to be prepared for 19.11.2013

Exercise 25. Let p be a prime, $m \in \mathbb{N}$. Let K and L be the finite fields $K = GF(p)$, $L = GF(q)$ with $q = p^m$. Let

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

be a polynomial with coefficients in L. Demonstrate that the following properties are equivalent.

1. $f(a) \in K$ for every $a \in L$.
2. $x^q - x$ divides $f(x)^p - f(x)$.

Exercise 26. Let R be a commutative ring with 1. Demonstrate that the following statements are equivalent:

1. Every ideal in R is generated by a finite set.
2. There are no infinite strictly ascending chains of ideals in R.
3. Every nonempty set S of ideals contains a maximal element (i.e. an ideal $a \in S$ such that $\forall b \in S$, if $a \subseteq b$ then $a = b$.

Exercise 27. The graduated reverse lexicographic ordering on power products of $x_1, \ldots, x_n$ $<_{\text{grlex}}$ is defined by

$$s <_{\text{grlex}} t \quad \text{iff} \quad \begin{array}{lll} \deg(s) < \deg(t) & \text{or} & \\ \deg(s) = \deg(t) & \text{and} & t <_{\text{lex},\pi} s; \end{array}$$

where π is the permutation on n letters given by $\pi(j) = n - j + 1$ and $<_{\text{lex},\pi}$ is the lexicographic order wrto. π. Prove that $<_{\text{grlex}}$ is an admissible ordering.

Exercise 28. Let $<_1$ be an admissible ordering on $X_1 = [x_1, \ldots, x_i]$ and $<_2$ an admissible ordering on $X_2 = [x_{i+1}, \ldots, x_n]$. Show that the product ordering $<_{prod,i,<_1,<_2}$ on $X = [x_1, \ldots, x_n]$ is an admissible ordering.

Exercise 29. $R[x_1, \ldots, x_n] = R[X]$ denote the polynomial ring in n indeterminates over a commutative ring with 1. Any admissible ordering $<$ on the monoid of power products $[X]$ induces a partial order $<<$ on $R[X]$ in the following way:

$$f << g \quad \text{iff} \quad \begin{array}{l} f = 0 \text{ and } g \neq 0 \text{ or} \\ f \neq 0,\ g \neq 0 \text{ and } \operatorname{lpp}(f) < \operatorname{lpp}(g) \text{ or} \\ f \neq 0,\ g \neq 0,\ \operatorname{lpp}(f) = \operatorname{lpp}(g) \text{ and } \operatorname{red}(f) << \operatorname{red}(g). \end{array}$$

Prove that $<<$ is a Noetherian partial order on $R[X]$.

Exercise 30. Consider the partial order $\leq_\pi$ on $\mathbb{N}^n$ defined as

$$(a_1, \ldots, a_n) \leq_\pi (b_1, \ldots, b_n) \iff a_i \leq b_i \ \forall i \in \{1, \ldots, n\}.$$

Prove that any set $X \subseteq \mathbb{N}^n$ contains a finite set $Y \subseteq X$ such that

$$\forall x \in X \exists y \in Y \text{ with } y \leq_\pi x.$$