Rewriting-Based Deduction. Completion

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Motivation

- Unrestricted use of the paramodulation rule can be very inefficient.
- Various methods have been proposed to restrict it without compromising the completeness.
- Term rewriting contributed essential techniques for refining paramodulation into a practical inference system.

Rewriting-Based Deduction for Unit Equalities

- We assume that the given set of clauses consists of unit equalities and one ground inequality.
- Goal: Design a calculus which works on such sets, restricts applications of the paramodulation rule, and is complete.
- Later this calculus can be extended to general clauses.

Equational Theory

- E: A set of equations.
- A: The set of equality axioms for E.
- $E \models s \approx t$ iff $I \models s \approx t$ for all interpretations I which is a model of $E \cup A$.
- Equational theory of E:

$$\approx_E := \{(s,t) \mid E \models s \approx t\}$$

- Notation: $s \approx_E t$ iff $(s,t) \in \approx_E$.

Basic Concepts in Term Rewriting

- A rewrite rule is an ordered pair of terms, written $l \to r$.
- Term rewriting system (TRS): a set of rewrite rules.

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What's this?

Problem and Solving Idea

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- In the course of completion, from time to time check whether s' and t' can be rewritten to the same term with the equations and rules constructed so far.

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- If yes, stop. You obtained a contradiction, which proves $s \approx_E t$.

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- In the course of completion, from time to time check whether s' and t' can be rewritten to the same term with the equations and rules constructed so far.
- If yes, stop. You obtained a contradiction, which proves $s \approx_E t$.
- If not, continue with completion. If this is not possible, then report: $s \approx_E t$ does not hold.

What We Need To Know

- What is rewriting?
- What is a ground convergent set of equations and rewrite rules?
- What is completion?

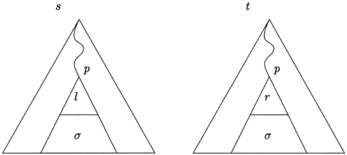
Basic Concepts in Term Rewriting

R: A term rewriting system.

- The rewrite relation induced by R, denoted $\to_R$, is a binary relation on terms defined as:

$s \to_R t$ iff

there exist $l \to r \in R$, a position p in s, a substitution σ such that $s|_p = l\sigma$ and $t = s[r\sigma]_p$.



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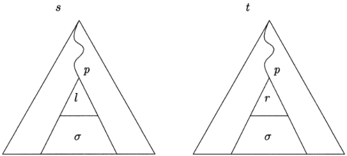

- Obviously $R \subseteq \to_R$.
- We may omit R when it is obvious from the context.

Basic Concepts in Term Rewriting

- s rewrites to t by R iff $s \to_R t$.
- $\leftarrow_R$ stands for the inverse and $\to_R^*$ for reflexive-transitive closure of $\to_R$.
- s is irreducible by R iff there is no t such that $s \to_R t$.
- t is a normal form of s by R iff $s \to_R^* t$ and t is irreducible by R.
- R is terminating iff $\to_R$ is well-founded, i.e., there is no infinite sequence of rewrite steps $s_1 \to_R s_2 \to_R s_3 \to_R \cdots$.

Basic Concepts in Term Rewriting

- R is confluent iff for all terms s, t_1, t_2, if

  $$s \to_R^* t_1 \quad \text{and} \quad s \to_R^* t_2,$$

  then there exists a term r such that

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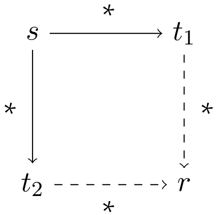
Graphically:



Basic Concepts in Term Rewriting

- t_1 and t_2 are joinable by R if there exists a term r such that

$$t_1 \to_R^* r \quad \text{and} \quad t_2 \to_R^* r.$$

- Notation: $t_1 \downarrow_R t_2$.

Basic Concepts in Term Rewriting

Example 1

Let $+$ be a binary (infix) function symbol, s a unary function symbol, 0 a constant.

$$R := \{0 + x \to x, \quad s(x) + y \to s(x + y)\}.$$

Then:

- $s(0) + s(s(0)) \to_R s(0 + s(s(0))) \to_R s(s(s(0)))$.
- $s(0) + s(s(0)) \to_R^* s(s(s(0)))$.
- $s(s(s(0)))$ is irreducible by R and, hence, is a normal form of $s(0) + s(s(0))$, of $s(0 + s(s(0)))$, and of $s(s(s(0)))$.

