

Logic Programming

Unification

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Unification algorithm: The heart of the computation model of logic programs.

Substitution

Definition (Substitution)

A *substitution* is a finite set of the form

$$\theta = \{v_1 \mapsto t_1, \dots, v_n \mapsto t_n\}$$

- ▶ v_i 's: distinct variables.
- ▶ t_i 's: terms with $t_i \neq v_i$.
- ▶ Binding: $v_i \mapsto t_i$.

Substitution Application

Definition (Substitution application)

Substitution $\theta = \{v_1 \mapsto t_1, \dots, v_n \mapsto t_n\}$ applied to an expression E ,

$$E\theta$$

Simultaneously replacing each occurrence of v_i in E with t_i .

$E\theta$ is called the *instance* of E wrt θ .

E_1 is more general than E_2 if E_2 is an instance of E_1 (wrt some substitution).

Substitution Application

Example (Application)

$$E = p(x, y, f(a)).$$

$$\theta = \{y \mapsto x, x \mapsto b\}.$$

$$E\theta = p(b, x, f(a)).$$

Note that x was not replaced second time.

Composition

Definition (Substitution Composition)

Given two substitutions

$$\begin{aligned}\theta &= \{v_1 \mapsto t_1, \dots, v_n \mapsto t_n\} \\ \sigma &= \{u_1 \mapsto s_1, \dots, u_m \mapsto s_m\},\end{aligned}$$

their *composition* $\theta\sigma$ is obtained from the set

$$\begin{aligned}\{v_1 \mapsto t_1\sigma, \dots, v_n \mapsto t_n\sigma, \\ u_1 \mapsto s_1, \dots, u_m \mapsto s_m\}\end{aligned}$$

by deleting

- ▶ all $u_i \mapsto s_i$'s with $u_i \in \{v_1, \dots, v_n\}$,
- ▶ all $v_j \mapsto t_j\sigma$'s with $v_j = t_j\sigma$.

Substitution Composition

Example (Composition)

$$\theta = \{x \mapsto f(y), y \mapsto z\}.$$

$$\sigma = \{x \mapsto a, y \mapsto b, z \mapsto y\}.$$

$$\theta\sigma = \{x \mapsto f(b), z \mapsto y\}.$$

Empty Substitution

Empty substitution, denoted ε :

- ▶ Empty set of bindings.
- ▶ $E\varepsilon = E$ for all expressions E .

Properties

Theorem

$$\theta\varepsilon = \varepsilon\theta = \theta.$$

$$(E\theta)\sigma = E(\theta\sigma).$$

$$(\theta\sigma)\lambda = \theta(\sigma\lambda).$$

Example (Properties)

Example

Given:

$$\theta = \{x \mapsto f(y), y \mapsto z\}.$$

$$\sigma = \{x \mapsto a, z \mapsto b\}.$$

$$E = p(x, y, g(z)).$$

Then

$$\theta\sigma = \{x \mapsto f(y), y \mapsto b, z \mapsto b\}.$$

$$E\theta = p(f(y), z, g(z)).$$

$$(E\theta)\sigma = p(f(y), b, g(b)).$$

$$E(\theta\sigma) = p(f(y), b, g(b)).$$

Renaming Substitution

Definition (Renaming Substitution)

$\{x_1 \mapsto y_1, \dots, x_n \mapsto y_n\}$ is a *renaming substitution* iff y_i 's are distinct variables.

Renaming an Expression

Definition (Renaming Substitution for an Expression)

Let V be the set of variables of an expression E .

A substitution

$$\theta = \{x_1 \mapsto y_1, \dots, x_n \mapsto y_n\}$$

is a *renaming substitution for E* iff

- ▶ θ is a renaming substitution, and
- ▶ $\{x_1, \dots, x_n\} \subseteq V$, and
- ▶ $(V \setminus \{x_1, \dots, x_n\}) \cap \{y_1, \dots, y_n\} = \emptyset$.

Renaming an Expression

Example

- ▶ $E = f(x, a, y, z)$
- ▶ $\sigma_1 = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_3\}$ is a renaming subst. for E .
- ▶ $\sigma_2 = \{x \mapsto u_1, y \mapsto u_2\}$ is a renaming subst. for E .
- ▶ $\sigma_3 = \{x \mapsto y, y \mapsto x, z \mapsto u\}$ is a renaming subst. for E .
- ▶ $\sigma_4 = \{x \mapsto y, z \mapsto u\}$ is **not** a renaming subst. for E .
- ▶ $\sigma_5 = \{x \mapsto u, y \mapsto u, z \mapsto u\}$ is **not** a renaming subst.

Variants

Definition (Variant)

Expression E and expression F are *variants* iff there exist substitutions θ and σ such that

- ▶ $E\theta = F$ and
- ▶ $F\sigma = E$.

Variants and Renaming

Theorem

*Expression E and expression F are variants iff there exist **renaming** substitutions θ and σ such that*

- ▶ $E\theta = F$ and
- ▶ $F\sigma = E$.

