to be prepared for 03.12.2013

Exercise 34. Let I be an ideal in $K[x_1, \ldots, x_n]$, $F \subset K[x_1, \ldots, x_n]$ with $\langle F \rangle = I$. Prove that F is a Gröbner basis for I iff every $h \in I$ can be written as a bounded linear combination of elements in F; i.e. there are $f_j \in F, h_j \in K[x_1, \ldots, x_n]$ such that $h = \sum_j h_j f_j$ with $\mathrm{lpp}(h_j f_j) \leq \mathrm{lpp}(h)$ for all j.

Exercise 35. Show that the result of applying the Euclidean Algorithm in $K[x]$ to any pair of polynomials f, g is a reduced Gröbner basis for $\langle f, g \rangle$.

Exercise 36. Consider linear polynomials in $K[x_1, \ldots, x_n]$

$$f_i = a_{i1}x_1 + \cdots + a_{i_n}x_n \qquad 1 \leq i \leq m$$

and let $A = (a_{ij})$ be the $m \times n$ matrix of their coefficients. Let B be the reduced row echelon matrix determined by A and let $g_1, \ldots, g_r$ be the linear polynomials coming from the nonzero rows of B. Use lex order with $x_1 > \cdots > x_n$ and show that $g_1, \ldots, g_r$ form the reduced Gröbner basis of $\langle f_1, \ldots, f_m \rangle$.

Exercise 37. Use Gröbner bases for solving over $\mathbb{C}$:

$$\begin{aligned} f_1(x, y, z) &= xz - xy^2 - 4x^2 - \frac{1}{4} = 0, \\ f_2(x, y, z) &= y^2z + 2x + \frac{1}{2} = 0, \\ f_3(x, y, z) &= x^2z + y^2 + \frac{1}{2}x = 0. \end{aligned}$$

Exercise 38. Consider the polynomials

$$\begin{aligned} f_1(x, y) &= x^2y + xy + 1, \\ f_2(x, y) &= y^2 + x + y \end{aligned}$$

in $\mathbb{Z}_3[x, y]$. Compute a Gröbner basis for the ideal $\langle f_1, f_2 \rangle$ w.r.t. the graduated lexicographical ordering with $x < y$. Show intermediate results.

Exercise 39. Use Gröbner bases to find the implicit representation of the parametrized surface

$$\begin{aligned} x &= 1 + s + t + st \\ y &= 2 + s + st + t^2 \\ z &= s + t + s^2 \end{aligned}$$