Basic Concepts in Term Rewriting

- A TRS R is convergent iff it is confluent and terminating.
- A convergent TRS provides a decision procedure for the underlying equational theory: Two terms are equivalent iff they reduce to the same normal form.
- Computation of normal forms by repeated reduction is a don't care non-deterministic process for convergent TRSs.

Basic Concepts in Term Rewriting

A strict order $>$ on terms is called a reduction order iff it is

1. monotonic: If $s > t$, then $r[s] > r[t]$ for all terms s, t, r;
2. stable: If $s > t$, then $s\sigma > t\sigma$ for all terms s, t and a substitution σ;
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Why are reduction orders interesting?

Theorem 1
A TRS R terminates iff there exists a reduction order $>$ that satisfies $l > r$ for all $l \to r \in R$.

Reduction Orders

Example 2

- $|t|$: The size of the term t.
- The order $>_1$: $s >_1 t$ iff $|s| > |t|$.

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- $|t|$: The size of the term t.
- The order $>_1$: $s >_1 t$ iff $|s| > |t|$.
- $>_1$ is monotonic and well-founded.
- However, $>_1$ is not a reduction order because it is not stable:

$$|f(f(x,x),y)| = 5 > 3 = |f(y,y)|$$

For $\sigma = \{y \mapsto f(x,x)\}$:

$$|\sigma(f(f(x,x),y))| = |f(f(x,x),f(x,x))| = 7,$$
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Reduction Orders

Example 2 (Cont.)

- $|t|_x$: The number of occurrences of x in t.
- The order $>_2$: $s >_2 t$ iff $|s| > |t|$ and $|s|_x \geq |t|_x$ for all x.

Reduction Orders

Example 2 (Cont.)

- $|t|_x$: The number of occurrences of x in t.
- The order $>_2$: $s >_2 t$ iff $|s| > |t|$ and $|s|_x \geq |t|_x$ for all x.
- $>_2$ is a reduction order.

Methods for Construction Reduction Orders

- Polynomial orders
- Simplification orders:
  - Recursive path orders
  - Knuth-Bendix orders

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Goal: Provide a variety of different reduction orders that can be used to show termination; not only by hand, but also automatically.

Lexicographic Path Order

Main idea behind recursive path orders:

- Two terms are compared by first comparing their root symbols.
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- Collections seen as tuples yields the lexicographic path order.

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- Collections seen as multisets yields the multiset path order. (Not considered in this course.)
- Collections seen as tuples yields the lexicographic path order.
- Combination of multisets and tuples yields the recursive path order with status. (Not considered in this course.)

Lexicographic Path Order

Definition 1

Let $\mathcal{F}$ be a finite signature and $>$ be a strict order on $\mathcal{F}$ (called the precedence). The lexicographic path order $>_{lpo}$ on $T(\mathcal{F}, \mathcal{V})$ induced by $>$ is defined as follows:

$s >_{lpo} t$ iff

(LPO1) $t \in \mathcal{V}ar(s)$ and $t \neq s$, or

(LPO2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

- (LPO2a) $s_i \geq_{lpo} t$ for some i, $1 \leq i \leq m$, or
- (LPO2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leq j \leq n$, or
- (LPO2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leq j \leq n$, and there exists i, $1 \leq i \leq m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

$\geq_{lpo}$ stands for the reflexive closure of $>_{lpo}$.

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Example 3

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

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- $f(x, e) >_{lpo} x$ by (LPO1)
- $i(e) >_{lpo} e$ by (LPO2a), because $e \geq_{lpo} e$.

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  - $i(f(x,y)) >_{lpo} i(y)$ is shown similarly.

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- $f(f(x,y),z) >^?_{lpo} f(x, f(y,z)))$. By (LPO2c) with $i = 1$:
  - $f(f(x,y),z) >_{lpo} x$ because of (LPO1).

