Sheet 8

Discussion on **Dec. 2, 2021**

Exercise 1 Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is continuous on its entire domain.

Exercise 2 Let $f : (a, b) \to \mathbb{R}$ be a continuous function with $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, and let $x \in (a, b)$. Prove that the following statements are equivalent:

a) f is continuous at x.

b) For all $\epsilon > 0$ there exists some $\delta > 0$ such that for all $\zeta \in (a, b)$:

$$|f(\zeta) - f(x)| < \epsilon, \text{ whenever } |\zeta - x| < \delta.$$

Exercise 3 Let $(h_n)_{n\geq 0}$ be an arbitrary but fixed sequence converging to 0 such that $h_n \neq 0$ for every $n \geq 0$. Show that

$$\lim_{n\to\infty} |h_n| \sin\left(\frac{1}{h_n}\right) = 0.$$

Exercise 4 Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ both be continuous in (a, b), and that $g(x) \neq 0$ for all $x \in (a, b)$. Show that f/g is continuous on (a, b).

Exercise 5 Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ both be continuous on their domains. Show that $g \circ f : (a, b) \to \mathbb{R}$, $g \circ f(x) = g(f(x))$ is continuous on (a, b).

Exercise 6 Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = 1.6x^3 - 1.76x^2 - 68.16x.$$

With a calculator, compute $f(x)$ for $x = 7 + \frac{n}{100}$ for $1 \leq n \leq 99$. Notice that this takes place for x in the interval $(7, 8)$. What can you say about $f(x)$ for x in this interval? Does $f(x)$ vanish? If so, where?

Exercise 7 Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$h(x) = \sin(x).$$

With a calculator, compute $h(x)$ for $x = 3 + \frac{n}{100}$ for $1 \leq n \leq 99$. Notice that this takes place for x in the interval $(3, 4)$. Estimate where $h(x)$ must vanish.

Exercise 8 Consider the function $w : \mathbb{R} \to \mathbb{R}$ defined by

$$w(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Is w a continuous function? *Hint: Is* $\mathbb{Q}$ *dense in* $\mathbb{R}$*?*