to be prepared for 07.01.2014

Exercise 45. Prove the following theorem:
If $p(y)$ is the minimal polynomial of α over the field k, *then* $\mathrm{norm}_{[\mathrm{k}(\alpha)/\mathrm{k}]}(h(x,\alpha))$ *and* $\mathrm{res}_y(h(x,y),p(y))$ *agree up to a non-zero multiplicative constant.*

Exercise 46. Consider polyomials $f, g \in k[x]$ of positive degrees m, n respectively. Let I denote the ideal in $k[x]$ generated by f, and let μ denote the multiplication map

$$\mu\colon k[x]/I \longrightarrow k[x]/I, \quad h + I \mapsto gh + I.$$

Demonstrate that $\mathrm{res}_x(f,g) = LC(f)^{\deg(g)} \det(\mu)$.

Exercise 47. Consider $f, g \in k[x]$ whose roots in $\overline{k}$ are $\zeta_1, \ldots, \zeta_m$ and $\eta_1, \ldots, \eta_n$ respectively. Prove that

$$\mathrm{res}_x(f,g) = LC(f)^{\deg(g)} \prod_{i=1}^{\deg(f)} g(\zeta_i) = (-1)^{\deg(f)\deg(g)} LC(g)^{\deg(f)} \prod_{j=1}^{\deg(g)} f(\eta_j)$$

Exercise 48. Given homogeneous polynomials

$$F = \sum_{i=0}^{m} a_i x^{m-i} y^i \text{ and } G = \sum_{j=0}^{n} b_j x^{n-j} y^j$$

in $k[x,y]$, their resultant $\mathrm{res}(F,G)$ is defined as the Sylvester determinant of the corresponding univariate polynomials (n rows of coefficients of $F(x,1)$, m rows of coefficients of $G(x,1)$; notice that the 'leading coefficients' of $F(x,1)$ and $G(x,1)$ are allowed to be 0).

Prove the following statements.

1. $\mathrm{res}(F,G)$ is an integer polynomial in the coefficients of F, G, i.e., there is a polynomial $\mathrm{Res}_{m,n} \in \mathbb{Z}[s_0, \ldots, s_m, t_0, \ldots, t_n]$ such that

   $$\mathrm{res}(F,G) = \mathrm{Res}_{m,n}(a_0, \ldots, a_m, b_0, \ldots, b_n)$$

   for all homogeneous polynomials F, G of degrees m, n.

2. $\mathrm{res}(F,G) = 0$ iff F and G have a common solution in $\mathbb{P}^1(\overline{k})$.

3. $\mathrm{res}(x^m, y^n) = 1$.