Due date: 9.11.2021

Exercise 10. Let I be an integral domain and consider polynomials $a, b \in I[x]$ such that $b \neq 0$ and $m := \deg(a) \geq \deg(b) =: n$. Show that there are uniquely defined polynomials $q, r \in I[x]$ such that $\operatorname{lc}(b)^{m-n+1}a = qb + r$ and either $r = 0$ or $\deg(r) < \deg(b)$.

Exercise 11. If R is a commutative ring (with unit), show that the polynomial ring $R[x]$ is a principal ideal domain if and only if R is a field. What conditions must R satisfy such that $R[x]$ is an Euclidean domain?

Exercise 12. Which of the following domains (with the usual addition and multiplication) are Euclidean domains and which are not? Justify your answer.

(a) $\mathbb{Z}$ together with the degree function $\deg(i) = |i|$, $i \in \mathbb{Z}^*$.

(b) $\mathbb{Q}$ together with the degree function $\deg(q) = |q|$, $q \in \mathbb{Q}^*$.

(c) $\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$ together with the degree function $\deg(z) = |z|^2$, $z \in \mathbb{Z}[i]^*$. Note that $\mathbb{Z}[i]$ is called the ring of *Gaussian integers*.

(d) The polynomial ring $K[x, y]$, where K is a field and the degree function is set to be the total degree of the polynomial.

Exercise 13. Solve the following Chinese remainder problems, i.e. find solutions $x, y \in \mathbb{Z}$ of the subsequent systems of simultaneous congruences:

$$\begin{aligned} x &\equiv 1 \bmod 8 \\ x &\equiv 2 \bmod 25 \\ x &\equiv 3 \bmod 81 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &\equiv 5 \bmod 8 \\ y &\equiv 12 \bmod 25. \\ y &\equiv 47 \bmod 81 \end{aligned}$$

Exercise 14. Consider the polynomials

$$\begin{aligned} f &= x^7 - 3x^5 - 2x^4 + 13x^3 - 15x^2 + 7x - 1, \\ g &= x^6 - 9x^5 + 18x^4 - 13x^3 + 2x^2 + 2x - 1. \end{aligned}$$

Find $\gcd(f, g)$ in $\mathbb{Z}[x]$ using the modular algorithm. Verify whether the integer factors of the resultant of f/h and g/h are unlucky primes in the modular approach to GCD computation.