{R}$$

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Special case ($g = 1$):

$$\|h\|_\infty \le \|h\|_2 \le \|h\|_1 \le 2^k\, \|f\|_2 \le (n+1)^{1/2} 2^k\, \|f\|_\infty\ .$$

Corollary. Let $f, g \in \mathbb{Z}[x]$ with $n = \deg(f) \geq \deg(g) \geq 1$ and $\|f\|_\infty, \|g\|_\infty \leq A$. Then

$$\|\gcd(f,g)\|_\infty \leq (n+1)^{1/2} 2^n A.$$

Lemma. Let $f, g \in \mathbb{Z}[x]$ with $\|f\|_\infty, \|g\|_\infty < \frac{p}{2}$. Then

$$\bar{f} = \bar{g} \quad \Leftrightarrow \quad f = g.$$

Lecture 10: December 15, 2022

Recall: **Algorithm modGCD** for $\mathbb{F}[x, y]$ (big prime version)
Input: primitive $f, g \in \mathbb{F}[x, y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $\deg_y(f), \deg_y(g) \leq d$ for some $d \in \mathbb{N}$.
Output: $h = \gcd(f, g) \in \mathbb{F}[x, y]$

1. Compute $b := \gcd_{\mathbb{F}[y]}(\mathrm{lc}_x(f), \mathrm{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$
2. repeat
3. choose a random monic irreducible $p \in \mathbb{F}[y]$ with $\deg_y(p) = \ell$
4. call the EEA for $\bar{f}, \bar{g} \in \mathbb{E}[x]$ over the field $\mathbb{E} = \mathbb{F}[y]/\langle p \rangle$ to get the monic $v \in R[x]$ with $\deg_y(v) < \ell$ such that $\bar{v} = \gcd(\bar{f}, \bar{g}) \in \mathbb{E}[x]$.
5. Compute $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ where
$$\bar{w} = \overline{b\, v}, \quad \bar{f^*} = \frac{\bar{f}}{\bar{v}}, \quad \bar{g^*} = \frac{\bar{g}}{\bar{v}}$$
6. until $\deg_y(f^* w) = \deg_y(bf)$ and $\deg_y(g^* w) = \deg_y(bg)$
7. return $\mathrm{pp}_x(w)$

Algorithm modGCD for $\mathbb{Z}[x]$ (big prime version)
Input: primitive $f, g \in \mathbb{Z}[x]$ with $n = \deg(f) \geq \deg(g) \geq 1$ and $\|f\|_\infty, \|g\|_\infty \leq A$ for some $A \in \mathbb{N}$.
Output: $h = \gcd(f, g) \in \mathbb{Z}[x]$

1. Compute $b := \gcd_{\mathbb{Z}}(\mathrm{lc}(f), \mathrm{lc}(g))$ and set $B = (n+1)^{1/2} 2^n A\, b$
2. repeat
3. choose a random prime p with $2B < p$
4. call the EEA for $\bar{f}, \bar{g} \in \mathbb{Z}_p[x]$ over the finite field $\mathbb{Z}_p$ to get the monic $v \in R[x]$ with $\|v\|_\infty < p/2$ such that $\bar{v} = \gcd(\bar{f}, \bar{g}) \in \mathbb{Z}_p[x]$.
5. Compute $w, f^*, g^* \in \mathbb{Z}[x]$ with $\|w\|_\infty, \|f^*\|_\infty, \|g^*\|_\infty < p/2$ where
$$\bar{w} = \overline{b\, v}, \quad \bar{f}^* = \frac{\bar{f}}{\bar{v}}, \quad \bar{g}^* = \frac{\bar{g}}{\bar{v}}$$
6. until $\|f^*\|_1 \|w\|_1 \leq B$ and $\|g^*\|_1 \|w\|_1 \leq B$
7. return pp(w)

Recall: **Theorem.** Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

1. $\deg(r) \leq 2nd$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $d \geq \deg_y(f), \deg_y(g)$.
2. $p \nmid_R r$ if and only if the halting condition holds.
3. If $p \nmid_R r$ then $h = \mathsf{pp}_x(w)$.

Theorem. Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \gcd_{\mathbb{Z}[x]}(f, g)$ and $r = \mathsf{res}(f/h, g/h) \in \mathbb{Z}$; note that $\mathrm{lc}(h) > 0$.
Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop. Then:

[2] There is the improved version $|r| \leq (n+1)^n A^{2n}$.