Lexicographic Path Order

$s >_{lpo} t$ iff

(LPO1) $t \in \mathcal{V}ar(s)$ and $t \neq s$, or

(LPO2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

(LPO2a) $s_i \geq_{lpo} t$ for some i, $1 \leq i \leq m$, or

(LPO2b) $f > g$ and $s >_{lpo} t_j$ for all j, $1 \leq j \leq n$, or

(LPO2c) $f = g$, $s >_{lpo} t_j$ for all j, $1 \leq j \leq n$, and there exists i, $1 \leq i \leq m$ such that $s_1 = t_1, \ldots s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Example 3 (Cont.)

$\mathcal{F} = \{f, i, e\}$, f is binary, i is unary, e is constant, with $i > f > e$.

- $f(f(x,y),z) >^?_{lpo} f(x, f(y,z)))$. By (LPO2c) with $i = 1$:
  - $f(f(x,y),z) >_{lpo} x$ because of (LPO1).
  - $f(f(x,y),z) >^?_{lpo} f(y,z)$: By (LPO2c) with $i = 1$:
    - $f(f(x,y),z) >_{lpo} y$ and $f(f(x,y),z) >_{lpo} z$ by (LPO1).
    - $f(x,y) >_{lpo} y$ by (LPO1).

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Example 3 (Cont.)

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    - $f(x,y) >_{lpo} y$ by (LPO1).
  - $f(x,y) >_{lpo} x$ by (LPO1).

Reduction Orders

- Reduction orders are not total for terms with variables.
- For instance, $f(x)$ and $f(y)$ can not be ordered.
- $f(x, y)$ and $f(y, x)$ can not be ordered either.
- However, many reduction orders are total on ground terms.
- Fortunately, in theorem proving applications one can often reason about non-ground formulas by considering the corresponding ground instances.
- In such situations, ordered rewriting techniques can be applied.

Ordered Rewriting

- Given: A reduction order $>$ and a set of equations E.
- The rewrite system $E^>$ is defined as

$$E^> := \{s\sigma \to r\sigma \mid (s \approx t \in E \text{ or } t \approx s \in E) \text{ and } s\sigma > t\sigma\}$$

- The rewrite relation $\to_{E^>}$ induced by $E^>$ represents ordered rewriting with respect to E and $>$.

Ordered Rewriting

Example 4

- If $>$ is a lexicographic path ordering with precedence $+ > a > b > c$, then $b + c > c + b > c$.
- Let $E := \{x + y \approx y + x\}$.
- We may use the commutativity equation for ordered rewriting.
- $(b + c) + c \to_{E^>} (c + b) + c \to_{E^>} c + (c + b)$.

Ordered Rewriting

- If $>$ is a reduction ordering total on ground terms, then $E^{>}$ contains all (non-trivial) ground instances of an equation $s \approx t \in E$, either as a rule $s\sigma \rightarrow t\sigma$ or a rule $t\sigma \rightarrow s\sigma$.
- A rewrite system R is called ground convergent if the induced ground rewrite relation (that is, the rewrite relation $\rightarrow_R$ restricted to pairs of ground terms) is terminating and confluent.
- A set of equations E is called ground convergent with respect to $>$ if $E^{>}$ is ground convergent.

Critical Pairs

Ordered rewriting leads to the inference rule, called superposition:

$$\frac{s \approx t \qquad r[u] \approx v}{(r[t] \approx v)\sigma},$$

where $\sigma = mgu(s, u)$, $t\sigma \not\succeq s\sigma$, $v\sigma \not\succeq r\sigma$, and u is not a variable.

The equation $(r[t] \approx v)\sigma$ is called an ordered critical pair (with overlapped term $r[u]\sigma$) between $s \approx t$ and $r[u] \approx v$.

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The equation $(r[t] \approx v)\sigma$ is called an ordered critical pair (with overlapped term $r[u]\sigma$) between $s \approx t$ and $r[u] \approx v$.

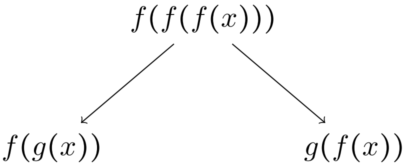
Lemma 1

Let $>$ be a ground total reduction ordering. A set E of equations is ground convergent with respect to $>$ iff for all ordered critical pairs $(r[t] \approx v)\sigma$ (with overlapped term $r[u]\sigma$) between equations in E and for all ground substitutions φ, if $r[u]\sigma\varphi > r[t]\sigma\varphi$ and $r[u]\sigma\varphi > v\sigma\varphi$, then $r[t]\sigma\varphi \downarrow_{E^>} v\sigma\varphi$.

