Due date: 12.10.2021

The exercises are meant as an invitation to try out different computer algebra systems.[1] Exercise sheets contain problems which have to be solved on a computer from time to time. Furthermore, most projects will contain explicit programming tasks, so it is a good idea to familiarize yourself with a CAS early on.

Here are three of the most popular and user-friendly systems. Any of these will be a perfect choice for the course.

1. Wolfram Mathematica (`https://www.wolfram.com/mathematica/`),
2. SageMath (`https://www.sagemath.org/`); downloadable for free at `http://www.sagemath.org/download.html`,
3. Maple (`https://www.maplesoft.com/products/maple/`).

Exercise 1. Consider the following system of equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 1 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + x_5 = 2 \\ x_4 + 2x_5 = 3 \\ 2x_4 + 3x_5 = 4 \\ 3x_4 + 4x_5 = 5 \end{cases}.$$

Use a CAS of your choice to find all solutions for $x_1, x_2, \dots, x_5$. In general, how would you solve such a system of linear equations? What are the commands in your CAS required for constructing and solving linear systems?

Exercise 2. Use a CAS to compute greatest common divisors (GCDs) in different domains.

1. Compute the GCD of the integers $a = 33600$, $b = 784080$, and $c = 214500$. Is the number 13860 contained in the ideal $\langle a, b, c\rangle \subseteq \mathbb{Z}$?
2. Compute the GCD of the polynomials $f, g \in \mathbb{Q}[x]$, where $f = 6x^5 + 2x^4 - 19x^3 - 6x^2 + 15x + 9$ and $g = 5x^4 - 4x^3 + 2x^2 - 2x - 2$. Is $\langle f, g\rangle \subseteq \mathbb{Q}[x]$ a proper ideal, i.e. is the ideal different form the entire polynomial ring?

   Hint: Both $\mathbb{Z}$ and $\mathbb{Q}[x]$ are so-called *principal ideal domains*. Every ideal of such a ring is principal, viz. generated by a single element. In our case, the single generator is determined by the GCD of the generating elements.

Exercise 3. Consider the polynomial $f = x^4 + 1 \in \mathbb{F}_p[x]$, where p is a prime. Find out how to factor polynomials over finite fields in your favourite CAS and compute the factorization of f in $\mathbb{F}_p[x]$ for the first 15 primes.

Bonus: How many factors does f have in $\mathbb{F}_p[x]$ for a given prime p? You should consider the cases $p = 2, 8k + 1, 8k + 3, 8k + 5, 8k + 7$, separately. Does f factor over $\mathbb{Q}$?

[1] In the sequel, we use the abbreviation CAS for computer algebra system.

Exercise 4. Consider the polynomial $f = x^5 - x^4 - 2x^3 + 3x^2 - 2$. Use a CAS to perform the following tasks:

1. Compute the roots of f numerically. You have influence on floating point precision if you want to.
2. Plot the graph of the function induced by f. Choose the plotting interval in such a way that you can "see" the real roots of f.
3. Compute the roots of f symbolically. What output does your CAS generate?
4. Repeat these steps for the polynomial $g = x^5 - x^4 + x^3 - x^2 + x - 2$.