to be prepared for 17.01.2017

Exercise 47. Consider the ideals I and J in $\mathbb{R}[x,y,z]$.

$$\begin{aligned} I &= \langle xs^2+xt^2+x-2s,\ ys^2+yt^2+y-2t,\ zs^2+zt^2+z-s^2-t^2+1\rangle \cap \mathbb{R}[x,y,z] \\ J &= \langle x^2+y^2+z^2-1\rangle \end{aligned}$$

Determine whether they are equal.

Exercise 48. Let $I = \langle x^\alpha \mid \alpha \in A\rangle$ be a monomial ideal, and consider the set S of exponents of all monomials in I. For any admissible order $<$, prove that the smallest element of S with respect to $<$ must lie in A.

Exercise 49. Let $I_1, \ldots, I_r, J \subseteq K[x_1, \ldots, x_n]$ be ideals. Show the following:

1. $(I_1 + I_2)J = I_1J + I_2J$
2. $(I_1 \cdots I_r)^m = I_1^m \cdots I_r^m \quad (m \in \mathbb{N})$

Exercise 50.

1. For ideals I, J prove that $\sqrt{IJ} = \sqrt{I \cap J}$.
2. Give an example to show that the product of radical ideals[1] need not be a radical.
3. Demonstrate that $\sqrt{IJ}$ can differ from $\sqrt{I}\sqrt{J}$.

Exercise 51. Let $I, J, K \subseteq k[x_1, \ldots, x_n]$ be ideals. Prove the following:

1. $I : k[x_1, \ldots, x_n] = I$
2. $IJ \subseteq K \iff I \subseteq K : J$
3. $J \subseteq I \iff I : J = k[x_1, \ldots, x_n]$

[1] An ideal I is called **radical** or a **radical ideal** if it coincides with its radical, that is $\sqrt{I} = I$.