Recall: **Theorem.** Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

1. $\deg(r) \leq 2nd$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $d \geq \deg_y(f), \deg_y(g)$.
2. $p \nmid_R r$ if and only if the halting condition holds.
3. If $p \nmid_R r$ then $h = \mathsf{pp}_x(w)$.

Theorem. Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \gcd_{\mathbb{Z}[x]}(f, g)$ and $r = \mathsf{res}(f/h, g/h) \in \mathbb{Z}$; note that $\mathrm{lc}(h) > 0$.
Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop. Then:

1. $|r| \leq (n+1)^n A^{2n} 4^n$ where[2] $n = \deg(f) \geq \deg(g) \geq 1$ and $A \geq \|f\|_\infty, \|g\|_\infty$.

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Algorithm modGCD for $\mathbb{F}[x, y]$ (small prime version)
Input: primitive $f, g \in \mathbb{F}[x, y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $\deg_y(f), \deg_y(g) \leq d$ for some $d \in \mathbb{N}$.
Output: $h = \gcd(f, g) \in \mathbb{F}[x, y]$

1. Compute $b := \gcd_{\mathbb{F}[y]}(\mathrm{lc}_x(f), \mathrm{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$
2. repeat
3. choose a set $S \subseteq \mathbb{F}$ of ℓ evaluation points u with $b(u) \neq 0$.
4. for each $u \in S$ call the EEA to get $v_u = \gcd_{\mathbb{F}[x]}(f(x, u), g(x, u))$

7. Compute by interpolation each coefficient in $\mathbb{F}[y]$ of the polynomials $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ such that for each $u \in S$ we have

$$w(x, u) = b(u)v_u, \quad f^*(x, u) = \frac{f(x,u)}{v_u}, \quad g^*(x, u) = \frac{g(x,u)}{v_u}.$$

8. until $\deg_y(f^* w) = \deg_y(bf)$ and $\deg_y(g^* w) = \deg_y(bg)$
9. return $\mathrm{pp}_x(w)$

Algorithm modGCD for $\mathbb{F}[x,y]$ (small prime version)
Input: primitive $f, g \in \mathbb{F}[x,y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $\deg_y(f), \deg_y(g) \leq d$ for some $d \in \mathbb{N}$.
Output: $h = \gcd(f,g) \in \mathbb{F}[x,y]$

1. Compute $b := \gcd_{\mathbb{F}[y]}(\mathrm{lc}_x(f), \mathrm{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$
2. repeat
3. choose a set $S \subseteq \mathbb{F}$ of 2ℓ evaluation points u with $b(u) \neq 0$.
4. for each $u \in S$ call the EEA to get $v_u = \gcd_{\mathbb{F}[x]}(f(x,u), g(x,u))$

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Input: primitive $f, g \in \mathbb{F}[x,y] = R[x]$ with $R = \mathbb{F}[y]$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $\deg_y(f), \deg_y(g) \leq d$ for some $d \in \mathbb{N}$.
Output: $h = \gcd(f,g) \in \mathbb{F}[x,y]$

1. Compute $b := \gcd_{\mathbb{F}[y]}(\mathrm{lc}_x(f), \mathrm{lc}_x(g)) \in \mathbb{F}[y]$ and set $\ell = d + 1 + \deg_y(b)$
2. repeat
3. choose a set $S \subseteq \mathbb{F}$ of 2ℓ evaluation points u with $b(u) \neq 0$.
4. for each $u \in S$ call the EEA to get $v_u = \gcd_{\mathbb{F}[x]}(f(x,u), g(x,u))$
5. $\lambda = \min\{\deg(v_u) \mid u \in S\}$ and refine $S := \{u \in S \mid \deg(v_u) = \lambda\}$
6. if $|S| \geq \ell$ then remove $|S| - \ell$ points from S else goto 3.
7. Compute by interpolation each coefficient in $\mathbb{F}[y]$ of the polynomials $w, f^*, g^* \in R[x]$ with $\deg_y(w), \deg_y(f^*), \deg_y(g^*) < \ell$ such that for each $u \in S$ we have
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8. until $\deg_y(f^*w) = \deg_y(bf)$ and $\deg_y(g^*w) = \deg_y(bg)$
9. return $\mathrm{pp}_x(w)$

Recall: **Theorem.** Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop. Then:

1. $\deg(r) \leq 2nd$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $d \geq \deg_y(f), \deg_y(g)$.
2. $p \nmid_R r$ if and only if the halting condition holds.
3. If $p \nmid_R r$ then $h = \mathsf{pp}_x(w)$.