Critical Pairs

Example 5

- Let $E := \{f(f(x)) \approx g(x)\}$ and $>$ be the LPO with $f > g$.
- Take a critical pair between the equation and its renamed copy, $f(f(x)) \approx g(x)$ and $f(f(y)) \approx g(y)$.



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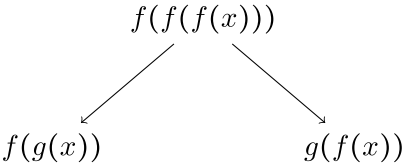

- $f(f(f(x))) > f(g(x))$ and $f(f(f(x))) > g(f(x))$, but $f(g(x)) \not\downarrow_{E^>} g(f(x))$.

Critical Pairs

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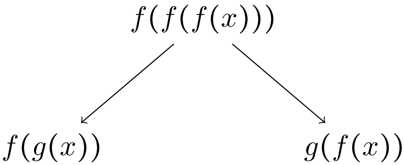

- $f(f(f(x))) > f(g(x))$ and $f(f(f(x))) > g(f(x))$, but $f(g(x)) \not\downarrow_{E^{>}} g(f(x))$.
- E is not ground convergent with respect to $>$.

Adding Critical Pairs to Equations

- Since critical pairs are equational consequences, adding a critical pair to the set of equations does not change the induced equational theory.
- If E' is obtained from E by adding a critical pair, then $\approx_E = \approx_{E'}$.
- The idea of adding a critical pair as a new equation is called "completion".

Convergence

Example 6

- Let $E' := \{f(f(x)) \approx g(x), f(g(x)) \approx g(f(x))\}$
- Let $>$ be the LPO with $f > g$.

Convergence

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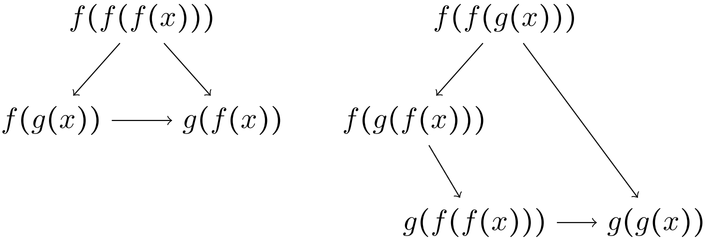
- Let $E' := \{f(f(x)) \approx g(x), f(g(x)) \approx g(f(x))\}$
- Let $>$ be the LPO with $f > g$.
- E' has two critical pairs. Both are joinable:



Convergence

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- Let $E' := \{f(f(x)) \approx g(x), f(g(x)) \approx g(f(x))\}$
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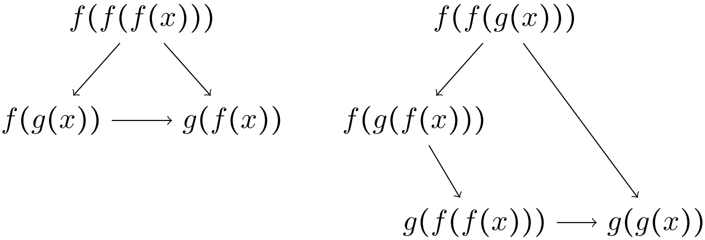

- E' is (ground) convergent.

Ordered Completion

- Described as a set of inference rules.
- Parametrized by a reduction ordering $>$.
- Works on pairs (E, R), where E is a set of equations and R is a set of rewrite rules.
- $E; R \vdash E'; R'$ means that $E'; R'$ can be obtained from $E; R$ by applying a completion inference.

Ordered Completion: Notions

- Derivation: A (finite or countably infinite) sequence $(E_0; R_0) \vdash (E_1; R_1) \cdots$.
- Usually, E_0 is the set of initial equations and $R_0 = \emptyset$.
- The limit of a derivation: the pair $E_\omega; R_\omega$, where

$$E_\omega := \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j \text{ and } R_\omega := \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j.$$

- Goal: to obtain a limit system that is ground convergent.