Instantiation Quasi-Ordering

Definition (More General Substitution)

A substitution θ is *more general* than a substitution σ , written $\theta \leq \sigma$, iff there exists a substitution λ such that

$$\theta\lambda = \sigma.$$

The relation \leq on substitutions is called the *instantiation quasi-ordering*.

Instantiation Quasi-Ordering

Example (More General)

Let θ and σ be the substitutions:

$$\theta = \{x \mapsto y, u \mapsto f(y, z)\},$$

$$\sigma = \{x \mapsto z, y \mapsto z, u \mapsto f(z, z)\}.$$

Then $\theta \leq \sigma$ because $\theta\lambda = \sigma$ where

$$\lambda = \{y \mapsto z\}.$$

Unifier

Definition (Unifier of Expressions)

A substitution θ is a *unifier* of expressions E and F iff

$$E\theta = F\theta.$$

Unifier

Example (Unifier of Expressions)

Let E and F be two expressions:

$$E = f(x, b, g(z)),$$

$$F = f(f(y), y, g(u)).$$

Then $\theta = \{x \mapsto f(b), y \mapsto b, z \mapsto u\}$ is a unifier of E and F :

$$E\theta = f(f(b), b, g(u)),$$

$$F\theta = f(f(b), b, g(u)).$$

Unification Problem, Unifier

Definition (Unification Problem)

Unification problem is a finite set of equations (expression pairs).

Definition (Unifier)

σ is a *unifier* of a unification problem

$$\{E_1 \stackrel{?}{=} F_1, \dots, E_n \stackrel{?}{=} F_n\}$$

iff σ is a unifier of E_i and F_i for each $1 \leq i \leq n$, i.e., iff

$$E_1\sigma = F_1\sigma,$$

$\dots,$

$$E_n\sigma = F_n\sigma$$

Most General Unifier

Definition (MGU)

A unifier θ of E and F is *most general* iff θ is more general than any other unifier of E and F .

Unifiers and MGU

Example (Unifiers)

Let E and F be two expressions:

$$E = f(x, b, g(z)),$$

$$F = f(f(y), y, g(u)).$$

Unifiers of E and F (infinitely many):

$$\theta_1 = \{x \mapsto f(b), y \mapsto b, z \mapsto u\},$$

$$\theta_2 = \{x \mapsto f(b), y \mapsto b, u \mapsto z\},$$

$$\theta_3 = \{x \mapsto f(b), y \mapsto b, z \mapsto a, u \mapsto a\},$$

$$\theta_4 = \{x \mapsto f(b), y \mapsto b, z \mapsto u, w \mapsto d\},$$

...

Unifiers and MGU

Example (MGU)

Let E and F be expressions from the previous example:

$$E = f(x, b, g(z)), \quad F = f(f(y), y, g(u)).$$

MGU's of E and F :

$$\theta_1 = \{x \mapsto f(b), y \mapsto b, z \mapsto u\},$$

$$\theta_2 = \{x \mapsto f(b), y \mapsto b, u \mapsto z\}.$$

$$\theta_1 \leq \theta_2: \quad \theta_2 = \theta_1 \lambda_1 \text{ with } \lambda_1 = \{u \mapsto z\}.$$

$$\theta_2 \leq \theta_1: \quad \theta_1 = \theta_2 \lambda_2 \text{ with } \lambda_2 = \{z \mapsto u\}.$$

Note: λ_1 and λ_2 are renaming substitutions.

Equivalence of mgu-s

Theorem

Most general unifier of two expressions is unique up to variable renaming

Unification Algorithm

Rule-based approach.

- ▶ General form of rules:

$$P; \sigma \implies Q; \theta \text{ or}$$

$$P; \sigma \implies \perp.$$

- ▶ \perp denotes failure.
- ▶ σ and θ are substitutions.
- ▶ P and Q are unification problems: $\{E_1 \stackrel{?}{=} F_1, \dots, E_n \stackrel{?}{=} F_n\}$.

Unification Rules

Trivial:

$$\{s \stackrel{?}{=} s\} \cup P'; \sigma \Longrightarrow P'; \sigma.$$

Decomposition:

$$\begin{aligned} \{f(s_1, \dots, s_n) \stackrel{?}{=} f(t_1, \dots, t_n)\} \cup P'; \sigma \Longrightarrow \\ \{s_1 \stackrel{?}{=} t_1, \dots, s_n \stackrel{?}{=} t_n\} \cup P'; \sigma. \end{aligned}$$

if $f(s_1, \dots, s_n) \neq f(t_1, \dots, t_n)$.

Symbol Clash:

$$\{f(s_1, \dots, s_n) \stackrel{?}{=} g(t_1, \dots, t_m)\} \cup P'; \sigma \Longrightarrow \perp.$$

if $f \neq g$.

Unification Rules (Contd.)