Theorem. Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop using the ℓ given points from S. Then:

Recall: **Theorem.** Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
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Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop using the ℓ given points from S. Then:

1. $\deg(r) \leq 2nd$ where $n = \deg_x(f) \geq \deg_x(g) \geq 1$ and $d \geq \deg_y(f), \deg_y(g)$.
2. $r(s) \neq 0$ for all $s \in S$ if and only if the halting condition holds.

Recall: **Theorem.** Let $f, g \in R$ be primitive where $R = \mathbb{F}[y]$. Let $h = \gcd_{R[x]}(f, g)$ and $r = \mathsf{res}_x(f/h, g/h) \in R$.
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Let $w \in \mathbb{F}[x]$ as calculated in the algorithm above after one loop using the ℓ given points from S. Then:

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3. If $r(s) \neq 0$ for all $s \in S$ then $h = \mathsf{pp}_x(w)$.

Lemma. Let $p \in R = \mathbb{F}[y]$ be a prime and $w \in R[x]$ as given in the algorithm above within one of the loops.
If $\mathsf{pp}(w) \mid f$ and $\mathsf{pp}(w) \mid g$ then $\gcd(f, g) = \mathsf{pp}(w)$.

Algorithm modGCD for $\mathbb{Z}[x]$ (small prime version)
Input: primitive $f, g \in \mathbb{Z}[x]$ with $n = \deg(f) \geq \deg(g) \geq 1$
and $\|f\|_\infty, \|g\|_\infty \leq A$ for some $A \in \mathbb{N}$.
Output: $h = \gcd(f, g) \in \mathbb{Z}[x]$

1. Compute $b := \gcd_{\mathbb{Z}}(\mathrm{lc}(f), \mathrm{lc}(g))$ and set $B = (n+1)^{1/2} 2^n A\, b$.

   Take $\ell = \log_2(2B+1)$
2. repeat
3. choose a set S of 2ℓ primes p with $p \nmid b$.
4. for each $p \in S$ call the EEA to get the monic $v_p \in \mathbb{Z}[x]$ where the coefficients are from $\{0, \ldots, p-1\}$ with $\bar{v}_p = \gcd_{\mathbb{Z}_p[x]}(\bar{f}, \bar{g})$
5. $\lambda = \min\{\deg(v_p) \mid p \in S\}$ and refine $S := \{p \in S \mid \deg(v_p) = \lambda\}$
6. if $|S| \geq \ell$ then remove $|S| - \ell$ points from S else goto 3.
7. Compute by CRA the coefficients of the polynomials $w, f^*, g^* \in \mathbb{Z}[x]$ with $\|w\|_\infty, \|f^*\|_\infty, \|g^*\|_\infty < (\prod_{p \in S} p)/2$ s.t. for each $p \in S$ we have

   $$\bar{w} = \overline{b\, v_p}, \quad \bar{f^*} = \frac{\bar{f}}{\bar{v_p}}, \quad \bar{g^*} = \frac{\bar{g}}{\bar{v_p}} \quad \text{(reduction mod } p\text{)}$$
8. until $\|f^*\|_1 \|w\|_1 \leq B$ and $\|g^*\|_1 \|w\|_1 \leq B$
9. return pp(w)

Recall: **Theorem.** Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \gcd_{\mathbb{Z}[x]}(f, g)$ and $r = \mathrm{res}(f/h, g/h) \in \mathbb{Z}$; note that $\mathrm{lc}(h) > 0$.
Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop. Then:

1. $|r| \leq (n+1)^n A^{2n}$ where $n = \deg(f) \geq \deg(g) \geq 1$ and $A \geq \|f\|_\infty, \|g\|_\infty$.
2. $p \nmid_{\mathbb{Z}} r$ if and only if the halting condition holds.
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Theorem. Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \gcd_{\mathbb{Z}[x]}(f, g)$ and $r = \mathrm{res}(f/h, g/h) \in \mathbb{Z}$; note that $\mathrm{lc}(h) > 0$.
Let $w \in \mathbb{Z}[x]$ as calculated in the algorithm above after one loop using the ℓ primes given in S. Then:

Recall: **Theorem.** Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \gcd_{\mathbb{Z}[x]}(f, g)$ and $r = \text{res}(f/h, g/h) \in \mathbb{Z}$; note that $\mathrm{lc}(h) > 0$.
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Recall: **Theorem.** Let $f, g \in \mathbb{Z}[x]$ be primitive. Let $h = \gcd_{\mathbb{Z}[x]}(f, g)$ and $r = \text{res}(f/h, g/h) \in \mathbb{Z}$; note that $\text{lc}(h) > 0$.
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2. $p \nmid_{\mathbb{Z}} r$ for all $p \in S$ if and only if the halting condition holds.
3. If $p \nmid_{\mathbb{Z}} r$ for all $p \in S$ then $h = \text{pp}(w)$.

Lemma. Let $p \in \mathbb{N}$ be a prime and $w \in \mathbb{Z}[x]$ as given in the algorithm above within one of the loops.
If $\mathsf{pp}(w) \mid f$ and $\mathsf{pp}(w) \mid g$ then $\gcd(f, g) = \mathsf{pp}(w)$.

Lecture 12: January 12, 2023

Definition. A monomial order on $\mathbb{F}[\mathbf{x}]$ is a relation $<$ on $\mathbb{N}^n$ such that

1. $<$ is a total order;
2. For all $\alpha, \beta, \gamma \in \mathbb{N}^n$:

$$\alpha < \beta \qquad \Rightarrow \qquad \alpha + \gamma < \beta + \gamma$$

3. $<$ is well-ordered, i.e.,

$$\forall S \subseteq \mathbb{N}^n \; \exists m \in S \; \forall s \in S : \; m \leq s$$

$$\Updownarrow$$

$$\forall s \in \mathbb{N}^n : \; s \geq \mathbf{0}.$$

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Lemma Deg. Let $<$ be a monomial order on $\mathbb{F}[\mathbf{x}]$ and $f, g \in \mathbb{F}[\mathbf{x}]^*$. Then:

1. $\deg(f\,g) = \deg(f) + \deg(g)$;
2. If $f + g \neq 0$ then

$$\deg(f + g) \leq \max(\deg(f), \deg(g));$$

   equality holds if $\deg(f) \neq \deg(g)$.

Algorithm PolynomialReduce

Input: $f, g_1, \dots, g_s \in \mathbb{F}[x_1, \dots, x_n] =: R$ with a monomial order $<$.

Output: $r, q_1, \dots, q_s \in R$ with $f = r + q_1\, g_1 + \dots + q_s\, g_s$

1. $r = 0$, $p = f$, $q_i = 0$ for $1 \le i \le s$
2. while $p \neq 0$ do
3. if $\mathrm{lt}(g_i) \mid \mathrm{lt}(p)$ for some $1 \le i \le s$ then
4. choose such an i and set $q_i = q_i + \frac{\mathrm{lt}(p_i)}{\mathrm{lt}(g_i)}$
   $$p = p - \frac{\mathrm{lt}(p_i)}{\mathrm{lt}(f_i)} g_i$$
5. else
6. $r = r + \mathrm{lt}(p)$, $p = p - \mathrm{lt}(p)$
7. fi
8. od
9. return $q_1, \dots, q_s$, r

Remark: If $s = n = 1$ then $q_1 =$ quot$\,(f, g_1)$ and $r =$ rem$\,(f, g_1)$

Algorithm PolynomialReduce

Input: $f, g_1, \ldots, g_s \in \mathbb{F}[x_1, \ldots, x_n] =: R$ with a monomial order $<$.

Output: $r, q_1, \ldots, q_s \in R$ with $f = r + q_1 g_1 + \cdots + q_s g_s$ where no monomial in r is divisible by $\mathrm{lt}(g_i)$ for all $1 \le i \le s$.