Ordered Completion: Notation

- $\uplus$: Disjoint union
- $s \rhd t$: Strict encompassment relation. An instance of t is a subterm of s, but not vice versa.
- $s \approxeq t$ stands for $s \approx t$ or $t \approx s$.
- $CP_{>}(E \cup R)$: The set of all ordered critical pairs, with the ordering $>$, generated by equations in E and rewrite rules in R treated as equations.

Ordered Completion: Rules

DEDUCTION: $E; R \vdash E \cup \{s \approx t\}; R$
if $s \approx t \in CP_{>}(E \cup R)$.

ORIENTATION: $E \uplus \{s \mathrel{\dot{\approx}} t\}; R \vdash E; R \cup \{s \to t\}$, if $s > t$.

DELETION: $E \uplus \{s \approx s\}; R \vdash E; R$.

COMPOSITION: $E; R \uplus \{s \to t\} \vdash E; R \cup \{s \to r\}$,
if $t \to_{R \cup E^{>}} r$.

Ordered Completion: Rules

SIMPLIFICATION: $E \cup \{s \approxeq t\}; R \vdash E \cup \{u \approx t\}; R,$

if $s \to_R u$ or $s \to_{E^>} u$ with $l\sigma \to r\sigma$

for $l \approxeq r \in E, s \rhd l$.

COLLAPSE: $E; R \uplus \{s \to t\} \vdash E \cup \{u \approx t\}; R,$

if $s \to_R u$ or $s \to_{E^>} u$ with $l\sigma \to r\sigma$

for $l \approxeq r \in E, s \rhd l$.

Ordered Completion: Properties

Theorem 2

Let $(E_0; R_0), (E_1; R_1), \ldots$ be an ordered completion derivation where all critical pairs are eventually generated (a fair derivation). Then these three properties are equivalent for all ground terms s and t:

(1) $E_0 \models s \approx t$.

(2) $s \downarrow_{E_\omega^> \cup R_\omega} t$.

(3) $s \downarrow_{E_i^> \cup R_i} t$ *for some* $i \geq 0$.

This theorem, in particular, asserts the refutational completeness of ordered completion.

Proving by Ordered Completion: Example

Given:

1. $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$.
2. $x \cdot e \approx x$.
3. $x \cdot i(x) \approx e$.
4. $x \cdot x \approx e$.

Prove

Goal: $x \cdot y \approx y \cdot x$.

Proving by Ordered Completion: Example

Proof by ordered completion:

- Skolemize the goal: $a \cdot b \approx b \cdot a$.
- Take LPO as the reduction ordering with the precedence $i > f > e > a > b$
- $E_0 := \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot e \approx x,\ x \cdot i(x) \approx e,\ x \cdot x \approx e\}$
- $R_0 := \varnothing$
- Start applying the rules.

Proving by Ordered Completion: Example

$$E_0 = \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot e \approx x,\ x \cdot i(x) \approx e,\ x \cdot x \approx e\}$$
$$R_0 = \varnothing$$

Apply ORIENT 4 times:

$$E_4 = \varnothing$$
$$R_4 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e\}$$

Proving by Ordered Completion: Example

$$E_0 = \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot e \approx x,\ x \cdot i(x) \approx e,\ x \cdot x \approx e\}$$
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Apply ORIENT 4 times:

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$$R_4 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e\}$$

Apply DEDUCE with the rules $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ and $x \cdot e \to x$ to the overlapping term $(x \cdot e) \cdot z$, and then ORIENT:

$$E_6 = \varnothing$$
$$R_6 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_6 = \varnothing$$

$$R_6 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2\}$$

Apply Deduce with the rules $x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2$ and $x \cdot i(x) \to e$ to the overlapping term $x_1 \cdot (e \cdot i(e))$:

$$E_7 = \{x_1 \cdot i(e) \approx x_1 \cdot e\}$$

$$R_7 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_7 = \{x_1 \cdot i(e) \approx x_1 \cdot e\}$$
$$R_7 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2\}$$

Apply Orient to $x_1 \cdot i(e) \approx x_1 \cdot e$ and then Composition with the rule $x \cdot e \to x$:

$$E_9 = \varnothing$$
$$R_9 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x\}$$

Proving by Ordered Completion: Example

$$E_9 = \varnothing$$
$$R_9 = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x\}$$

