Due date: 10.11.2020

19. Exercise

Given non-constant polynomials $f, g \in I[x]$ over an integral domain I such that $\deg(f) = m$ and $\deg(g) = n$.

(a) Prove that $\operatorname{res}_x(f, g) = (-1)^{m\,n} \operatorname{res}_x(g, f)$.

(b) Let $\lambda, \mu \in I \setminus \{0\}$. Show that $\operatorname{res}_x(\lambda\, f, \mu\, g) = \lambda^n \mu^m \operatorname{res}_x(f, g)$.

(c) Given a subring $R \subseteq I$. If $f, g \in R[x]$, can we conclude that $\operatorname{res}_x(f, g) \in R$?

(d) Let $p, q \in \mathbb{Q}[x]$, where $p = x^5 - 3\,x^4 - 2\,x^3 + 3\,x^2 + 7\,x + 6$ and $q = x^4 + x^2 + 1$. Construct the Sylvester matrix $\operatorname{Syl}_x(p, q)$ and compute the resultant $\operatorname{res}_x(p, q)$. What does the resultant tell you about common factors of p and q in $\mathbb{Q}[x]$?

20. Exercise

Let $f, g \in F[x]$ be non-constant polynomials over a field F. Perform division with remainder of f by g, i.e. $f = q\,g + r$ such that $q, r \in F[x]$ and $r = 0$ or $\deg(r) < \deg(g)$. Assume that r is non-constant and let $\deg(f) = m$, $\deg(g) = n$ and $\deg(r) = o$. Show that in this case

$$\operatorname{res}_x(f, g) = (-1)^{m\,n} \operatorname{lc}(g)^{m-o} \operatorname{res}_x(g, r).$$

21. Exercise

In this exercise you should prove the following claim from the lecture notes: Consider non-constant polynomials $f, g \in F[x]$ over the field F such that $\deg(f) = m$ and $\deg(g) = n$. Denote by $\zeta_1, \ldots, \zeta_m$ and $\eta_1, \ldots, \eta_n$ the roots of f and g in a common splitting field, respectively. Now show that

$$\operatorname{res}_x(f, g) \;=\; \operatorname{lc}(f)^n \operatorname{lc}(g)^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\zeta_i - \eta_j).$$

22. Exercise

Given, as usual, non-constant polynomials $f, g \in F[x]$ over a field F such that $\deg(f) = m$ and $\deg(g) = n$. Let I denote the ideal $\langle f \rangle \subseteq F[x]$ and define the multiplication map

$$\begin{aligned} \varphi : F[x] \,/\, I &\to F[x] \,/\, I, \\ h + I &\mapsto g\,h + I. \end{aligned}$$

Prove that $\operatorname{res}_x(f, g) = \operatorname{lc}(f)^n \det(\varphi)$.

Note: The quotient ring $F[x] \,/\, I$ is an F-vector space of dimension m by interpreting the equivalence classes as remainders on division by f. It is easy to show that φ is a linear map. Recall that the determinant of a linear map is defined to be the determinant of any matrix representing this map (with respect to some basis of course).