Due date: 15.12.2020

37. Exercise

Implement Buchberger's algorithm GROEBNER_B for computing Gröbner bases in polynomial rings over $\mathbb{Q}$. Use your implementation to compute a Gröbner basis of the ideal $\langle f_1, f_2 \rangle \subseteq \mathbb{Q}[x, y]$ with respect to the lexicographic ordering with $x < y$, where

$$f_1 = x\,y^2 + x^2 + x$$
$$f_2 = x^2\,y + x.$$

Remark: You should use the command for computing the normal form of a polynomial offered by your CAS.[1]

38. Exercise

Let $R = F[x_1, \dots, x_n]$ be the polynomial ring over the field F. Consider linear polynomials

$$f_i = a_{i1}\,x_1 + \cdots + a_{in}\,x_n \in R,$$

where $1 \leq i \leq m$, $m \in \mathbb{Z}^+$. Denote by $A = \big(a_{ij}\big)$ the $m \times n$-matrix whose rows are formed by coefficients of the f_i. Let $B = \big(b_{ij}\big)$ be the reduced row echelon matrix obtained from A. The goal of this exercise is to prove that the non-zero polynomials obtained from the rows of B constitute the normed reduced Gröbner basis[2] of the ideal $I = \langle f_1, \dots, f_m \rangle_R$ with respect to the lexicographic ordering with $x_1 > \cdots > x_n$.

(a) Let l be the number of non-zero rows in B and consider the polynomials $g_1, \dots, g_l$, where

$$g_k = b_{k1}\,x_1 + \cdots + b_{kn}\,x_n$$

for $1 \leq k \leq l$. Show that $\langle g_1, \dots, g_l \rangle_R = I$.

(b) Show that $G = \{g_1, \dots, g_l\}$ is a Gröbner basis of the ideal I with respect to the aforementioned admissible ordering.

(c) Explain why the Gröbner basis G is normed and reduced.

[1] For *Mathematica* users: use the command `PolynomialReduce`. *Maple* users: use `Groebner:-NormalForm`. Finally, for *SageMath* the command is: `f.reduce(S)`, where f is a polynomial and S is a list of polynomials.

[2] Cf. Definition 4.2.24.

39. Exercise

Let F be a field. Show that the result of applying the Euclidean algorithm in $F[x]$ to any pair of polynomials f, g is a reduced Gröbner basis for $\langle f, g\rangle \subseteq F[x]$.

40. Exercise

Let $R = F[x_1, \dots, x_n]$ be the polynomial ring over the field F, $I \subseteq R$ an ideal and $G \subseteq I$ a Gröbner basis of I. Prove the following statements for $f, g \in G$ with $f \neq g$:[3]

(a) If $\mathrm{lpp}(f) \mid \mathrm{lpp}(g)$, then $G \setminus \{g\}$ is a Gröbner basis of I.

(b) If $g \longrightarrow_f g'$, then $(G \setminus \{g\}) \cup \{g'\}$ is a Gröbner basis of I.

41. Exercise

Let R be as in the previous exercise. Given a subset $G \subseteq F[x_1, \dots, x_n]$ and let $I = \langle G\rangle_R$. Show that G is a Gröbner basis of I (with respect to a fixed admissible ordering $>$) if and only if[4]

$$\langle \mathrm{initial}_>(G)\rangle_R = \langle \mathrm{initial}_>(I)\rangle_R.$$

As an early Christmas gift: This is the last exercise sheet!

Good luck with the projects!

[3]Cf. Theorem 4.2.23 in the lecture notes.

[4]This will prove the equivalence of Theorem 4.2.22 (f) and gives another characterization of Gröbner bases which is quite useful for theoretical proofs.