Due date: 14.12.2021

Exercise 28. Let K be a field and let $a \in K^*$, $s \in [X]$, $g_1, g_2 \in K[X]$, and $F \subseteq K[X]$. Prove the remaining parts of Lemma 4.2.14 from the lecture notes, i.e. show that

(a) $\longrightarrow_F \subseteq \gg$,

(b) $\longrightarrow_F$ is Noetherian, and

(c) if $g_1 \longrightarrow_F g_2$, then $a \cdot s \cdot g_1 \longrightarrow_F a \cdot s \cdot g_2$.

Exercise 29. Provide an example of a locally confluent reduction relation which is not confluent.

Exercise 30. Fix an admissible ordering and consider the polynomial ring $R = K[x_1, \dots, x_n]$ over the field K. Let $f \in R$ and $I \subseteq R$ be an ideal. Show that f can be written in the form $f = g + r$, where $g \in I$, $r \in R$ and no term of r is divisible by any element of $\operatorname{lpp}(I)$. Is this representation unique?

Exercise 31. Make yourself familiar with Buchberger's algorithm `GRÖBNER_B` from the lecture notes and compute a Gröbner basis for the following ideals:

(a) $\langle xy^2 + y^2 - xy - 1, y^2 - 1 \rangle \subseteq \mathbb{Q}[x, y]$, and

(b) $\langle u^2 v + w + 1, u^2 w + v \rangle \subseteq \mathbb{Q}[u, v, w]$.

Use the lexicographic ordering with $x > y$ and $u > v > w$, respectively.

Exercise 32. Consider the polynomials $f = y - x^2$ and $g = z - x^3$. Prove or disprove: $\{f, g\}$ is a Gröbner basis of the ideal $\langle f, g \rangle \subseteq \mathbb{Q}[x, y, z]$ with respect to the lexicographic ordering with $x > y > z$. What would the result be if we choose a permutation of the variables, where x is not greater than y and z?

This is the last exercise sheet!

Good luck with the projects!