Name: .... 18 March 2022

Studienkennzahl: ....

Matrikelnummer: ....

Final Exam / Klausur
Computer Algebra (326.105)
(no books / ohne Unterlagen)

You may give answers either in English or in German.
Man kann auf Englisch oder Deutsch antworten.

Explain all your answers. Simply giving the result or "yes/no" is not enough.
Begründe alle Antworten. Nur das Ergebnis oder "ja/nein" genügt nicht.

(1) (a) Give a definition of a Euclidean domain.
(b) Prove: *In a Euclidean domain E every ideal I is generated by one element (i.e., E is a principal ideal domain).*
Complete the following proof:
If $I = \langle 0 \rangle$, then ...
Otherwise let $a \in I^*$ s.t. $\deg(a)$ is minimal. .....

(2) Prove:
Let K be a field, and $a(x), b(x) \in K[x] \setminus K$.
$a(x)$ and $b(x)$ have a factor $c(x)$ of degree greater than 0
if and only if
there are polynomials $p(x), q(x) \in K[x]$, not both equal to 0, with $\deg(p) < \deg(b)$, $\deg(q) < \deg(a)$ and $p(x)a(x) + q(x)b(x) = 0$.

(3) For a polynomial $f(x) \in \mathbb{Z}[x]$ we denote the image of f modulo the prime p by $f_{(p)}(x)$.
(a) Consider $a(x) = x^3 + 2x^2 - 1$, $b(x) = x^3 - 3x - 2$ in $\mathbb{Z}[x]$.
The gcd of $a_{(3)}(x), b_{(3)}(x)$ in $\mathbb{Z}_3[x]$ is $x + 1$.
The gcd of $a_{(5)}(x), b_{(5)}(x)$ in $\mathbb{Z}_5[x]$ is $x^2 + 4x + 3$.
Can the gcd of $a(x), b(x)$ in $\mathbb{Z}[x]$ have degree 2?
(b) Consider $a(x) = x^5 - x^4 + x^3 - x^2 - 3x - 1$, $b(x) = 3x^5 - 3x^4 - 4x^3 + 4x^2 - 2x - 3$ in $\mathbb{Z}[x]$.
The greatest common divisor of a and b is $c(x) = x^2 - x - 1$.
The resultant of a/c and b/c is 343.
Can the gcd of a and b in $\mathbb{Z}_{17}[x]$ have degree 3?

(4) Consider $G = \{3x^2 + 2y^2 - 2y,\ xy,\ y^3 - y^2\}$ in $\mathbb{Q}[x, y]$.
Is G a Gröbner basis for the corresponding ideal w.r.t. the lexicographic ordering with $x > y$?

(5) Consider G as in Problem (4). Let I be the ideal generated by G in $\mathbb{Q}[x, y]$.
(a) Do the polynomials in I have a common root (in $\overline{\mathbb{Q}}^2$) ?
(b) Is the set of common roots of polynomials in I finite?
(c) What is the vector space dimension of $\mathbb{Q}[x, y]_{/I}$ over the field $\mathbb{Q}$?