to be prepared for 26.11.2013

Exercise 31. Prove the following theorem.
Let $F \subseteq k[x_1, \dots, x_n]$. The ideal congruence modulo $\langle F \rangle$ equals the reflexive-transitive-symmetric closure of the reduction relation $\longrightarrow_F$, i.e., $\equiv_{\langle F \rangle} = \longleftrightarrow^{\star}_F$.

Exercise 32. Let I be an ideal in $K[x_1, \dots, x_n]$, $F \subset K[x_1, \dots, x_n]$ with $\langle F \rangle = I$. Prove the equivalence of the following statements.

1. F is a Gröbner basis for I.
2. $\forall f \in I$ we have that $f \longrightarrow^{\star}_F 0$.
3. $f \longrightarrow_F$ for every $f \in I \setminus 0$.
4. $\forall g \in I\ \forall h \in K[x_1, \dots, x_n]$: if $g \longrightarrow^{\star}_F \underline{h}$ then $h = 0$.
5. $\forall g, h_1, h_2 \in K[x_1, \dots, x_n]$: if $g \longrightarrow^{\star}_F \underline{h_1}$ and $g \longrightarrow^{\star}_F \underline{h_2}$ then $h_1 = h_2$.
6. $\langle \mathrm{in}(F) \rangle = \langle \mathrm{in}(I) \rangle$.

Exercise 33. Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal and G a Gröbner basis for I. Let $g, h \in G$ with $g \neq h$. Prove the following statements.

1. If $\mathrm{lpp}(g) | \mathrm{lpp}(h)$ then $G \setminus \{h\}$ is a Gröbner basis for I.
2. If $h \longrightarrow_g h'$ then $(G \setminus \{h\}) \cup \{h'\}$ is a Gröbner basis for I.

Exercise 34. Use Gröbner bases for solving over $\mathbb{C}$:

$$\begin{aligned}
f_1(x, y, z) &= xz - xy^2 - 4x^2 - \frac{1}{4} = 0, \\
f_2(x, y, z) &= y^2 z + 2x + \frac{1}{2} = 0, \\
f_3(x, y, z) &= x^2 z + y^2 + \frac{1}{2} x = 0.
\end{aligned}$$

Exercise 35. Consider the polynomials

$$\begin{aligned}
f_1(x, y) &= x^2 y + xy + 1, \\
f_2(x, y) &= y^2 + x + y
\end{aligned}$$

in $\mathbb{Z}_3[x, y]$. Compute a Gröbner basis for the ideal $\langle f_1, f_2 \rangle$ w.r.t. the graduated lexicographical ordering with $x < y$. Show intermediate results.