to be prepared for 13.12.2016

Exercise 36. Verify the statement of Lemma 2.3.14 (d) on the basis of the following example: $R = K[x, y]$, $F = \{x^2y^2 + y - 1, x^2y + x\}$, $g_1 = x^5y^5$, $h = x^3y^3$.

Exercise 37. Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal and G a Gröbner basis for I. Let $g, h \in G$ with $g \neq h$. Prove the following statements.

1. If $\mathrm{lpp}(g)|\mathrm{lpp}(h)$ then $G \setminus \{h\}$ is a Gröbner basis for I.
2. If $h \longrightarrow_g h'$ then $(G \setminus \{h\}) \cup \{h'\}$ is a Gröbner basis for I.

Exercise 38. Show that the result of applying the Euclidean Algorithm in $K[x]$ to any pair of polynomials f, g is a reduced Gröbner basis for $\langle f, g\rangle$.

Exercise 39. Consider linear polynomials in $K[x_1, \dots, x_n]$

$$f_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad 1 \leq i \leq m$$

and let $A = (a_{ij})$ be the $m \times n$ matrix of their coefficients. Let B be the reduced row echelon matrix determined by A and let $g_1, \dots, g_r$ be the linear polynomials coming from the nonzero rows of B. Use lex order with $x_1 > \cdots > x_n$ and show that $\{g_1, \dots, g_r\}$ is the reduced Groebner basis of $\langle f_1, \dots, f_m\rangle$.

Exercise 40. Let $I = \langle x^\alpha \mid \alpha \in A\rangle$ be a monomial ideal, and consider the set S of exponents of all monomials in I. For any admissible order $<$, prove that the smallest element of S with respect to $<$ must lie in A.

Exercise 41. Determine whether the polynomial $f = xy^3 - z^2 + y^5 - z^3$ is in the ideal $I = \langle -x^3 + y, x^2y - z\rangle$.