

Automated Reasoning - WS 2013

Lecture 3: Examples

Mădălina Erăscu and Tudor Jebelean
 Research Institute for Symbolic Computation,
 Johannes Kepler University, Linz, Austria
 {merascu,tjebelea}@risc.jku.at

October 23, 2013

Example 1 (Semantics) Let

$$F : \iff \forall_{x,y} \exists x + y > c$$

$$I : \begin{cases} D = \{0, 1\} \\ c_I = 0 \\ +_I \rightarrow +_{\mathbb{Z}} \\ >_I \rightarrow >_{\mathbb{Z}} \end{cases}$$

Solution. $\langle F \rangle_I = \mathbb{T}$ iff for each $d \in D$:

$$\left\langle \exists_y x + y > c \right\rangle_{\sigma \uplus \{x \rightarrow d\}}^I = \mathbb{T}$$

- **Case $x \rightarrow 0$.** We have

$$\left\langle \exists_y x + y > c \right\rangle_{\{x \rightarrow 0\}}^I = \mathbb{T} \text{ iff for some } d \in D : \langle x + y > c \rangle_{\{x \rightarrow 0\} \uplus \{y \rightarrow d\}}^I = \mathbb{T}$$

- **Case $y \rightarrow 0$.** We have

$$\begin{aligned} & \langle x + y > c \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \\ \rightsquigarrow & >_{\mathbb{Z}} \left[\langle x + y \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I, \langle c \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \right] \\ \rightsquigarrow & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I, \langle y \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \right], 0 \right] \\ \rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [0, 0], 0] \\ \rightsquigarrow & >_{\mathbb{Z}} [0, 0] \\ \rightsquigarrow & \mathbb{F} \end{aligned}$$

– Case $y \rightarrow 1$. We have

$$\begin{aligned}
& \langle x + y > c \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \\
\rightsquigarrow & >_{\mathbb{Z}} \left[\langle x + y \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I, \langle c \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \right] \\
\rightsquigarrow & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I, \langle y \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \right], 0 \right] \\
\rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [0, 1], 0] \\
\rightsquigarrow & >_{\mathbb{Z}} [1, 0] \\
\rightsquigarrow & \mathbb{T}
\end{aligned}$$

• Case $x \rightarrow 1$. We have

$$\left\langle \exists_y x + y > c \right\rangle_{\{x \rightarrow 1\}}^I = \mathbb{T} \text{ iff for some } d \in D : \langle x + y > c \rangle_{\{x \rightarrow 1\} \cup \{y \rightarrow d\}}^I = \mathbb{T}$$

– Case $y \rightarrow 0$. We have

$$\begin{aligned}
& \langle x + y > c \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \\
\rightsquigarrow & >_{\mathbb{Z}} \left[\langle x + y \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I, \langle c \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \right] \\
\rightsquigarrow & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I, \langle y \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \right], 0 \right] \\
\rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [1, 0], 0] \\
\rightsquigarrow & >_{\mathbb{Z}} [1, 0] \\
\rightsquigarrow & \mathbb{T}
\end{aligned}$$

– Case $y \rightarrow 1$. We have ...

Example 2 (CNF) Prove the following by bringing the formulas into CNF

$$\left(\forall_x P[x] \right) \Rightarrow Q \equiv \exists_x (P[x] \Rightarrow Q).$$

Solution. We have

$$\left(\forall_x P[x] \right) \Rightarrow Q \equiv \neg \left(\forall_x P[x] \right) \vee Q \equiv \left(\exists_x \neg P[x] \right) \vee Q \equiv \exists_x (\neg P[x] \vee Q)$$

Further we have

$$\exists_x (P[x] \Rightarrow Q) \equiv \exists_x (\neg P[x] \vee Q)$$

Example 3 (Skolem Standard Form) Bring the following formula into Skolem Standard Form

$$\forall_x \exists_{y,z} ((\neg P[x, y] \wedge Q[x, z]) \vee R[x, y, z])$$

Solution.

$$\begin{aligned}
& \forall x \exists y, z ((\neg P[x, y] \wedge Q[x, z]) \vee R[x, y, z]) \\
\iff & \forall x \exists y, z ((\neg P[x, y] \vee R[x, y, z]) \wedge (Q[x, z] \vee R[x, y, z])) \\
\rightsquigarrow & \forall x ((\neg P[x, f[x]] \vee R[x, f[x], g[x]]) \wedge (Q[x, g[x]] \vee R[x, f[x], g[x]]))
\end{aligned}$$

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Example 4 (Skolem Standard Form) Bring the following formula into Skolem Standard Form

$$\forall x, y \left(\exists z P[x, z] \wedge P[y, z] \right) \Rightarrow \exists u Q[x, y, u]$$

Solution.

$$\begin{aligned}
& \forall x, y \left(\exists z P[x, z] \wedge P[y, z] \right) \Rightarrow \exists u Q[x, y, u] \\
\iff & \forall x, y \neg \left(\exists z P[x, z] \wedge P[y, z] \right) \vee \exists u Q[x, y, u] \\
\iff & \forall x, y \left(\forall z \neg P[x, z] \vee \neg P[y, z] \right) \vee \exists u Q[x, y, u] \\
\iff & \forall x, y, z \left(\neg P[x, z] \vee \neg P[y, z] \right) \vee \exists u Q[x, y, u] \\
\iff & \forall x, y, z \exists u \left(\neg P[x, z] \vee \neg P[y, z] \right) \vee Q[x, y, u] \\
\rightsquigarrow & \forall x, y, z \neg P[x, z] \vee \neg P[y, z] \vee Q[x, y, f[x, y, z]]
\end{aligned}$$

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Example 5 ((Un)Satisfiability & (In)Validity) Prove that the formula

$$\forall x P[x] \wedge \exists y \neg P[y]$$

is inconsistent.

Solution. We have

$$\forall x P[x] \wedge \exists y \neg P[y] \equiv \forall x P[x] \wedge \neg \left(\forall y P[y] \right) \equiv \forall x P[x] \wedge \neg \left(\forall x P[x] \right) \equiv \mathbb{F}$$

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Example 6 Prove that the formula

$$\forall x P[x] \Rightarrow \exists y P[y]$$

is valid.

Solution. We assume that the formula is invalid and derive a contradiction. Hence, it exists an interpretation I under which the formula is false. That is

$$\begin{aligned} \langle \forall_x P[x] \rangle^I &= \mathbb{T} \\ \wedge \\ \langle \exists_y P[y] \rangle^I &= \mathbb{F} \rightsquigarrow \langle \forall_y \neg P[y] \rangle^I = \mathbb{T} \end{aligned}$$

From the above we obtain that

$$\langle \forall_x P[x] \rangle^I = \langle \forall_y \neg P[y] \rangle^I \text{ which is a contradiction}$$

