Sheet 12

Discussion on **Jan. 27, 2022**

Exercise 1 Prove the following extension of Theorem 7.21 (l'Hospital's Rule): Given two functions $f, g : (0, \infty) \to \mathbb{R}$ which are both differentiable on $(0, \infty)$,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Hint: Consider substituting $1/x$ *in for* x*, and taking* $\lim_{x\to 0^+}$.

Exercise 2 Let $m, n \in \mathbb{N}$, and $a_n, a_{n-1}, ..., a_1, a_0, b_m, b_{m-1}, ..., b_1, b_0 \in \mathbb{R}$. With l'Hospital's Rule in mind, justify our familiar rule that

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0} = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \\ \infty, & n > m \end{cases}.$$

Exercise 3 Using l'Hospital's Rule, compute

$$\lim_{x\to\infty} \frac{x^5}{e^x}.$$

Does this result depend on the fact that the power of x in the numerator is 5? What if it was 7, or 10, or 100? What does this tell you about how fast e^x grows, compared to polynomials in x?

Exercise 4 Using l'Hospital's Rule, and remembering that $x^{1/n} = \sqrt[n]{x}$ compute

$$\lim_{x\to\infty} \frac{\sqrt[3]{x}}{\ln(x)}.$$

Does this result depend on the fact that the root of x in the numerator is 3? What if it was 5, or 55, or 1000? What does this tell you about how fast $\ln(x)$ grows, compared to roots of x?

Exercise 5 Compute the following antiderivatives.

a) $\int (9x^3 - 7x^2 + 1)\, dx$
b) $\int \sin(x)^2 dx$ *Hint: Trigonometric identities involving* $\sin(x)^2$
c) $\int \sec(x)^2 dx$

Exercise 6 Compute the following antiderivatives.
a) $\int x \ln(x) dx$
b) $\int (-18x^{-1} + 23x^{-2})\, dx$
c) $\int (18x^{-4} - 2x^{-3} + 3x^{-1})\, dx$

Exercise 7 Consider the function

$$f(x) = \sqrt{x}$$

on the interval $(0, 3)$.
a) Compute the Taylor polynomial $T_n(x)$ centered on $x_0 = 1$, as given in Theorem 7.28. What is the coefficient of $(x - 1)^k$?
b) Consider the Taylor polynomial $T_n(2)$. What does the error term $R_{n+1}(2, 1)$ look like?
c) Give a bound on how large n must be in order to approximate $\sqrt{2}$ to d decimal places. It doesn't have to be an optimal bound.

Exercise 8 This problem is to show how Theorem 7.25 can be used to estimate the size of an atom. The energy of an electron in a hydrogen atom can be estimated by

$$E = -\underbrace{ke^2}_{\text{a}} \cdot \frac{1}{x} + \underbrace{\hbar^2/2m}_{\text{b}} \cdot \frac{1}{x^2}.$$

The term $-\frac{ke^2}{x}$ gives the classical energy of the electron to the nucleus, while the term $\frac{\hbar^2}{2mx^2}$ is a quantum mechanical effect. Here, e is the charge of the electron, m is the mass of the electron, k and $\hbar$ are certain physical constants, and x is the distance of the electron from the nucleus. Note: $e, m, k, \hbar$ are all *positive* constants.
a) Compute the first derivative $E'(x)$. *Hint: do not be intimidated by the fancy symbols. Think of the equation E as $E(x) = -a \cdot x^{-1} + b \cdot x^{-2}$, with $a, b > 0$ as positive constants.*
b) Set $E'(x) = 0$ and solve for x.
c) Denote x_0 as the solution to (b) above. Compute $E''(x_0)$. What can you say about the energy at $x = x_0$?
d) In physics, particles tend to occupy the minimal energy state. With this in mind, what does $x = x_0$ represent?

e) The constants $\hbar, k, m, e$ can be estimated by the following:

$$\begin{aligned}
\hbar &= 1.0546 \times 10^{-34}\,\mathrm{J\cdot s},\\
k &= 8.988 \times 10^{9}\,\frac{\mathrm{kg\cdot m^3}}{\mathrm{s^2C^2}},\\
m &= 9.109 \times 10^{-31}\,\mathrm{kg},\\
e &= 1.602 \times 10^{-19}\,\mathrm{C}.
\end{aligned}$$

Substitute these numbers for the solution x_0 in (b) in order to give a numerical estimation of x_0. *Hint: This is a straightforward computation. Don't worry about the units on the right of each number, e.g.,* $\mathrm{J\cdot s}$ *or* $\frac{\mathrm{kg\cdot m^3}}{\mathrm{s^2C^2}}$. *These are there for reasons connected with the underlying physics.*