Due date: 1.12.2020

32. Exercise

Consider the partial order $\leq_\pi$ on $\mathbb{N}^n$ defined in the following way:

$$(a_1, \dots, a_n) \leq_\pi (b_1, \dots, b_n) \quad \Leftrightarrow \quad a_i \leq b_i \text{ for all } i \in \{1, \dots, n\}.$$

Show that any set $S \subseteq \mathbb{N}^n$ contains a finite subset $T \subseteq S$ such that $\forall s \in S \, \exists t \in T : t \leq_\pi s$.

33. Exercise

Let K be a field, $a \in K \setminus \{0\}$, $s \in [X]$, $g_1, g_2, h \in K[X]$ and $F \subseteq K[X]$. Prove the remaining parts of Lemma 4.2.14 from the lecture notes, i.e. show that

(a) $\longrightarrow_F \subseteq \gg$,

(b) $\longrightarrow_F$ is Noetherian,

(c) if $g_1 \longrightarrow_F g_2$ then $a\,s\,g_1 \longrightarrow_F a\,s\,g_2$.

34. Exercise

Verify the statement of Lemma 4.2.14 (d) on the basis of the following example: Let $R = \mathbb{Q}[x, y]$, $F = \{x^2 y^2 + y - 1, x^2 y + x\} \subseteq R$ and $g_1, h \in R$ with $g_1 = x^5 y^5$ and $h = x^3 y^3$.

35. Exercise

Give an example of a locally confluent reduction relation which is not confluent.

36. Exercise

Fix an admissible ordering and consider the polynomial ring $R = K[x_1, \dots, x_n]$ over the field K. Let $f \in R$ and $I \subseteq R$ be an ideal.

(a) Show that f can be written in the form $f = g + r$, where $g \in I$, $r \in R$ and no term of r is divisible by any element of $\mathrm{lpp}(I)$.

(b) Given two such expressions $f = g_1 + r_1$ and $f = g_2 + r_2$. Prove that $g_1 = g_2$ and $r_1 = r_2$, i.e. this representation is unique.