Due date: 7.12.2021

We will (finally) discuss Exercise 19 as well.

Exercise 24. In the lecture notes, some of the most popular orderings of the commutative monoid of power products are described and it is claimed that these are admissible orderings. Verify this claim for the subsequent cases, i.e. show that

(a) the graduated reverse lexicographic ordering from Example 4.2.4 (c),

(b) the product ordering from Example 4.2.4 (d)

are admissible orderings.

For the next exercise you should make yourself familiar with the notation of Definition 4.2.6 in the lecture notes. In particular, you should know what the terms $\operatorname{lpp}(\cdot)$, $\operatorname{lc}(\cdot)$, and $\operatorname{initial}(\cdot)$ of a non-zero polynomial with respect to an admissible ordering are.

Exercise 25. An admissible ordering of the commutative monoid $[x, y, z]$ induces an ordering of the terms of a polynomial. Rewrite each of the polynomials

$$f = 2x + 3y + z + x^2 - z^2 + x^3 \quad \text{and} \quad g = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4,$$

such that the terms are ordered with respect to (choose $x_1 = x$, $x_2 = y$, and $x_3 = z$)

(a) the lexicographic ordering with $\pi = \mathrm{id}$,

(b) the graduated lexicographic ordering with $\pi = \mathrm{id}$ and weight function 1_{const},

(c) the graduated reverse lexicographic ordering.

What are the leading power product, leading coefficient, and initial in each case? Repeat (a)–(c) with the variable permutation $x_1 = z$, $x_2 = y$, and $x_3 = x$.

Exercise 26. Let K be a field and $[X] = [x_1, \dots, x_n]$ be the commutative monoid of power products in n variables. Prove or disprove the subsequent claims.

(a) If $f, g \in K[X]$ are non-zero polynomials, then $\operatorname{initial}(fg) = \operatorname{initial}(f) \cdot \operatorname{initial}(g)$.

(b) If $f_i, g_i \in K[X]$ with $i = 1, \dots, m$ are non-zero polynomials, then

$$\operatorname{lpp}\left(\sum_{j=1}^{m} f_j g_j\right) = \operatorname{lpp}(f_k g_k)$$

for some $1 \le k \le m$.

(c) If $n = 1$, then there exists a unique admissible ordering on $[X]$.

(d) Consider the polynomial $f = x_1^2 + 2x_1x_2 - x_2^2 \in K[X]$, where $n = 2$. There exist admissible orderings $<_1$, $<_2$, and $<_3$ such that $\operatorname{lpp}_{<_1}(f) = x_1^2$, $\operatorname{lpp}_{<_2}(f) = x_1 x_2$, and $\operatorname{lpp}_{<_3}(f) = x_2^2$.

Exercise 27. Let R be a commutative ring with 1. Show that R is a Noetherian ring if and only if every non-empty set S of ideals in R contains a maximal element, i.e. an ideal $I \in S$ such that $\forall J \in S : I \subseteq J \Rightarrow I = J$. Find an example for R which is not a Noetherian ring.