Orient:

$$\{t \stackrel{?}{=} x\} \cup P'; \sigma \Longrightarrow \{x \stackrel{?}{=} t\} \cup P'; \sigma,$$

if t is not a variable.

Occurs Check:

$$\{x \stackrel{?}{=} t\} \cup P'; \sigma \Longrightarrow \perp,$$

if x occurs in t and $x \neq t$.

Variable Elimination:

$$\{x \stackrel{?}{=} t\} \cup P'; \sigma \Longrightarrow P'\theta; \sigma\theta,$$

if x does not occur in t , and $\theta = \{x \mapsto t\}$.

Unification Algorithm

In order to unify expressions E_1 and E_2 :

1. Create initial system $\{E_1 \stackrel{?}{=} E_2\}; \varepsilon$.
2. Apply successively unification rules.

Termination

Theorem (Termination)

The unification algorithm terminates either with \perp or with $\emptyset; \sigma$.

Soundness

Theorem (Soundness)

If $P; \varepsilon \Longrightarrow^+ \emptyset; \sigma$ then σ is a unifier of P .

Completeness

Theorem (Completeness)

For any unifier θ of P the unification algorithm finds a unifier σ of P such that $\sigma \leq \theta$.

Major Result

Theorem (Main Theorem)

If two expressions are unifiable then the unification algorithm computes their MGU.

Examples

Example (Failure)

Unify $p(f(a), g(x))$ and $p(y, y)$.

$$\{p(f(a), g(x)) \stackrel{?}{=} p(y, y)\}; \varepsilon \Longrightarrow_{\text{Dec}}$$

$$\{f(a) \stackrel{?}{=} y, g(x) \stackrel{?}{=} y\}; \varepsilon \Longrightarrow_{\text{Or}}$$

$$\{y \stackrel{?}{=} f(a), g(x) \stackrel{?}{=} y\}; \varepsilon \Longrightarrow_{\text{VarEl}}$$

$$\{g(x) \stackrel{?}{=} f(a)\}; \{y \mapsto f(a)\} \Longrightarrow_{\text{SymCl}}$$

\perp

Examples

Example (Success)

Unify $p(a, x, h(g(z)))$ and $p(z, h(y), h(y))$.

$$\begin{aligned} & \{p(a, x, h(g(z))) \stackrel{?}{=} p(z, h(y), h(y))\}; \varepsilon \Longrightarrow \text{Dec} \\ & \{a \stackrel{?}{=} z, x \stackrel{?}{=} h(y), h(g(z)) \stackrel{?}{=} h(y)\}; \varepsilon \Longrightarrow \text{Or} \\ & \{z \stackrel{?}{=} a, x \stackrel{?}{=} h(y), h(g(z)) \stackrel{?}{=} h(y)\}; \varepsilon \Longrightarrow \text{VarEl} \\ & \{x \stackrel{?}{=} h(y), h(g(a)) \stackrel{?}{=} h(y)\}; \{z \mapsto a\} \Longrightarrow \text{VarEl} \\ & \{h(g(a)) \stackrel{?}{=} h(y)\}; \{z \mapsto a, x \mapsto h(y)\} \Longrightarrow \text{Dec} \\ & \{g(a) \stackrel{?}{=} y\}; \{z \mapsto a, x \mapsto h(y)\} \Longrightarrow \text{Or} \\ & \{y \stackrel{?}{=} g(a)\}; \{z \mapsto a, x \mapsto h(y)\} \Longrightarrow \text{VarEl} \\ & \emptyset; \{z \mapsto a, x \mapsto h(g(a)), y \mapsto g(a)\}. \end{aligned}$$

Examples

Example (Failure)

Unify $p(x, x)$ and $p(y, f(y))$.

$$\begin{aligned} \{p(x, x) \stackrel{?}{=} p(y, f(y))\}; \varepsilon &\Longrightarrow_{\text{Dec}} \\ \{x \stackrel{?}{=} y, x \stackrel{?}{=} f(y)\}; \varepsilon &\Longrightarrow_{\text{VarEl}} \\ \{y \stackrel{?}{=} f(y)\}; \{x \mapsto y\} &\Longrightarrow_{\text{OccCh}} \\ &\perp \end{aligned}$$

Previous Example on PROLOG

Example (Infinite Terms)

?- p(X,X)=p(Y,f(Y)).

X = f(**), Y = f(**).

In some versions of PROLOG output looks like this:

X = f(f(f(f(f(f(f(f(f(f(...))))))))))

Y = f(f(f(f(f(f(f(f(f(f(...))))))))))

Occurrence Check

PROLOG unification algorithm skips Occurrence Check.

Reason: Occurrence Check can be expensive.

Justification: Most of the time this rule is not needed.

Drawback: Sometimes might lead to unexpected answers.

Occurrence Check

Example

```
less(X,s(X)).
```

```
foo:-less(s(Y),Y).
```

```
?- foo.
```

Yes