Apply DEDUCE with the rules $x \cdot x \to e$ and $x \cdot i(e) \to x$ to the overlapping term $e \cdot i(e)$, and then ORIENT:

$$E_{11} = \varnothing$$
$$R_{11} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x,\ i(e) \to e\}$$

Proving by Ordered Completion: Example

$$E_{11} = \varnothing$$

$$R_{11} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ x \cdot i(e) \to x,\ i(e) \to e\}$$

Apply COLLAPSE to $x \cdot i(e) \to x$ with $i(e) \to e$:

$$E_{12} = \{x \cdot e \approx x\}$$

$$R_{12} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Proving by Ordered Completion: Example

$$E_{12} = \{x \cdot e \approx x\}$$

$$R_{12} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Apply SIMPLIFICATION to $x \cdot e \approx x$ with $x \cdot e \to x$ and then DELETE to the obtained $x \approx x$:

$$E_{14} = \varnothing$$

$$R_{14} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Proving by Ordered Completion: Example

$$E_{14} = \varnothing$$

$$R_{14} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e\}$$

Apply DEDUCE to $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ and $x \cdot i(x) \to e$ with the overlapping term $(x \cdot i(x)) \cdot z$ and then ORIENT:

$$E_{16} = \varnothing$$

$$R_{16} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{16} = \emptyset$$

$$R_{16} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Apply DEDUCE to $x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2$ and $x \cdot x \to e$ with the overlapping term $x_1 \cdot (i(x_1) \cdot i(x_1))$:

$$E_{17} = \{e \cdot i(x) \approx x \cdot e\}$$

$$R_{17} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{17} = \{e \cdot i(x) \approx x \cdot e\}$$
$$R_{17} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2\}$$

Apply Simplification to $e \cdot i(x) \approx x \cdot e$ with $x \cdot e \to x$ and then Orient:

$$E_{19} = \emptyset$$
$$R_{19} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot i(x) \to x\}$$

Proving by Ordered Completion: Example

$$E_{19} = \varnothing$$

$$R_{19} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot i(x) \to x\}$$

Apply DEDUCE to $x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2$ and $e \cdot i(x) \to x$ with the overlapping term $x_1 \cdot (e \cdot i(x_2))$ and then ORIENT:

$$E_{21} = \varnothing$$

$$R_{21} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,\ x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{21} = \emptyset$$
$$R_{21} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot i(x) \to e,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Applying COLLAPSE, SIMPLIFICATION, and DELETE, we get rid of $x \cdot i(x) \to e$:

$$E_{24} = \emptyset$$
$$R_{24} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,$$
$$e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}$$

Proving by Ordered Completion: Example

$$E_{24} = \emptyset$$

$$\begin{aligned} R_{24} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ &e \cdot i(x) \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\} \end{aligned}$$

Applying Collapse and Orient, we replace $e \cdot i(x) \to x$ with $e \cdot x \to x$:

$$E_{26} = \emptyset$$

$$\begin{aligned} R_{26} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ &e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{26} = \emptyset$$

$$\begin{aligned} R_{26} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ & e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2 \} \end{aligned}$$

Applying COLLAPSE and DELETE, we get rid of $x_1 \cdot (e \cdot x_2) \to x_1 \cdot x_2$:

$$E_{28} = \emptyset$$

$$\begin{aligned} R_{28} = \{ & (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ & i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ & e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2 \} \end{aligned}$$

Proving by Ordered Completion: Example

$$
\begin{aligned}
E_{28} &= \varnothing \\
R_{28} &= \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\
&\qquad i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\
&\qquad e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}
\end{aligned}
$$

Apply DEDUCE to $e \cdot x \to x$ and $x_1 \cdot i(x_2) \to x_1 \cdot x_2$ with the overlapping term $e \cdot i(x_2)$:

$$
\begin{aligned}
E_{29} &= \{i(x_1) \approx e \cdot x_2\} \\
R_{29} &= \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\
&\qquad i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\
&\qquad e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\}
\end{aligned}
$$

Proving by Ordered Completion: Example

$$E_{29} = \{i(x_2) \approx e \cdot x_2\}$$

$$\begin{aligned} R_{29} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ &e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2\} \end{aligned}$$