1. $r = 0$, $p = f$, $q_i = 0$ for $1 \le i \le s$
2. while $p \neq 0$ do
3. if $\mathrm{lt}(g_i) \mid \mathrm{lt}(p)$ for some $1 \le i \le s$ then
4. choose such an i and set $q_i = q_i + \frac{\mathrm{lt}(p_i)}{\mathrm{lt}(g_i)}$
   $$p = p - \frac{\mathrm{lt}(p_i)}{\mathrm{lt}(f_i)} g_i$$
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6. $r = r + \mathrm{lt}(p)$, $p = p - \mathrm{lt}(p)$
7. fi
8. od
9. return $q_1, \ldots, q_s$, r

Remark: If $s = n = 1$ then $q_1 = \mathrm{quot}\,(f, g_1)$ and $r = \mathrm{rem}\,(f, g_1)$

Lemma Mon. Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ that is generated by a set M of monomials, and let h be a monomial. Then:

$$h \in I \qquad \Leftrightarrow \qquad \exists m \in M : \, m \mid h.$$

Lecture 13: January 19, 2023

Definition. Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$, $G \subseteq I$ finite, $<$ monomial order.

$$\langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle \qquad :\Leftrightarrow \quad G \text{ is a GB of I}$$

Proposition Red. Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$, $G \subseteq I$ finite, $<$ monomial order. Then:

$$\langle \mathsf{LT}(I) \rangle = \langle \mathsf{LT}(G) \rangle \qquad \text{(i.e., } G \text{ is a GB of I)}$$
$$\Updownarrow$$
$$\forall p \in I \ \exists g \in G : \ \mathsf{lt}(g) \mid \mathsf{lt}(p).$$

Proposition Red. Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$, $G \subseteq I$ finite, $<$ monomial order. Then:

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$$f = \sum_{i=1}^{s} c_i\, \mathbf{x}^{\alpha_i} g_i \in \mathbb{F}[\mathbf{x}] \qquad \begin{array}{l} g_1, \dots, g_s \in \mathbb{F}[\mathbf{x}] \\ \alpha_1, \dots, \alpha_s \in \mathbb{N}^n \\ c_1, \dots, c_s \in \mathbb{F}^* \end{array}$$

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(c) There exist $c_{i,j} \in \mathbb{F}$ such that

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Algorithm GetGroebnerBasis (Buchberger's algorithm)
Input: $f_1, \ldots, f_s \in \mathbb{F}[x_1, \ldots, x_n]$ with a monomial order $<$.
Output: A Gröbner basis G of $\langle f_1, \ldots, f_s \rangle$ w.r.t. $<$.

1. Set $G = \{f_1, \ldots, f_s\}$
2. repeat do
3. $S = \{\}$
       (*let $G = \{g_1, \ldots, g_\sigma\}$*)
4.   for all i, j with $1 \leq i < j < \sigma$ do
5. $r =$ PolynomialReduce$(S(g_i, g_j), G) = S(g_i, g_j)$ rem G
6.   if $r \neq 0$ then $S = S \cup \{r\}$ fi
7.   od
8.   if $S = \{\}$ then return G fi
9. $G = G \cup S$
10. od
11. return G

Lecture 14: January 26, 2023

Theorem-Summary. $I \trianglelefteq \mathbb{F}[\mathbf{x}]$, $G = \{g_1, \ldots, g_s\} \subseteq I$, $<$ monomial order. Then the following statements are equivalent.

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Definition. Let G be a GB of $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ w.r.t. $<$. G is a reduced GB of I iff

1. $\mathrm{lc}(g) = 1$ for all $g \in G$.
2. for all $g \in G$ no monomial of g lies in $\langle \mathsf{LT}(G \setminus \{g\}) \rangle$.

Theorem UniqueGB.
Let G_1 and G_2 be two reduced GB of $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ w.r.t. $<$. Then $G_1 = G_2$.

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5. $\forall 1 \leq i < j \leq s :\ S(g_i, g_j) \,\mathsf{rem}\, G = 0$
6. $B = \{b + I \mid b \in \hat{B}\}$ forms a basis of the $\mathbb{F}$-vector space $\mathbb{F}[\mathbf{x}]/I$ with
$$\hat{B} = \{m \in [\mathbf{x}] \mid m \,\mathsf{rem}\, G = m\}.$$

Applications

1. Computation in the quotient ring $R = \mathbb{F}[\mathbf{x}]/I$
2. Ideal membership
3. Test ideal equality
4. Radical ideal membership
5. Elimination property
6. Finding zeros
7. Ideal operations (and the corresponding operations of varieties)
   (a) sum of ideals
   (b) product of ideals
   (c) intersection of ideals

$\vdots$