Due date: 17.11.2020

23. Exercise

Solve the subsequent system of algebraic equations over the field $\mathbb{C}$ using resultants:

$$\begin{cases} x\,z - x\,y^2 - 4\,x^2 - 1/4 & = 0 \\ y^2\,z + 2\,x + 1/2 & = 0 \\ x^2\,z + y^2 + x/2 & = 0. \end{cases}$$

24. Exercise

Given the polynomial ring $R = F[x_1, \dots, x_n]$ over an algebraically closed field F with $n \in \mathbb{Z}^+$. Consider the hypersurface $H \subseteq F^n$ defined by the zero locus of a polynomial $f \in R \setminus F$, i.e.

$$H = \mathbf{Z}(\{f\}) = \{(a_1, \dots, a_n) \in F^n \mid f(a_1, \dots, a_n) = 0\}.$$

Let $f = f_1^{e_1} \cdots f_k^{e_k}$ be the factorization of f into distinct irreducible factors and $\mathbf{I}(H)$ be the vanishing ideal of H, i.e.

$$\mathbf{I}(H) = \{g \in R \mid g(a_1, \dots, a_n) = 0 \text{ for all } (a_1, \dots, a_n) \in H\}.$$

Show that $\mathbf{I}(H) = \langle f_1 \cdots f_k \rangle_R$. In other words, prove that the vanishing ideal is generated by the square-free part of f.[1]

25. Exercise

Use resultants to find an implicit description of the subsequent parametric surface. Eliminate the variables s and t to obtain a single polynomial $f \in \mathbb{Q}[x, y, z]$ which describes the surface

$$\begin{cases} x = \frac{s}{3}\left(1 - \frac{s^2}{3} + t^2\right) \\ y = \frac{t}{3}\left(1 - \frac{t^2}{3} + s^2\right) \\ z = \frac{1}{3}\left(s^2 - t^2\right). \end{cases} \tag{1}$$

Now answer the following questions:

[1] Cf. Exercise 16(a).

(a) Do you obtain different results by changing the elimination order of the variables or by using other pairs of polynomials in the elimination steps? If yes, how are the results related?

(b) Are there extraneous factors in f? In other words, are there factors f_i of f such that $f_i(x, y, z) \neq 0$ when x, y and z are substituted by the parametrization?

(c) If we compute the square-free factorization of f, is the polynomial describing this surface guaranteed to be among those factors?

Note: The surface which corresponds to this parametrization is known as the *Enneper surface* and is implicitly described by a polynomial of total degree 9.

Bonus:[2] Compute a Gröbner basis of the ideal in $\mathbb{Q}[s, t, x, y, z]$ generated by the parametric equations (1). Use the lexicographic ordering with $s > t > x > y > z$. From this basis, eliminate all polynomials containing the variables s and t. You should be left with a single polynomial in $\mathbb{Q}[x, y, z]$. Compare this remaining polynomial to the results from the resultant computations.

26. Exercise

Implicitization is also possible for hypersurfaces given by a rational parametrization. Consider the plane curve described by the parametrization

$$x = \frac{3\,t}{1 + t^3} \quad \text{and} \quad y = \frac{3\,t^2}{1 + t^3}. \tag{2}$$

The naive approach to implicitization is to clear the denominators of the system (2) and then to eliminate the parameter t in the resulting algebraic equations as usual. Use resultants to find the implicit equation of the curve described by the rational parametrization (2) and plot the curve. Do you get extraneous factors?

27. Exercise

In this exercise you should implement an algorithm for computing resultants by polynomial division with remainder. Consider non-constant polynomials $f, g \in F[x]$ over a field F and recall from Exercise 20 that

$$\operatorname{res}_x(f, g) = (-1)^{\deg(f)\deg(g)} \operatorname{lc}(g)^{\deg(f)-\deg(r)} \operatorname{res}_x(g, r),$$

where r is the remainder on division of f by g. Similarly, $\operatorname{res}_x(g, r)$ can be computed by a division of g by r. Continuing in this way, we will get to the point where the current remainder, viz. the second argument of the resultant, becomes a constant.

[2]For exceptionally motivated students.

In the lecture notes the resultant is defined for non-constant polynomials only. In this exercise we will extend this definition as follows. If $f, g \in F[x] \setminus \{0\}$ and $p, q \in F \setminus \{0\}$, then define[3]

$$\begin{aligned} \operatorname{res}_x(f, q) &:= q^{\deg(f)}, \\ \operatorname{res}_x(p, g) &:= p^{\deg(g)}, \\ \operatorname{res}_x(f, 0) &:= 0 =: \operatorname{res}_x(0, g). \end{aligned}$$

Now use the antecedent ideas and definitions to implement an algorithm that computes the resultant of two non-constant polynomials in $\mathbb{Q}[x]$ by successive polynomial division with remainder. You are **not** allowed to use the resultant method of your CAS. Test your implementation on several examples.

Remark: Your algorithm should be somewhat similar to the Euclidean algorithm.

[3]From the first two lines it follows that $\operatorname{res}_x(p, q) = 1$.