Due date: 24.11.2020

Make yourself familiar with the notations of Definition 4.2.6, lecture notes p. 35. In particular, you should know what the terms lpp(·), lc(·) and initial(·) of a non-zero polynomial with respect to an admissible ordering are.

28. Exercise

An admissible ordering of the commutative monoid $[x, y, z]$ induces an ordering of the terms of a polynomial. Let $x_1 = x$, $x_2 = y$ and $x_3 = z$. Rewrite each of the polynomials

$$f = 2x + 3y + z + x^2 - z^2 + x^3 \quad \text{and} \quad g = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4,$$

such that the terms are ordered with respect to

(a) the lexicographic ordering with $\pi = \mathrm{id}$,

(b) the graduated lexicographic ordering with $\pi = \mathrm{id}$ and weight function 1_{const},

(c) the graduated reverse lexicographic ordering.

What are the leading power product, leading coefficient and initial in each case? Repeat (a)–(c) with the variable permutation $x_1 = z$, $x_2 = y$ and $x_3 = x$.

Remark: You can do the exercise by hand or use a CAS.

29. Exercise

Example 4.2.4, lecture notes p. 33/34, lists some well-known orderings of the commutative monoid of power products and claims that these are admissible orderings. Verify this claim for the subsequent cases.

(a) Prove that the graduated reverse lexicographic ordering from Example 4.2.4 (c) is an admissible ordering.

(b) Show that the product ordering from Example 4.2.4 (d) is an admissible ordering.

30. Exercise

Let F be a field and $[X] = [x_1, \dots, x_n]$, $n \in \mathbb{Z}^+$. Prove or disprove the subsequent claims.

(a) If $f, g \in F[X]$ are non-zero polynomials, then $\operatorname{initial}(f\,g) = \operatorname{initial}(f)\operatorname{initial}(g)$.

(b) Given non-zero polynomials $f_i, g_i \in F[X]$ with $i \in \{1, \dots, s\}$, $s \in \mathbb{Z}^+$. Then

$$\operatorname{lpp}\left(\sum_{j=1}^{s} f_j\,g_j\right) = \operatorname{lpp}(f_i\,g_i)$$

for some i.

(c) If $[X] = [x_1]$, then there exists a unique admissible ordering on this commutative monoid of power products.

(d) Let $[X] = [x_1, x_2]$ and $f = x_1^2 + 2\,x_1\,x_2 - x_2^2$. There exist admissible orderings $<_1$, $<_2$ and $<_3$ such that $\operatorname{lpp}_{<_1}(f) = x_1^2$, $\operatorname{lpp}_{<_2}(f) = x_1\,x_2$ and $\operatorname{lpp}_{<_3}(f) = x_2^2$.

31. Exercise

Let R be a commutative ring with 1. Show that the following conditions are equivalent:

(a) Every ideal in R is finitely generated.

(b) There are no infinite strictly ascending chains of ideals in R.

(c) Every non-empty set S of ideals in R contains a maximal element, i.e. an ideal $I \in S$ such that $\forall J \in S : I \subseteq J \Rightarrow I = J$.