Apply SIMPLIFICATION to $i(x_1) \approx e \cdot x_2$ with $e \cdot x \to x$ and then ORIENT:

$$E_{31} = \emptyset$$

$$\begin{aligned} R_{31} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2, \\ &e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{31} = \varnothing$$

$$R_{31} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\ i(e) \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}$$

Apply COLLAPSE and DELETE, we get rid of $i(e) \to e$:

$$E_{33} = \varnothing$$

$$R_{33} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}$$

Proving by Ordered Completion: Example

$$E_{33} = \emptyset$$

$$\begin{aligned} R_{33} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2,\ e \cdot x \to x, \\ &x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\} \end{aligned}$$

Applying Composition, we replace $x_1 \cdot (i(x_1) \cdot x_2) \to e \cdot x_2$ by $x_1 \cdot (i(x_1) \cdot x_2) \to x_2$:

$$E_{34} = \emptyset$$

$$\begin{aligned} R_{34} = \{&(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\ &x_1 \cdot (i(x_1) \cdot x_2) \to x_2,\ e \cdot x \to x, \\ &x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\} \end{aligned}$$

Proving by Ordered Completion: Example

$$E_{34} = \varnothing$$

$$R_{34} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (i(x_1) \cdot x_2) \to x_2,\ e \cdot x \to x,$$
$$x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}$$

Applying SIMPLIFICATION and ORIENT, we replace $x_1 \cdot (i(x_1) \cdot x_2) \to x_2$ by $x_1 \cdot (x_1 \cdot x_2) \to x_2$:

$$E_{36} = \varnothing$$

$$R_{36} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,$$
$$x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}$$

Proving by Ordered Completion: Example

$$E_{36} = \varnothing$$
$$R_{36} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\ x_1 \cdot (i(x_1) \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x\}$$

Apply DEDUCE to $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ and $x \cdot x \to e$ with the overlapping term $(x_1 \cdot x_2) \cdot (x_1 \cdot x_2)$, then ORIENT:

$$E_{37} = \varnothing$$
$$R_{37} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\ x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e\}$$

Proving by Ordered Completion: Example

$$E_{37} = \emptyset$$

$$R_{37} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,$$
$$i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e\}$$

Apply DEDUCE to $x_1 \cdot (x_1 \cdot x_2) \to x_2$ and $x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e$ with the overlapping term $x_1 \cdot (x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)))$, then ORIENT:

$$E_{39} = \emptyset$$

$$R_{39} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,$$
$$i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e\}$$

Proving by Ordered Completion: Example

$$E_{39} = \emptyset$$

$$R_{39} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,$$
$$i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e\}$$

Apply Composition to $x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e$ with $x \cdot e \to x$:

$$E_{40} = \emptyset$$

$$R_{40} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,$$
$$x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,$$
$$i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1\}$$

Proving by Ordered Completion: Example

$$E_{41} = \emptyset$$

$$R_{41} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\ x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1\}$$

Apply Deduce to $x_1 \cdot (x_1 \cdot x_2) \to x_2$ and $x_2 \cdot (x_1 \cdot x_2) \to x_1$ with the overlapping term $x_2 \cdot (x_2 \cdot (x_1 \cdot x_2))$:

$$E_{42} = \{x_1 \cdot x_2 \approx x_2 \cdot x_1\}$$

$$R_{42} = \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e,\ x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2,\ i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e\}$$

Proving by Ordered Completion: Example

$$
\begin{aligned}
E_{42} &= \{x_1 \cdot x_2 \approx x_2 \cdot x_1\} \\
R_{42} &= \{(x \cdot y) \cdot z \to x \cdot (y \cdot z),\ x \cdot e \to x,\ x \cdot x \to e, \\
&\qquad x_1 \cdot (x_1 \cdot x_2) \to x_2,\ e \cdot x \to x,\ x_1 \cdot i(x_2) \to x_1 \cdot x_2, \\
&\qquad i(x) \to x,\ x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) \to e,\ x_2 \cdot (x_1 \cdot x_2) \to x_1 \cdot e\}
\end{aligned}
$$

The equation $x_1 \cdot x_2 \approx x_2 \cdot x_1$ joins the goal $a \cdot b \approx b \cdot a$. Hence, the goal is proved.