Due date: 30.11.2021

We will discuss Exercise 19 as well.

Exercise 20. In this exercise you should prove a claim from the lecture notes. Consider non-constant polynomials $f, g \in K[x]$ over the field K such that $\deg(f) = m$ and $\deg(g) = n$. Denote by $\zeta_1, \dots, \zeta_m$ and $\eta_1, \dots, \eta_n$ the roots of f and g in a common splitting field, respectively. Show that

(a) $\operatorname{res}_x(f, g) = \operatorname{lc}(f)^n \operatorname{lc}(g)^m \prod_{i=1}^m \prod_{j=1}^n (\zeta_i - \eta_j)$.

(b) $\operatorname{res}_x(f, g) = \operatorname{lc}(f)^n \prod_{i=1}^m g(\zeta_i) = (-1)^{mn} \operatorname{lc}(g)^m \prod_{j=1}^n f(\eta_j)$.

Exercise 21. Solve the subsequent polynomial system over the field $\mathbb{C}$ by using resultants:

$$\begin{cases} xz - xy^2 - 4x^2 - \frac{1}{4} = 0 \\ y^2 z + 2x + \frac{1}{2} = 0 \\ x^2 z + y^2 + \frac{x}{2} = 0. \end{cases}$$

Exercise 22. Use resultants to find an implicit description of the subsequent parametric surface. Eliminate the variables s and t to obtain a single polynomial $f \in \mathbb{Q}[x, y, z]$ which describes the surface

$$\begin{cases} x = \frac{s}{3}(1 - \frac{s^2}{3} + t^2) \\ y = \frac{t}{3}(1 - \frac{t^2}{3} + s^2) \\ z = \frac{1}{3}(s^2 - t^2). \end{cases} \tag{1}$$

Answer the following questions:

(a) Do you obtain different results by changing the elimination order of the variables or by using other pairs of polynomials in the elimination steps? If yes, how are the results related?

(b) Are there extraneous factors in f? In other words, are there factors f_i of f such that $f_i(x, y, z) \neq 0$ for certain values of x, y, and z?

Bonus:[1] Find out how to compute Gröbner bases in your favourite CAS and find a Gröbner basis of the ideal in $\mathbb{Q}[s, t, x, y, z]$ generated by the parametric equations (1). Use the lexicographic ordering with $s > t > x > y > z$. From this basis, eliminate all polynomials containing the variables s and t. You should be left with a single polynomial in $\mathbb{Q}[x, y, z]$. Compare this remaining polynomial to the results from the resultant computations.

[1]For highly motivated students. Does not count as an additional exercise.

Exercise 23. Implicitization is also possible for hypersurfaces given by a rational parametrization. Consider the plane curve described by the parametrization

$$\left\{ x = \frac{3t}{1+t^3}\,,\ y = \frac{3t^2}{1+t^3} \right\}. \tag{2}$$

The naive approach to implicitization is to clear the denominators of the system (2) and then to eliminate the parameter t in the resulting polynomial equations as usual. Use resultants to find the implicit equation of the curve described by the rational parametrization (2) and plot the curve. Do you get extraneous factors?